Math 32A Midterm 2 Review  
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We’ll spend this week working examples to prepare for the midterm. A good starting point for examples is the homework, but here are some additional practice problems. The questions and solutions are listed separately so that you can work each problem before checking your work. It’s important to understand that these problems were not vetted by Dr. Chayes, and should not be considered an indication of the problems you’ll see on the exam.

Examples

1. Let $E$ be the ellipse in the $xy$-plane determined by the equation
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \]
   where $a$ and $b$ are positive. At a point $(x, y)$ on $E$, the **osculating circle** is the unique circle $C$ which passes through $(x, y)$, has the same curvature as $E$ at $(x, y)$, and whose center lies in the direction of the standard unit normal vector to $E$. Using the parametrization
   \[ r(t) = (a \cos t, b \sin t), \quad 0 \leq t \leq 2\pi \]
   for $E$, give a parametrization $c(t)$ of the center of the osculating circle to $E$ at $r(t)$.

2. Let $L(t, x, v)$ be a twice continuously differentiable function of three variables, which we call the **Lagrangian** of our problem. The variables $x$ and $v$ are both single-variable functions of $t$, and $x'(t) = v(t)$. Suppose $x(t)$ satisfies the **Euler-Lagrange equation** of $L$:
   \[ \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}, \]
   and introduce a new variable $y := \frac{\partial L}{\partial x}(x, v)$. Show that the **Hamiltonian**, defined by
   \[ H(t, x, y) := y(t)v(t) - L(t, x, v), \]
   satisfies the **Hamiltonian differential equations**:
   \[ x'(t) = \frac{\partial H}{\partial y}, \quad y'(t) = -\frac{\partial H}{\partial x}. \]
Solutions

1. If we can find a parametrization $n(t)$ for the standard unit normal to $E$ at $r(t)$ and also a parametrization $R(t)$ for the radius of the osculating circle at $r(t)$, then the center of this circle will be given by

$$c(t) = r(t) + R(t)n(t).$$

Recall that the curvature of a circle is constant, and in particular is equal to $1/R$, where $R$ is the radius of the circle. Since we want $C$ to have the same curvature as $E$, we see that $R(t) = 1/\kappa(t)$, where $\kappa(t)$ is the curvature of $E$ at $r(t)$. Towards finding $\kappa(t)$ we compute

$$r'(t) = \langle -a \sin t, b \cos t \rangle \quad \text{and} \quad r''(t) = \langle -a \cos t, -b \sin t \rangle.$$

Then we have

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -a \sin t & b \cos t & 0 \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = \langle 0, 0, ab \rangle,$$

so $\|r'(t) \times r''(t)\| = ab$. We also see that

$$\|r'(t)\| = (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2},$$

so

$$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}.$$

Then the radius $R(t)$ of the osculating circle $C$ at $r(t)$ is given by

$$R(t) = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab}.$$

Next we must find $n(t)$. First we have the unit tangent vector

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle -a \sin t, b \cos t \rangle}{(a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}}.$$

Now the unit normal vector is defined by $n(t) = T'(t)/\|T'(t)\|$. Rather than compute this relatively messy derivative, we’ll cheat a little bit. Because of our familiarity with the parametrization $r(t)$, we know that the unit tangent vector $T(t)$ is always turning to the left. In particular, $T'(t)$ is always to the left of $T(t)$, and thus we can obtain $n(t)$ by rotating $T(t)$ $90^\circ$ counterclockwise. We know how to make this rotation component-wise, so we have

$$n(t) = \frac{\langle -b \cos t, -a \sin t \rangle}{(a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}}.$$
Finally we have
\[
c(t) = r(t) + R(t)n(t) = \langle a \cos t, b \sin t \rangle + \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab} \langle -b \cos t, -a \sin t \rangle \]
\[
= \langle a \cos t, b \sin t \rangle + \frac{a^2 \sin^2 t + b^2 \cos^2 t}{ab} \langle -b \cos t, -a \sin t \rangle \]
\[
= \left\langle \frac{a^2}{a} \cos t, \frac{b^2}{b} \sin t \right\rangle - \left\langle \frac{a^2 \sin^2 t + b^2 \cos^2 t}{a} \cos t, \frac{a^2 \sin^2 t + b^2 \cos^2 t}{b} \sin t \right\rangle \]
\[
= \left\langle \frac{a^2 - b^2}{a} \cos^3 t, \frac{b^2 - a^2}{b} \sin^3 t \right\rangle.
\]

To see our parametrization of the center in action, have a look at this video.

2. We’ll start by computing the partial derivatives on the right sides of the equations in (1). First we have
\[
\frac{\partial H}{\partial y} = v(t) = x'(t),
\]
giving us the first of the Hamiltonian differential equations. Next we have
\[
\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x}.
\]
But since \(x(t)\) satisfies the Euler-Lagrange equation,
\[
\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} = -\frac{d}{dt} \frac{\partial L}{\partial v} = -\frac{d}{dt} v = -y'(t),
\]
and we have the second Hamiltonian differential equation. This transformation of the Euler-Lagrange equation into the pair of Hamiltonian equations is known as the Legendre transformation, and is important in the calculus of variations and Hamiltonian mechanics.