Below are some practice questions that I think are relevant for the final exam, but be warned that these should not be taken as an indication of what you’ll see on exam day. In particular, all of these problems come from chapter 15 material — you are still responsible for older material.

Examples

1. (a) Let \( f(x, y) = x^2 - y^2 \). Draw the level curve \( k = f(x, y) \) for \( k = -2, -1, 0, 1, 2 \).
   (b) Let \( g(x, y, z) = (x - 2)^2 + y^2 + z^2 \). Describe the level surfaces of \( g \) in words. Find an equation of the level surface passing through \( (2, 3, 4) \).

2. (a) Evaluate \( \lim_{(x, y) \to (0, 0)} \frac{x^2 y^2}{x^2 + y^2} \). (Hint: Convert to polar coordinates.)
   (b) Evaluate \( \lim_{(x, y) \to (0, 1)} \tan^{-1}\left(\frac{x^2 - 1}{x^2 + (y - 1)^2}\right) \).

3. According to the ideal gas law, the pressure, temperature, and volume of a gas are related by \( P = kT/V \), where \( k \) is a constant of proportionality. Suppose that \( V \) is measured in cubic inches, \( T \) is measured in kelvins, and that for a certain gas the constant of proportionality is \( k = 10 \text{ in} \cdot \text{lb}/K \).
   (a) Find the instantaneous rate of change of pressure with respect to temperature if the temperature is 80 K and the volume remains fixed at 50 in\(^3\).
   (b) Find the instantaneous rate of change of volume with respect to pressure if the volume is 50 in\(^3\) and the temperature remains fixed at 80 K.
   (c) (Unrelated to other parts.) Approximate the percentage change in pressure if the temperature of a gas\(^1\) is increased by 3% and the volume is increased by 5%.

4. Recall that a function \( f(x, y) \) is called harmonic if
   \[
   \Delta f(x, y) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
   \]
   Suppose \( u(x, y) \) and \( v(x, y) \) each satisfy the hypotheses of Clairaut’s theorem, and that \( u \) and \( v \) satisfy the Cauchy-Riemann equations:
   \[
   \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
   \]
   Show that \( u, v, \) and \( u + v \) are harmonic.

5. Consider the production function \( f(x, y) = 60x^{3/4}y^{1/4} \), which gives the number of units of goods produced when \( x \) units of labor and \( y \) units of capital are used. Find the production level when 81 units of labor and 16 units of capital are used, and use this to approximate the production level when 80 units of labor and 18 units of capital are used.

\(^1\)Any gas, so forget the previous constant of proportionality.
6. Suppose \( u(x,y) \) and \( v(x,y) \) satisfy the Cauchy-Reimann equations (1). If \( x = r \cos \theta \) and \( y = r \sin \theta \), use the chain rule to show that
\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.
\]

This is the polar form of the Cauchy-Riemann equations.

7. Find the directional derivative of \( f(x,y,z) = \frac{z-x}{z+y} \) at \( P = (1,0,-3) \) in the direction of \( \mathbf{v} = (-6,3,-2) \).

8. Find the directional derivative of \( f(x,y) = \sqrt{xy} \) at \( P = (1,4) \) in the direction of a vector making counterclockwise angle \( \pi/3 \) with the positive \( x \)-axis.

9. On a certain mountain, the elevation \( z \) above a point \( (x,y) \) in an \( xy \)-plane at sea level is \( z = 2000 - 0.02x^2 - 0.04y^2 \), where \( x \), \( y \), and \( z \) are in meters. The positive \( x \)-axis points east, and the positive \( y \)-axis north. A climber is at the point \( ( -20, 5, 1991 ) \).
   (a) If the climber walks due west, will she begin to ascend or descend?
   (b) If the climber walks northeast, will she ascend or descend? At what rate (meters per meter)?
   (c) In what directions could the climber begin walking to travel a level path? (There are two.)

10. Prove: If the function \( f \) is differentiable at the point \( (x,y) \) and if \( D_u f(x,y) = 0 \) in two nonparallel directions, then \( D_u f(x,y) = 0 \) in all directions.

11. Show that the surfaces
\[
z = \sqrt{x^2 + y^2} \quad \text{and} \quad z = \frac{1}{10} (x^2 + y^2) + \frac{5}{2}
\]
intersect at \( (3,4,5) \) and have a common tangent plane at that point.

12. Consider the sphere \( x^2 + y^2 + z^2 = a^2 \) and the cone \( z^2 = x^2 + y^2 \). Show that at any point where these surfaces intersect, their respective normal vectors are perpendicular.

13. Locate all relative maxima, relative minima, and saddle points of
\[
f(x,y) = x^3 + y^3 - 3x - 3y,
\]
if there are any\(^2\).

\(^2\)I now know that optimization didn’t make the cut for final exam material, so maybe don’t worry about this problem.
Figure 1: Some level curves of the function $f(x, y) = x^2 - y^2$.

Solutions

1. (Solution) (a) The level curves $k = x^2 - y^2$ are the hyperbolas seen in Figure 1. The $k = 0$ level curve is the X shape given by $x^2 = y^2$. The $k = 1$ and $k = 2$ hyperbolas open horizontally, while the $k = -1$ and $k = -2$ hyperbolas open vertically.

(b) Level surfaces of $g(x, y, z)$ will have equations of the form
   
   $$c = (x - 2)^2 + y^2 + z^2,$$

   and will thus be spheres of radius $\sqrt{c}$ centered at $(2, 0, 0)$. The level surface passing through $(2, 3, 4)$ will have value
   
   $$g(2, 3, 4) = (2 - 2)^2 + 3^2 + 4^2 = 25,$$

   and thus will be determined by the equation
   
   $$25 = (x - 2)^2 + y^2 + z^2.$$

   This is the sphere of radius 5 centered at $(2, 3, 4)$.

2. (Solution) (a) As hinted, let $x = r \cos \theta$ and $y = r \sin \theta$. Letting $(x, y)$ approach $(0, 0)$ amounts to letting $r$ approach 0, so we have

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} = \lim_{r \to 0} \frac{(r^2 \cos^2 \theta)(r^2 \sin^2 \theta)}{\sqrt{r^2}} = \lim_{r \to 0} r^3 \cos^2 \theta \sin^2 \theta = 0.$$ 

   Notice that the limit is zero no matter the value of $\theta$. Indeed, if the limit depended on $\theta$ then it would depend on the path we use to approach $(0, 0)$ — so in this case the limit would not exist.

(b) Let’s consider first the limit

   $$\lim_{(x,y)\to(0,1)} \frac{x^2 - 1}{x^2 + (y - 1)^2}.$$
Notice that the numerator approaches \(-1\), and that the denominator is always non-negative. Because the denominator approaches 0 through positive numbers while the numerator approaches \(-1\), the quotient diverges to \(-\infty\). So

\[
\lim_{(x,y)\to(0,1)} \tan^{-1} \left( \frac{x^2 - 1}{x^2 + (y - 1)^2} \right) = \lim_{s \to -\infty} \tan^{-1}(s) = -\pi/2.
\]

3. (Solution) (a) We have \(P = P(T,V)\) as a function of temperature and volume, and we’re being asked to find \(\partial P/\partial T\). We have

\[
\frac{\partial P}{\partial T}(T,V) = \frac{k}{V},
\]

so \(\frac{\partial P}{\partial T}(80,50) = \frac{10}{50} = 0.2\) (lbs/in\(^2\))/K. The units here are pounds-per-square inch (the unit in which pressure is measured), per kelvin (the scale on which temperature is measured).

(b) Probably the easiest way to find \(\partial V/\partial P\) is to first write volume as a function of temperature and pressure. We have

\[
V = V(T) = \frac{kT}{P},
\]

so

\[
\frac{\partial V}{\partial P}(T,P) = -\frac{kT}{P^2}.
\]

Now when \(V = 50\) and \(T = 80\), the pressure is given by

\[
P = (10)(80)/(50) = 16\) lbs/in\(^2\),
\]

so we care about

\[
\frac{\partial V}{\partial P}(80,16) = -\frac{(10)(80)}{256} = -\frac{25}{8} = -3.125\) in\(^3\)/(lbs/in\(^2\)).
\]

Because this derivative is negative, an increase in pressure corresponds to a decrease in volume.

(c) The local linear approximation for \(P(T,V)\) near \((T_0,V_0)\) has the form

\[
P(T,V) \approx P(T_0,V_0) + P_T(T_0,V_0)(T - T_0) + P_V(T_0,V_0)(V - V_0).
\]

If we subtract the \(P(T_0,V_0)\) term over, we can consider \(\Delta T = T - T_0, \Delta V = V - V_0,\) and likewise for \(\Delta P\). Then

\[
\Delta P \approx P_T(T_0,V_0)\Delta T + P_V(T_0,V_0)\Delta V = \frac{k}{V_0}\Delta T - \frac{kT_0}{V_0^2}\Delta V.
\]

Now the percentage change in pressure is \(\Delta P/P_0\), and we have

\[
\frac{\Delta P}{P_0} \approx \frac{(k/V_0)}{(kT_0/V_0^2)}\Delta T - \frac{(kT_0/V_0^2)}{(kT_0/V_0)}\Delta V = \frac{\Delta T}{T_0} - \frac{\Delta V}{V_0}.
\]

So the percentage change in pressure is just the difference in the percentage changes in temperature and volume. In particular, if we increase temperature by 3% and also increase volume by 5%, we will decrease pressure by approximately 2%.

\(\diamondsuit\)
4. (Solution) We have
\[
\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial x} \right),
\]
with the last equality coming from the Cauchy-Riemann equations. The last expression is zero because, by Clairaut’s theorem, \(v_{yx} = v_{xy}\). So \(u\) is harmonic. A very similar computation will show that \(v\) is harmonic. Then
\[
\Delta (u + v) = \frac{\partial^2}{\partial x^2}(u + v) + \frac{\partial^2}{\partial y^2}(u + v) = \Delta u + \Delta v = 0,
\]
so \(u + v\) is also harmonic. ♦

5. (Solution) We start by computing \(f(81, 16)\). We have
\[
f(81, 16) = 60(81)^{3/4}(16)^{1/4} = 60(3)^3(2)^1 = 60 \cdot 27 \cdot 2 = 3,240 \text{ units.}
\]
To approximate \(f(80, 18)\), we’ll need to compute \(f_x(81, 16)\) and \(f_y(81, 16)\). (These quantities are known as the marginal productivity of labor and the marginal productivity of capital, respectively.) We have
\[
\frac{\partial f}{\partial x} = 60 \cdot \frac{3}{4} x^{-1/4} y^{1/4},
\]
so
\[
\frac{\partial f}{\partial x}(81, 16) = 60 \cdot \frac{3}{4}(81)^{-1/4}(16)^{1/4} = 60 \cdot \frac{3}{4} \cdot \frac{1}{3} \cdot 2 = 30.
\]
Similarly,
\[
\frac{\partial f}{\partial y} = 60 \cdot \frac{1}{4} x^{3/4} y^{-3/4},
\]
so
\[
\frac{\partial f}{\partial y}(81, 16) = 60 \cdot \frac{1}{4}(81)^{3/4}(16)^{-3/4} = 60 \cdot \frac{1}{4} (3)^3 (2)^{-3} = 15 \cdot \frac{27}{8} = 50.625.
\]
Then
\[
f(80, 18) - f(81, 16) \approx (-1)(30) + (2)(50.625) = -30 + 101.25 = 71.25.
\]
So this change to our labor-capital arrangement will see an increase in production of about 71 units of goods. In particular, the production level will rise to about 3,311.25 units. ♦

6. (Solution) Since \(u\) is a function of \(x\) and \(y\) and each of these depend on \(r\), we’ll have
\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}.
\]
We immediately have \(x_r = \cos \theta\) and \(y_r = \sin \theta\), so
\[
\frac{\partial u}{\partial r} = \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y}.
\]
On the other hand,
\[
\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta}.
\]
Since \(x_\theta = -r \sin \theta\) and \(y_\theta = r \cos \theta\), this simplifies to
\[
\frac{\partial v}{\partial \theta} = r \cos \theta \frac{\partial v}{\partial y} - r \sin \theta \frac{\partial v}{\partial x} = r \left( \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y} \right) = r \frac{\partial u}{\partial r},
\]
where the second-to-last equality comes from the Cauchy-Riemann equation \(u_y = -v_x\). Dividing by \(r\) gives us
\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta},
\]
as desired. Next,
\[
\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial v}{\partial x} + \sin \theta \frac{\partial v}{\partial y} = -\frac{1}{r} \left( r \cos \theta \frac{\partial u}{\partial y} - r \sin \theta \frac{\partial u}{\partial x} \right)
\]
which is the second of the polar Cauchy-Riemann equations.

7. (Solution) We have
\[
\nabla f(x, y, z) = \langle -\frac{1}{z+y}, -\frac{z-x}{(z+y)^2}, \frac{x+y}{(z+y)^2} \rangle,
\]
so \(\nabla f(P) = \langle \frac{1}{3}, \frac{4}{9}, \frac{1}{9} \rangle\). The unit vector in the direction of \(v\) is \(u = \frac{1}{7}v\), so
\[
D_v f(P) = \nabla f(P) \cdot u = \frac{1}{7} \left( \frac{1}{3}, \frac{4}{9}, \frac{1}{9} \right) \cdot \langle -6, 3, -2 \rangle = \frac{1}{7} \left( -\frac{6}{3} + \frac{12}{9} - \frac{2}{9} \right) = -\frac{8}{63}.
\]

8. (Solution) We start by computing the gradient:
\[
\nabla f(x, y) = \langle \sqrt{y/x/2}, \sqrt{x/y/2} \rangle,
\]
so \(\nabla f(P) = \langle 1, 1/4 \rangle\). Now let \(u\) be the unit vector in the \(xy\)-plane making counterclockwise angle \(\pi/3\) with the positive \(x\)-axis. Then
\[
\mathbf{u} = \langle \cos(\pi/3), \sin(\pi/3) \rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle,
\]
so, since \(u\) is a unit vector,
\[
D_u f(P) = \nabla f(P) \cdot u = \frac{1}{2} + \frac{\sqrt{3}}{8} = \frac{4 + \sqrt{3}}{8}.
\]
So the directional derivative of \(f\) at \(P\) in the \(\pi/3\)-direction is \((4 + \sqrt{3})/8\).
9. (Solution) (a) Let’s compute the directional derivative of \( f(x, y) = z \) in the direction of \( \mathbf{w} = (-1, 0) \), which points due west. We have
\[
\nabla f(x, y) = (-0.04x, -0.08y),
\]
so \( \nabla f(-20, 5) = (0.8, -0.4) \). This means that
\[
D_{\mathbf{w}} f(-20, 5) = \nabla f(-20, 5) \cdot \mathbf{w} = -0.8,
\]
since \( \mathbf{w} \) is a unit vector. So the elevation \( z = f(x, y) \) will decrease as we head west, with an instantaneous slope of \(-0.8\).

(b) Walking northeast means walking in the direction of \( \mathbf{v} = (1, 1) \). Notice that \( \mathbf{v} \) is not a unit vector, so we have
\[
D_{\mathbf{v}} f(-20, 5) = \nabla f(-20, 5) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}} (0.8 - 0.4) = \frac{0.4}{\sqrt{2}},
\]
So the climber will ascend if she walks northeast, and will do so at a rate of \( 0.4/\sqrt{2} \approx 0.283 \) meters per meter walked.

(c) To walk along a level path the climber wants her directional derivative to be zero. So if her velocity vector is \( \mathbf{v} \) she wants
\[
D_{\mathbf{v}} f(-20, 5) = \nabla f(-20, 5) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = 0.
\]
This will be the case precisely when \( \mathbf{v} \) is perpendicular to \( \nabla f(-20, 5) \). We can easily produce a vector perpendicular to \( \nabla f(-20, 5) \) by swapping the components and negating the first component: \( \mathbf{v}_1 = (0.4, 0.8) \) is a level direction. Walking in the opposite direction is also perpendicular to \( \nabla f \), so \( \mathbf{v}_2 = (-0.4, -0.8) \) is also a level direction. In terms of unit vectors, the vectors
\[
\mathbf{u}_1 = \left(\frac{\sqrt{0.8}}{2}, \frac{\sqrt{0.8}}{2}\right) \quad \text{and} \quad \mathbf{u}_2 = \left(-\frac{\sqrt{0.8}}{2}, -\frac{\sqrt{0.8}}{2}\right)
\]
are tangent to the height-1991 level curve.

10. (Solution) Suppose \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^2 \) are nonparallel, nonzero vectors, and that \( D_{\mathbf{v}} f(x, y) = D_{\mathbf{w}} f(x, y) = 0 \). Take any vector \( \mathbf{u} \in \mathbb{R}^2 \). Because \( \mathbf{v} \) and \( \mathbf{w} \) are nonparallel, so are the unit vectors
\[
\frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \text{and} \quad \frac{\mathbf{w}}{\|\mathbf{w}\|},
\]
so there are scalars \( a, b \in \mathbb{R} \) so that
\[
\frac{\mathbf{u}}{\|\mathbf{u}\|} = a \frac{\mathbf{v}}{\|\mathbf{v}\|} + b \frac{\mathbf{w}}{\|\mathbf{w}\|}.
\]
Then
\[
D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} = \nabla f(x, y) \cdot \left( a \frac{\mathbf{v}}{\|\mathbf{v}\|} + b \frac{\mathbf{w}}{\|\mathbf{w}\|}\right) = a D_{\mathbf{v}} f(x, y) + b D_{\mathbf{w}} f(x, y) = 0.
\]
Figure 2: The sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$ meet at right angles.

11. (Solution) Showing that the surfaces intersect at the given point just means verifying that $(3, 4, 5)$ satisfies both equations. To show that the two surfaces have a common tangent plane, we show that their normal vectors at this point are parallel. To this end, consider the functions

$$f(x, y, z) = \sqrt{x^2 + y^2} - z \quad \text{and} \quad g(x, y, z) = \frac{1}{10}(x^2 + y^2) + \frac{5}{2} - z.$$ 

Our surfaces are the level-0 surfaces of these two functions, so the gradient $\nabla f$ will be perpendicular to the first surface, and $\nabla g$ will be perpendicular to the second. We have

$$\nabla f(x, y, z) = \langle x(x^2 + y^2)^{-1/2}, y(x^2 + y^2)^{-1/2}, -1 \rangle \quad \text{and} \quad \nabla g(x, y, z) = \langle x/5, y/5, -1 \rangle.$$ 

At the point $(3, 4, 5)$ these vectors are

$$\nabla f(3, 4, 5) = \langle 3/5, 4/5, -1 \rangle \quad \text{and} \quad \nabla g(3, 4, 5) = \langle 3/5, 4/5, -1 \rangle.$$ 

So the vector $\langle 3/5, 4/5, -1 \rangle$ is perpendicular to both surfaces at the point $(3, 4, 5)$ where they intersect, meaning that the surfaces have a common tangent plane at this point.

12. (Solution) First let’s find normal vectors to these two surfaces. The sphere $x^2 + y^2 + z^2 = a^2$ is a level surface for the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$ 

Because the gradient of a three-variable function is perpendicular to the function’s level surface, the vector

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$$

is everywhere perpendicular to our sphere. On the other hand, the cone $z^2 = x^2 + y^2$ is a level surface of the function

$$g(x, y, z) = x^2 + y^2 - z^2,$$
so it has a normal vector given by
\[ \nabla g(x, y, z) = \langle 2x, 2y, -2z \rangle. \]

Now suppose \((x, y, z)\) is in the intersection of the sphere and the cone, so that
\[ x^2 + y^2 + z^2 = a^2 \quad \text{and} \quad z = x^2 + y^2. \]

Then
\[ \nabla f(x, y, z) \cdot \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle \cdot \langle 2x, 2y, -2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4(x^2 + y^2 - z^2) = 0, \]
so the sphere and cone meet perpendicularly. The \(a = 1\) case can be seen in Figure 2. 

13. \((\text{Solution})\) We begin by identifying the critical points of \(f\) — the points where both \(f_x\) and \(f_y\) vanish. We have
\[ \frac{\partial f}{\partial x} = 3x^2 - 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3, \]
so setting \(f_x(x, y) = 0\) gives \(3x^2 - 3 = 0\), which means \(x = \pm 1\). Similarly, setting \(f_y(x, y) = 0\) gives \(y = \pm 1\). This means that \(f_x\) and \(f_y\) are both zero when \(x = \pm 1\) and \(y = \pm 1\). This means that we have four critical points:
\[ (-1, -1), \quad (-1, 1), \quad (1, -1), \quad (1, 1). \]

Next we compute our second partial derivatives:
\[ \frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = 0. \]

This means that
\[ D(x, y) = (6x)(6y) - 0 = 36xy. \]

From this we may determine whether each of our critical points is a relative maximum, relative minimum, or saddle point:
The conclusions are made by applying the second derivative test. Notice that when $D < 0$ the second derivative test does not require us to compute $f_{xx}$. We can verify our conclusions by consulting Figure 3.

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
Critical Point & D & $f_{xx}$ & Conclusion \\
\hline
$(-1, -1)$ & $36 > 0$ & $-6 < 0$ & relative maximum \\
\hline
$(-1, 1)$ & $-36 < 0$ & N/A & saddle point \\
\hline
$(1, -1)$ & $-36 < 0$ & N/A & saddle point \\
\hline
$(1, 1)$ & $36 > 0$ & $6 > 0$ & relative minimum \\
\hline
\end{tabular}
\end{table}