This week we’ll discuss two new topics: we’ll finish up basic vector geometry by talking about planes, and then we’ll make our first step towards vector calculus by introducing a small library of parametrized curves.

Planes
Say we choose a point \( P_0 = (x_0, y_0, z_0) \) in \( \mathbb{R}^3 \) and also a vector \( \mathbf{n} = \langle a, b, c \rangle \) and we allow ourselves to move from \( P_0 \) to any other point \( Q \in \mathbb{R}^3 \), so long as the vector \( \overrightarrow{P_0Q} \) is perpendicular to \( \mathbf{n} \). The set of all such points \( Q \) is evidently a plane in 3-space. If we write \( Q \) as \( Q = (x, y, z) \) then the requirement that \( \overrightarrow{P_0Q} \) be perpendicular to \( \mathbf{n} \) may be written
\[
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.
\]
That is,
\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,
\]
so
\[
ax + by + cz = d, \tag{1}
\]
where \( d = ax_0 + by_0 + cz_0 \). Notice the similarity between this last expression and the standard equation \( Ax + By = C \) for a line in \( \mathbb{R}^2 \).

Planes are similar to lines in that they are determined (in \( \mathbb{R}^3 \)) by a point and a direction. If we want to specify a line by giving points through which it passes, two suffice. That is, given two distinct points in \( \mathbb{R}^3 \), there is a unique line passing through those points. Three distinct, non-collinear points in \( \mathbb{R}^3 \) uniquely determine a plane. For example, say we have the following points:
\[
P = (3, 0, 0), \quad Q = (0, 2, 0), \quad \text{and} \quad R = (0, 0, 5).
\]
These three points determine the plane \( \mathcal{P} \) seen in Figure 1. To produce an equation of the form (1) for this plane, we need to find a vector \( \mathbf{n} \) which is normal to \( \mathcal{P} \). We can do this by observing that \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are non-parallel vectors in the plane \( \mathcal{P} \). Since \( \overrightarrow{PQ} \times \overrightarrow{PR} \) is perpendicular to both of these vectors, \( \overrightarrow{PQ} \times \overrightarrow{PR} \) is perpendicular to \( \mathcal{P} \), and thus serves as a perfectly fine normal vector. We have
\[
\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 2 & 0 \\ -3 & 0 & 5 \end{vmatrix} = (10, 15, 6).
\]
Now if \( S = (x, y, z) \) is a point on \( \mathcal{P} \) we should have
\[
0 = \mathbf{n} \cdot \overrightarrow{PS} = (10, 15, 6) \cdot (x - 3, y, z) = 10(x - 3) + 15y + 6z.
\]
We can rearrange the above equation to obtain
\[
10x + 15y + 6z = 30,
\]
an equation for our plane in standard form. Notice that the normal vector $\langle 10, 15, 6 \rangle$ can be read off from the coefficients of this equation.

Certainly if $\mathbf{n}$ is perpendicular to our plane, then so is $-\mathbf{n}$. Also, using $Q$ or $R$ as the basepoint in the above calculations shouldn’t change our plane $P$. It might be a good exercise to see what happens to our final equation if you replace $\mathbf{n}$ with $-\mathbf{n}$ or $P$ with $Q$.

As is the case with lines in $\mathbb{R}^2$ (but not $\mathbb{R}^3$), planes in $\mathbb{R}^3$ are either parallel or intersect. We say that a pair of planes is parallel if they have the same direction vectors — that is, if their normal vectors are parallel. For instance, the planes determined by the equations

\[ 12x + 15y + 6z = 21 \quad \text{and} \quad 4x + 5y + 2z = 9 \]

are parallel, since their normal vectors are $\langle 12, 15, 6 \rangle$ and $\langle 4, 5, 2 \rangle$, respectively. Notice that these two planes will never intersect, since their equations could never be simultaneously true.

Remember that when two non-parallel lines (1-dimensional objects) in $\mathbb{R}^2$ intersect, their intersection is a point (a 0-dimensional object). When a pair of non-parallel planes (2-dimensional objects) in $\mathbb{R}^3$ intersects, the result is a line (a 1-dimensional object). For instance, consider the planes

\[ ax + by + cz = d \quad \text{and} \quad z = 0, \]

where at least one of $a$ and $b$ is not zero. A point $(x, y, z)$ lies in both of these planes if it satisfies the two equations simultaneously. This will occur if $z = 0$ and $ax + by = d$ — but this is just the equation of a line in the $xy$-plane.

We recently developed a preferred way of working with lines in $\mathbb{R}^3$ — vector parametrizations and parametric equations. For this reason, a not uncommon problem is one where we need to parametrize the line that lies at the intersection of two planes.

**Example 1.** (§13.5, Exercise 65 of the textbook) Let $\mathcal{L}$ denote the intersection of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. Find parametric equations for the line $\mathcal{L}$. 

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(Solution) Let’s call the plane determined by \( x - y - z = 1 \), \( P_1 \), and \( P_2 \) will be the plane determined by \( 2x + 3y + z = 2 \). The normal vectors of \( P_1 \) and \( P_2 \) will be \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \), respectively. Notice that

\[
\mathbf{n}_1 = (1, -1, -1) \quad \text{and} \quad \mathbf{n}_2 = (2, 3, 1).
\]

Now a vector parametrization of \( \mathcal{L} \) has the form

\[
\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},
\]

and because \( \mathcal{L} \) lies at the intersection of the two planes, \( \mathbf{v} \) is parallel to \( P_1 \) and \( P_2 \). But this means that \( \mathbf{v} \) is perpendicular to both \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \). We have a quick way to produce a vector perpendicular to a pair of vectors: let’s set \( \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 \). That is,

\[
\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 3 & 1 \end{vmatrix} = (2, -3, 5).
\]

So \( \mathcal{L} \) is parametrized by

\[
\mathbf{r}(t) = \mathbf{r}_0 + t(2, -3, 5),
\]

where \( \mathbf{r}_0 \) is any point on \( \mathcal{L} \). To find such a point, we must find a point satisfying the equations of \( P_1 \) and \( P_2 \). For instance, we can declare that the \( z \)-coordinate of \( \mathbf{r}_0 \) should be 0 (or any other value we want), reducing the equations to

\[
x - y = 1 \quad \text{and} \quad 2x + 3y = 2.
\]

This system of equations is satisfied when \( x = 1 \) and \( y = 0 \), so we see that the point \( (1, 0, 0) \) lies in the intersection of \( P_1 \) and \( P_2 \), and thus on \( \mathcal{L} \). Finally we can write

\[
\mathbf{r}(t) = (1, 0, 0) + t(2, -3, 5)
\]

as a parametrization of \( \mathcal{L} \).
A Basic Library of Parametrized Curves

We recently gained some skills used to parametrize lines in \( \mathbb{R}^3 \). It would be nice if we could parametrize any curve we happen to come across in \( \mathbb{R}^3 \), but this goal is probably a bit too lofty. What we’ll do now is discuss some of the most basic parametrizations with which we ought to be familiar, and our hope is that from these we can build most of the parametrizations we might need later on.

Circles

We often want to parametrize the circle centered at some point \( P_0 \in \mathbb{R}^3 \) lying in a particular plane in \( \mathbb{R}^3 \). As with lines, we’ll develop our ability to parametrize circles by first tackling the \( \mathbb{R}^2 \) case. Remember that the circle in the \( xy \)-plane of radius \( r \) centered at the origin (as seen in Figure 2a) is the solution set of the equation

\[
x^2 + y^2 = r^2.
\]

Also remember that any point on the unit circle can be written as \((\cos t, \sin t)\), and thus any point on the radius-\( r \) circle can be written as \((r \cos t, r \sin t)\). So we have our first parametrization:

\[
r(t) = (r \cos t, r \sin t), \quad t \in [0, 2\pi].
\]  

(2)

Perhaps instead of being centered at the origin, our circle is centered at a point \((h, k) \in \mathbb{R}^2\), as in Figure 2b. Then each point on this new circle corresponds to a point on the original circle, but displaced by the vector \((h, k)\). We thus obtain a parametrization for this curve by adding the vector \((h, k)\) to (2):

\[
r(t) = (r \cos t, r \sin t) + (h, k), \quad t \in [0, 2\pi].
\]  

(3)

The question remains, though, of how to parametrize an arbitrary circle in \( \mathbb{R}^3 \). For this, we rethink the parametrization (3). If, for some reason, we really hated component notation, we could rewrite this parametrization as

\[
r(t) = r \cos ti + r \sin tj + p,
\]
where \( p = \langle h, k \rangle \) is the vector representing the point \((h, k)\). The defining characteristic of a circle centered at \((h, k)\) is that the distance from any point on the circle to \((h, k)\) is constant. Indeed, we have

\[
\|r(t) - p\|^2 = (r \cos t i + r \sin t j) \cdot (r \cos t i + r \sin t j) = r^2 \cos^2 t + r^2 \sin^2 t = r^2,
\]

because \(i\) and \(j\) are unit vectors with the property that \(i \cdot j = 0\). (We call such vectors orthonormal.) But \(i\) and \(j\) aren’t special in this regard: if \(v\) and \(w\) are any unit vectors in \(\mathbb{R}^3\) with \(v \cdot w = 0\), then

\[
r(t) = r \cos t v + r \sin t w + p, \quad t \in [0, 2\pi]
\]

parametrizes the circle of radius \(r\) centered at \(p \in \mathbb{R}^3\) and contained in the plane spanned by \(v\) and \(w\).

**Example 2.** Parametrize the circle of radius 2 centered at \((1, 1, 1)\) and parallel to the \(xz\)-plane. (See Figure 3.)

*(Solution)* If we hope to mimic the parametrization in (4) we need to produce orthonormal vectors \(v, w\) spanning the plane in which our circle lies. Since our circle is parallel to the \(xz\)-plane, we may choose

\[
v = \langle 1, 0, 0 \rangle \quad \text{and} \quad w = \langle 0, 0, 1 \rangle,
\]

which form an orthonormal basis for the \(xz\)-plane. Because our center lies at \(p = \langle 1, 1, 1 \rangle\), we have

\[
r(t) = 2 \cos t \langle 1, 0, 0 \rangle + 2 \sin t \langle 0, 0, 1 \rangle + \langle 1, 1, 1 \rangle = \langle 2 \cos t + 1, 1, 2 \sin t + 1 \rangle, \quad t \in [0, 2\pi]
\]

as our parametrization.

\[\quad\]

**Ellipses**

A generalization of circles is found in ellipses. Remember that an ellipse with axes on the \(x\)- and \(y\)-axes satisfies the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

for some \(a, b > 0\). For the purposes of finding a parametrization of an ellipse, we’ll forget the nice analytic definition and think of an ellipse as a “stretched circle.” In the \(x\)-direction, we stretch our circle by a factor of \(a\), and by a factor of \(b\) in the \(y\)-direction. For this reason, the ellipse whose equation is given above (and which can be seen in Figure 4a) has parametrization

\[
r(t) = \langle a \cos t, b \sin t \rangle, \quad t \in [0, 2\pi].
\]
Figure 5: The intersection of the surfaces $y^2/4 + z^2/9 = 1$ and $x = y + z$.

As with the circle, we can displace our ellipse (see Figure 4b) so that it has center $(h, k)$ by adding the corresponding vector to our parametrization:

$$\mathbf{r}(t) = (a \cos t, b \sin t) + (h, k), \quad t \in [0, 2\pi].$$

Also as with the circle, the vectors $\mathbf{i}$ and $\mathbf{j}$ aren’t particularly special among orthonormal vectors in $\mathbb{R}^3$. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be any pair of orthonormal vectors, and let $\mathbf{p} \in \mathbb{R}^3$ be arbitrary. Then the ellipse centered at $\mathbf{p}$ with semi-minor and semi-major axes\(^1\) of lengths $a, b > 0$ along $\mathbf{v}, \mathbf{w}$ has parametrization

$$\mathbf{r}(t) = a \cos t \mathbf{v} + b \sin t \mathbf{w} + \mathbf{p}, \quad t \in [0, 2\pi].$$

(5)

**Example 3.** Parametrize the intersection of the surfaces

$$\frac{y^2}{4} + \frac{z^2}{9} = 1 \quad \text{and} \quad x = y + z.$$ 

See Figure 5.

*(Solution)* We immediately notice that the first surface has the equation of an ellipse in the $yz$-plane; the surface will therefore be a cylinder in the $x$-direction over an ellipse in the $yz$-plane. Let’s start by parametrizing the relevant ellipse. From our above discussion we should be able to identify the parametric equations

$$y(t) = 2 \cos t \quad \text{and} \quad z(t) = 3 \sin t$$

as outlining the ellipse in question, where $t \in [0, 2\pi]$. It remains to find the corresponding parametric equation for $x$. But since points on our curve must satisfy $x = y + z$, we have

$$x(t) = 2 \cos t + 3 \sin t.$$ 

\(^1\)Don’t worry about the minor/major-ness of the axes — the axis in the $\mathbf{v}$ direction has length $2a$ and the axis in the $\mathbf{w}$ direction has length $2b$. 
Hyperbolas

The last basic curve we’ll parametrize is a close cousin of the ellipse. Remember that a hyperbola with vertices on the \( x \)-axis satisfies the equation

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]  
(6)

for some real numbers \( a, b \). We were able to parametrize ellipses using trigonometric functions by taking advantage of the identity

\[
\cos^2 t + \sin^2 t = 1
\]  
(7)

for all \( t \). The hyperbolic cosine and hyperbolic sine functions satisfy a similar equality that is more useful in the present case:

\[
\cosh^2 t - \sinh^2 t = 1
\]  
(8)

We can express the hyperbolic sine and cosine functions using exponential functions:

\[
\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2},
\]

but we won’t often have a need for this. Because of the similarity between the identities (7) and (8), we quickly see that we can parametrize (part of) the hyperbola determined by (6) as follows:

\[
r(t) = (a \cosh t, b \sinh t), \quad t \in \mathbb{R}.
\]

Some important observations: Because the hyperbola consists of two disconnected curves, we can only parametrize one component at a time. Since \( \cosh t > 0 \) for all values of \( t \), the above parametrization gives the component on the right side of the \( y \)-axis. To get the other curve, negate...
the $x$-component of the parametrization. Another thing to notice is that our parameter $t$ must assume all real values in order to trace out our curve, unlike the bounded parameter situations above.

If the vertices of our hyperbola instead lie on the $y$-axis, our equation has the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$ 

In this case, we simply swap the roles of $x$ and $y$ in the parametrization given above and write

$$\mathbf{r}(t) = (b \sinh t, a \cosh t), \quad t \in \mathbb{R}.$$ 

The generalization to three dimensions follows the pattern exhibited above.

**Example 4.** Parametrize the portion of the intersection of the surfaces

$$\frac{(y - 2)^2}{9} - \frac{x^2}{16} = 1 \quad \text{and} \quad z = x^2 - y^2$$

which satisfies $y > 0$. See Figure 7.

*(Solution)* We begin by observing that $(y - 2)^2/9 - x^2/16 = 1$ is the equation of a hyperbola in the $xy$-plane. This hyperbola opens in the $y$-direction and is centered at $(0, 2)$, so we have parametric equations

$$x(t) = 4 \sinh t \quad \text{and} \quad y(t) = 3 \cosh t + 2.$$ 

Because $\cosh t > 0$, these equations give the portion of the hyperbola with $y > 0$. To determine a parametric equation for $z$, we simply notice that we must satisfy $z = x^2 - y^2$, so we have

$$z(t) = 16 \sinh^2 t - 9 \cosh^2 t - 12 \cosh t - 4 = 7 \sinh^2 t - 12 \cosh t - 13.$$ 

We then arrive at the parametrization

$$\mathbf{r}(t) = (4 \sinh t, 3 \cosh t + 2, 7 \sinh^2 t - 12 \cosh t - 13),$$

with $-\infty < t < \infty$. $\diamondsuit$