Example 1. Suppose you are hiking on a terrain modeled by \( z = xy + y^3 - x^2 \), and you are at the point \((2, 1, -1)\).

(a) Determine the slope you would encounter if you headed due west from your position. What angle of inclination does that correspond to?

(b) Determine the slope you would encounter if you headed northwest from your position? What angle of inclination does that correspond to?

(c) Determine the steepest descent you could encounter from your position. In what direction must you travel to experience this slope?

(Solution) The surface \( z = xy + y^3 - x^2 \) is the graph of the function \( f(x, y) = xy + y^3 - x^2 \), so each part of the example will make use of the gradient \( \nabla f \):

\[
\nabla f = \langle y - 2x, x + 3y^2 \rangle.
\]

In particular we care about \( \nabla f(2, 1) = \langle -3, 5 \rangle \), the gradient at \( P = (2, 1) \).

(a) If we head due west, our velocity vector \( \mathbf{v} \) (in two dimensions) will be some (positive) scalar multiple of \( \mathbf{u} = \langle -1, 0 \rangle \). The slope that we experience is given by the directional derivative in this direction:

\[
D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \langle -3, 5 \rangle \cdot \langle -1, 0 \rangle = 3.
\]

So if \( x, y \) and \( z \) are all measured in, say, meters, then our elevation will increase at a rate of 3 meters per meter if we walk due west. Notice that we didn’t have to scale \( \mathbf{u} \) when we computed the directional derivative because \( \mathbf{u} \) is already a unit vector. We find the angle of inclination for the path that heads due west as follows: \( \theta = \arctan(m) \), where \( m \) is the slope we experience and \( \theta \) is the angle of inclination. So we have

\[
\theta = \arctan(3) \approx 71.57^\circ,
\]

meaning that we’re ascending at an angle of roughly 72°.

\(^1\) Why should this equation be true? Look at the following figure:

The slope is “rise over run,” which in this case means “change in elevation over change in (planar) distance traveled.” But rise over run is clearly also how we compute \( \tan \theta \).
(b) This part will work similarly, except this time our velocity vector is parallel to \( \mathbf{v} = \langle -1, 1 \rangle \), since we’re heading northwest. The unit vector pointing in this direction is \( \mathbf{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle \), so we have

\[
D_v f(P) = D_u f(P) = \nabla f(P) \cdot \mathbf{u} = \langle -3, 5 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{8}{\sqrt{2}} = 4\sqrt{2}.
\]

The corresponding angle of inclination is

\[\theta = \arctan(4\sqrt{2}) \approx 79.98^\circ,\]

so in the northwestern direction we’re ascending at about 80°. (This is closer to being a wall than it is to being a hill...)

(c) Remember that the gradient points in the direction of greatest increase, and that \(-\nabla f\) points in the direction of greatest decrease. So from \(P\) the direction of greatest decrease is \(-\nabla f(P) = \langle 3, -5 \rangle\). The unit vector pointing in this direction is

\[\mathbf{u} = \frac{1}{\sqrt{34}} \langle 3, -5 \rangle,\]

so we have

\[
D_u f(P) = \nabla f(P) \cdot \mathbf{u} = \frac{1}{34} \langle -3, 5 \rangle \cdot \langle 3, -5 \rangle = \frac{-9 - 25}{\sqrt{34}} = -\sqrt{34}.
\]

Notice that the slope in this direction is exactly \(-\|\nabla f(P)\|\) — this is always the case. The angle of inclination in this direction of greatest decrease is

\[\arctan(-\sqrt{34}) \approx -80.27^\circ,\]

so don’t slip!

\[\diamond\]

Example 2. Let \( f(x, y) = (y/x)^3 \), considering only the part of the plane where \( x > 0 \).

(a) Show that all lines of the form \( y = mx \) are level curves for \( f \).

(b) Verify that at any point \( P = (a, b) \) in the domain, \( \nabla f(P) \) is perpendicular to the level curve passing through \( P \).

(Solution) (a) Notice that for any \( m \) we have

\[
f(m, mx) = \left( \frac{mx}{x} \right)^3 = m^3,
\]

so the value of \( f \) along \( y = mx \) depends only on \( m \). That is, \( f \) has the same value at every point on the line \( y = mx \), so \( y = mx \) is a level curve.
(b) The gradient $\nabla f$ is given by
$$\nabla f(x, y) = \left\langle -\frac{3y^3}{x^4}, \frac{3y^2}{x^3} \right\rangle,$$
so $\nabla f(P) = \langle -3b^3/a^4, 3b^2/a^3 \rangle$. On the other hand, the level curve passing through $P$ has the form $y = mx$ for some $m$. Since this line passes through $P$ we must have $b = ma$, so $m = b/a$ (we can divide by $a$, since $x$-values aren’t allowed to be zero). So the level curve passing through $P$ is the line
$$y = (b/a)x.$$ Notice that this line contains the vector $(1, b/a)$. We have
$$\nabla f(P) \cdot (1, b/a) = \left\langle -\frac{3b^3}{a^4}, \frac{3b^2}{a^3} \right\rangle \cdot \left\langle 1, \frac{b}{a} \right\rangle = -\frac{3b^3}{a^4} + \frac{3b^3}{a^4} = 0,$$ so $\nabla f(P)$ is perpendicular to $(1, b/a)$, and thus to the level curve containing $P$.

Example 3. For $x$ near 2, there is a unique value $y = r(x)$ that solves the equation $y^5 + 6xy = 13$.

(a) Show that $dy/dx = -6y/(5y^4 + 6x)$.

(b) Let $g(x) = f(x, r(x))$, where $f(x, y)$ is some function satisfying $f_x(2, 1) = 9$, $f_y(2, 1) = 17$.

Calculate $g'(2)$.

(Solution) (a) We use implicit differentiation. We can write the equation $y^5 + 6xy = 13$ as $r(x)^5 + 6xr(x) = 13$ and then differentiate with respect to $x$:
$$5r(x)^4r'(x) + 6r(x) + 6xr'(x) = 0.$$ Solving for $r'(x)$ yields $r'(x) = -6r(x)/(5r(x)^4 + 6x)$, and making the substitution $y = r(x)$ gives us $dy/dx = -6y/(5y^4 + 6x)$, as desired.

(b) According to the chain rule we have
$$g'(x) = \frac{d}{dx} (f(x, r(x))) = f_x(x, r(x)) \cdot \frac{d}{dx} (x) + f_y(x, r(x)) \cdot \frac{d}{dx} (r(x)) = f_x(x, r(x)) + f_y(x, r(x))r'(x).$$ At $x = 2$ this gives us
$$g'(2) = f_x(2, 1) + f_y(2, 1)r'(2) = 9 + 17 \left( -\frac{6}{5 + 12} \right) = 9 - 6 = 3.$$ Here we’re using the fact that $r(2) = 1$ (since the point $(2, 1)$ satisfies the original equation), and thus
$$r'(2) = -\frac{6r(2)}{5r(2)^4 + 6(2)} = -\frac{6}{17}.$$