Math 31A Discussion Session
Week 2 Notes
January 12 and 14, 2016

This week we'll discuss a powerful tool for computing limits, called the squeeze theorem. Following this, we may also mention limits at infinity, whose computation sometimes requires different methods. Finally, we will give a geometric motivation for the derivative, and investigate some of its properties.

The Squeeze Theorem

As useful as the limit laws are, there are many limits which simply will not fall to these simple rules. One helpful tool in tackling some of the more complicated limits is the **Squeeze Theorem**:

Theorem 1. Suppose f, g, and h are functions so that

$$f(x) \leq g(x) \leq h(x)$$

near a, with the exception that this inequality might not hold when $x = a$. Then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x),$$

if these three limits exist. In particular, if $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then

$$\lim_{x\to a} g(x) = L.$$

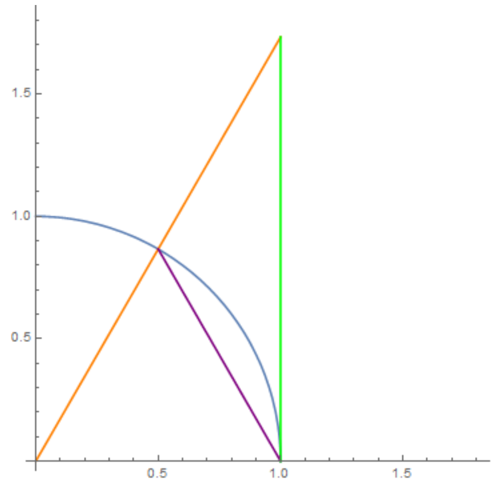
Example.

1. Evaluate the limit $\lim_{x\to 0}(x \cdot \cos(1/x))$, if it exists.

   (*Solution*) We know that $-1 \leq \cos(1/x) \leq 1$ for all $x \neq 0$. Then $-x \leq x \cdot \cos(1/x) \leq x$, so
   $$\lim_{x\to 0}(-x) \leq \lim_{x\to 0}(x \cdot \cos(1/x)) \leq \lim_{x\to 0} x.$$
   Since $\lim_{x\to 0}(-x) = 0 = \lim_{x\to 0} x$, we see that
   $$\lim_{x\to 0}(x \cdot \cos(1/x)) = 0.$$

2. Use the Squeeze Theorem to evaluate $\lim_{\theta\to 0}\left(\dfrac{\sin\theta}{\theta}\right)$, if it exists.

   (*Solution*) The following figure will prove to be useful in evaluating this limit:



In the above figure, the blue curve is the portion of the unit circle which lies in the first quadrant, and the orange ray makes an angle of θ with the origin, where $0 < \theta < \frac{\pi}{2}$. The green line is the line $x = 1$, so the intersection of the orange and green lines is the point $(1, \tan(\theta))$. Consider the right triangle made by the orange line, the green line, and the x-axis; this triangle has height $\tan\theta$ and width 1, so its area is

$$A_1 = \frac{\tan\theta}{2}.$$

Next consider the sector of the unit circle that lies between the x-axis and the orange ray. This sector has angle θ, so its area is

$$A_2 = \frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}.$$

Finally, consider the triangle formed by the orange line, the purple line, and the x-axis. This triangle has height $\sin\theta$ and width 1, so its area is

$$A_3 = \frac{\sin\theta}{2}.$$

Now since this triangle lies entirely inside the sector, and the sector lies entirely inside the first triangle, we have

$$A_1 \geq A_2 \geq A_3 \qquad \Rightarrow \qquad \frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}.$$

We can multiply through this inequality by 2; taking the reciprocal of each part will then reverse the inequality:

$$\frac{\cos\theta}{\sin\theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin\theta}.$$

Finally, multiply through by $\sin\theta$ to obtain

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1. \tag{1}$$

(Since $0 < \theta < \frac{\pi}{2}$, $\sin\theta > 0$, so the inequality will not reverse.) We have only seen that inequality (1) holds when $0 < \theta < \frac{\pi}{2}$, but a similar argument will show that it also holds when $-\frac{\pi}{2} < \theta < 0$, so, since $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, the Squeeze Theorem allows us to conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

3. Use the previous example to evaluate

$$\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)},$$

if this limit exists.

(*Solution*) For $x \neq 0$ we can rewrite this quotient as

$$\frac{\sin(4x)}{\sin(6x)} = \frac{x\cdot\sin(4x)}{x\cdot\sin(6x)} = \left(\frac{4}{6}\right)\left(\frac{6x}{\sin(6x)}\right)\left(\frac{\sin(4x)}{4x}\right).$$

Then

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)} &= \frac{4}{6}\left(\lim_{x\to 0}\frac{6x}{\sin(6x)}\right)\left(\lim_{x\to 0}\frac{\sin(4x)}{4x}\right)\\ &= \frac{4}{6}\left(\lim_{x\to 0}\frac{\sin(6x)}{6x}\right)^{-1}\left(\lim_{x\to 0}\frac{\sin(4x)}{4x}\right)\\ &= \frac{4}{6}\cdot 1\cdot 1 = \frac{2}{3}.\end{aligned}$$

Limits at Infinity

We'll carry out two illustrative examples of limits at infinity.

Example.

1. Find $\displaystyle\lim_{x\to\infty}\frac{8x^5+3x^2-4}{4-9x^5}$, if it exists.

(*Solution*) Neither $\lim_{x\to\infty}(8x^5+3x^2-4)$ nor $\lim_{x\to\infty}(4-9x^5)$ exists, so we cannot very well consider a ratio of these limits. What we can do, however, is rewrite this quotient so that the numerator and denominator limits exist. For $x \neq 0$, we have

$$\frac{8x^5+3x^2-4}{4-9x^5} = \frac{8x^5+3x^2-4}{4-9x^5}\cdot\frac{x^{-5}}{x^{-5}} = \frac{8+3x^{-3}-4x^{-5}}{4x^{-5}-9},$$

so

$$\lim_{x\to\infty} \frac{8x^5+3x^2-4}{4-9x^5} = \lim_{x\to\infty} \frac{8+3x^{-3}-4x^{-5}}{4x^{-5}-9} = \frac{\lim_{x\to\infty}(8+3x^{-3}-4x^{-5})}{\lim_{x\to\infty}(4x^{-5}-9)}$$
$$= \frac{8}{-9} = -\frac{8}{9}.$$

This technique of writing the denominator as a constant term plus terms with negative exponents is a good general strategy for determining the end behavior of rational functions.

2. Consider $f(x) = \dfrac{\sin(2x+7)\cos(x^2)+\cos^2(4-x^3)}{x}$. Find $\lim_{x\to\infty} f(x)$, if this limit exists.

(*Solution*) This limit may look daunting, but we need only recall that the sine and cosine functions are bounded. Since sine and cosine take values between -1 and 1, the values of the product $\sin(2x+7)\cos(x^2)$ will be between -1 and 1. That is,

$$-1 \le \sin(2x+7) \le 1 \text{ and } -1 \le \cos(x^2) \le 1, \qquad \text{so} \qquad -1 \le \sin(2x+7)\cos(x^2) \le 1.$$

Similarly, since $-1 \le \cos(4-x^3) \le 1$, $0 \le \cos^2(4-x^3) \le 1$. Adding these inequalities together,

$$-1 \le \sin(2x+7)\cos(x^2) + \cos^2(4-x^3) \le 2.$$

Putting these inequalities over x, we have

$$\frac{-1}{x} \le \frac{\sin(2x+7)\cos(x^2)+\cos^2(4-x^3)}{x} \le \frac{2}{x}.$$

Since the terms on each end will tend to zero as x tends to ∞, our limit is zero.

The Derivative

Given a continuous function f, suppose we want to find the equation of the line which lies tangent to the graph of $y = f(x)$ at the point $(a, f(a))$. At first glance this seems like a difficult problem to approach naively (that is, without derivatives). But we can tackle it with relative ease by considering **secant lines**.

Definition. Given a continuous function f, the **secant line** passing through $(a, f(a))$ and $(b, f(b))$ (where $a \neq b$) to the graph of $y = f(x)$ is defined by

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a). \tag{2}$$

Intuitively, we can see that the secant line passing through $(a, f(a))$ and $(b, f(b))$ should begin to look more and more like the tangent line at $(a, f(a))$ as b gets closer to a. So we should be able to obtain the equation of the tangent line by taking the limit of equation (2).

Example. Find the equation of the tangent line to the graph of $y = x^2 + 4$ at $(3, 13)$.

(*Solution*) For some $b \neq 3$, the equation of the secant line passing through $(3, 13)$ and $(b, f(b))$ is given by

$$y = \frac{f(b) - f(3)}{b - 3}(x - 3) + 13 = \frac{(b^2 + 4) - 13}{b - 3}(x - 3) + 13. \tag{3}$$

Notice that the only part of this equation which depends on b is the slope, $\frac{b^2-9}{b-3}$. So we should be able to find the tangent line by finding the limit of this slope as b approaches 3, and replacing the slope in (3) with this limit. We have

$$\lim_{b \to 3} \frac{b^2 - 9}{b - 3} = \lim_{b \to 3} \frac{(b - 3)(b + 3)}{b - 3} = \lim_{b \to 3}(b + 3) = 6,$$

so the equation of the tangent line is given by

$$y = 6(x - 3) + 13 = 6x - 5.$$

Frequently, we're just interested in the *slope* of the tangent line to a curve. To determine this, first notice that the slope of the line determined in (2) is given by

$$\frac{f(b) - f(a)}{b - a}.$$

Then the slope of the tangent line should be the slope we obtain by letting b get close to a:

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

Another way to consider this is as follows. Suppose we want to find the slope of the tangent line to $y = f(x)$ at the point $(x, f(x))$. If we change our x-value by a small amount — say, by h — then the coordinates of the point on the graph will be $(x+h, f(x+h))$. So the slope of the secant line between this two points is

$$\frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.$$

As before, we want to see what happens to this quantity as our second point approaches our first. That is, we want to see what happens when our displacement h gets closer to 0, so we consider

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

When this limit exists, we call the resulting value the **derivative** of f at x, and denote this variously as

$$\frac{d}{dx}(f(x)), \qquad f'(x), \qquad \dot{f}(x).$$

Example. Compute the derivative of $f(x) = x^3$ at an arbitrary point x.

(*Solution*) According to our above definition, the derivative will be given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\ &= \lim_{h\to 0} (3x^2 + 3hx + h^2) = 3x^2, \end{aligned}$$

whenever this limit exists. But this limit exists for all x-values, so f is everywhere differentiable, and $f'(x) = 3x^2$ for all x values.

One thing we notice immediately is that constant functions have derivative 0. To see this, notice that if $g(x) = c$, then

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

This fits with our understanding of the derivative as an instantaneous slope, since the graph of a constant function is a horizontal line. Because it is defined as a limit, the derivative also has the following pleasant arithmetic properties:

Proposition 2. Suppose $f'(x)$ and $g'(x)$ exist and c, d are real numbers. Then

1. $\frac{d}{dx}(cf(x) + dg(x)) = cf'(x) + dg'(x)$;
2. $\frac{d}{dx}(cf(x) - dg(x)) = cf'(x) - dg'(x)$.

The derivatives of products and quotients aren't quite as straightforward, but we'll discuss them soon. Finally, we introduce what we'll call the **power rule** for computing derivatives.

Proposition 3. If $f(x) = x^n$ for a positive integer n, then $f'(x) = nx^{n-1}$.

Proof. Using our definition, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} h^k x^{n-k} - x^n}{h} = \lim_{h\to 0} \frac{\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} h^k x^{n-k}}{h} \\ &= \lim_{h\to 0} \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} h^{k-1} x^{n-k} = nx^{n-1}. \end{aligned}$$

□

Example. Find the derivative of $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ as a function of x.

(*Solution*) We can use the fact that the derivative splits over sums and the power rule to find that

$$\begin{aligned}
p'(x) &= \frac{d}{dx}(a_n x^n) + \frac{d}{dx}(a_{n-1}x^{n-1}) + \cdots + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_0) \\
&= n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.
\end{aligned}$$