This week we’re discussing two important topics: limits and continuity. We won’t give a completely rigorous definition of either, but we’ll develop some tools for computing limits and determining whether or not functions are continuous.

Limit Laws

We begin by recalling some basic limits.

**Proposition 1.** For real numbers \( a \) and \( b \),

\[
\lim_{x \to a} b = b, \quad \lim_{x \to a} x = a, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \to 0^+} = +\infty.
\]

The following limit laws, which we state informally, will be a vital part of our toolbox when computing limits.

**Proposition 2.** In each of the following rules, we assume that the relevant limits exist.

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a product is the product of the limits.
4. The limit of a quotient is the quotient of the limits, as long as the limit of the denominator is not zero.

**Example.** From these basic limits and simple rules, which all seem obvious enough, we can compute limits of any polynomials. Let \( p(x) := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial, and choose \( a \in \mathbb{R} \). Then

\[
\lim_{x \to a} p(x) = p(a).
\]

To see this, we apply our rules to find that

\[
\begin{align*}
\lim_{x \to a} p(x) &= \lim_{x \to a} (a_n x^n) + \lim_{x \to a} (a_{n-1} x^{n-1}) + \cdots + \lim_{x \to a} (a_1 x) + \lim_{x \to a} (a_0) \\
&= a_n \lim_{x \to a} x^n + a_{n-1} \lim_{x \to a} x^{n-1} + \cdots + a_1 \lim_{x \to a} x + \lim_{x \to a} a_0 \\
&= a_n \left( \lim_{x \to a} x \right)^n + a_{n-1} \left( \lim_{x \to a} x \right)^{n-1} + \cdots + a_1 \lim_{x \to a} x + a_0 \\
&= a_n \cdot a^n + a_{n-1} \cdot a^{n-1} + \cdots + a_1 \cdot a + a_0 \\
&= p(a).
\end{align*}
\]
Next, we have a limit law whose statement might sound complicated, but which is easy to use.

**Proposition 3.** If a function \( f(x) \) is non-zero when \( x \) is near the real number \( a \) (but not necessarily nonzero at \( a \)), then

\[
\lim_{x \to a} \frac{f(x)g(x)}{f(x)} = \lim_{x \to a} g(x).
\]

**Note.** In general, you won’t find yourself worrying about the “non-zero near \( a \)” bit of this rule. It’s primarily meant to keep us from concluding that

\[
\lim_{x \to a} 0 = \lim_{x \to a} \frac{0 \cdot g(x)}{0} = \lim_{x \to a} g(x)
\]

for some function \( g(x) \), and this is certainly a conclusion we want to avoid.

**Example.**

1. Evaluate \( \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} \), if this limit exists.

   \[ (Solution) \] According to our previous rule (and some factoring), we have

   \[
   \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \to 2} (x + 3) = 5.
   \]

2. Evaluate \( \lim_{x \to 0} \frac{3x^4 + 2x^3 + x^2}{3x^4 + 5x^3 + 7x^2} \), if this limit exists.

   \[ (Solution) \] Notice that we can’t use rule 4 from Proposition 2 just yet, because we immediately see that the denominator approaches 0 as \( x \) approaches 0. We must first do some cancellation. We have

   \[
   \lim_{x \to 0} \frac{3x^4 + 2x^3 + x^2}{3x^4 + 5x^3 + 7x^2} = \lim_{x \to 0} \frac{x^2(3x^2 + 2x + 1)}{3x^4 + 5x^3 + 7x^2} = \lim_{x \to 0} \frac{3x^2 + 2x + 1}{3x^2 + 5x + 7} = \frac{1}{7}.
   \]

3. Evaluate \( \lim_{\theta \to 0} \frac{\sin^2(\theta)}{1 - \cos(\theta)} \), if this limit exists.

   \[ (Solution) \] We can use the identity \( \sin^2(\theta) + \cos^2(\theta) = 1 \) to find

   \[
   \lim_{\theta \to 0} \frac{\sin^2(\theta)}{1 - \cos(\theta)} = \lim_{\theta \to 0} \frac{1 - \cos^2(\theta)}{1 - \cos(\theta)} = \lim_{\theta \to 0} \frac{(1 - \cos(\theta))(1 + \cos(\theta))}{1 - \cos(\theta)} = \lim_{\theta \to 0} (1 + \cos(\theta)) = 2.
   \]
4. Evaluate \( \lim_{x \to 0} \frac{|x|}{x} \), if this limit exists.

\((Solution)\) First, let’s rewrite \( |x|/x \). We have

\[
\frac{|x|}{x} = \begin{cases} 
-1, & x < 0 \\
1, & x > 0
\end{cases}
\]

and \( |x|/x \) is undefined when \( x = 0 \). So we see that

\[
\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -1 = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1.
\]

Since the left and right limits do not agree, \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist.

5. Evaluate \( \lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} \), if this limit exists.

\((Solution)\) Since both the numerator and denominator approach 0 as \( x \) approaches 1, we must find a way to rewrite the fraction. We do this by rationalizing the denominator:

\[
\frac{x - 1}{\sqrt{x} - 1} = \frac{x - 1}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{(x - 1)(\sqrt{x} + 1)}{x - 1}.
\]

Then, using Proposition 3,

\[
\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{(x - 1)(\sqrt{x} + 1)}{x - 1} = \lim_{x \to 1} (\sqrt{x} + 1) = 2.
\]

**Continuity**

With limits, we are able to compute the value that an expression or function should take, if it’s nice. Of course, there are many cases where an expression or function doesn’t take the value that we think it should, so we distinguish the better-behaved functions by giving them a special name.

**Definition.** We say that a function \( f \) is **continuous** at a point \( x = a \) if:

1. \( f(a) \) is defined;
2. \( \lim_{x \to a} f(x) \) exists;
3. \( \lim_{x \to a} f(x) = f(a) \).

We say that \( f \) is continuous on the interval \((c, d)\) if \( f \) is continuous at each point \( a \in (c, d) \).

**Example.**
1. The function \( f(x) := |x| \) is continuous at \( x = 0 \), but the function 

\[
f(x) := \begin{cases} 
|x|, & x \neq 0, \\
42, & x = 0 
\end{cases}
\]

is not. (Why?)

2. The function 

\[
g(x) := \begin{cases} 
\sin(1/x), & x \neq 0, \\
k, & x = 0 
\end{cases}
\]

is continuous on \((-\infty, 0)\) and \((0, +\infty)\), but is not continuous at \( x = 0 \), no matter the value of \( k \). To see this, let 

\[
x_n = \frac{2}{(2n + 1)\pi} \quad \text{and} \quad y_n = \frac{1}{n\pi}.
\]

Then \( g(x_n) = \sin((2n + 1)\pi/2) = 1 \) for all \( n \), and \( g(y_n) = \sin(n\pi) = 0 \) for all \( n \). Since \( x_n \to 0 \) and \( y_n \to 0 \), but \( g(x_n) \) and \( g(y_n) \) approach distinct values, we see that \( \lim_{x \to 0} g(x) \) does not exist, and thus \( g \) cannot be continuous at \( x = 0 \).

As with limits, we have some important arithmetic properties for continuous functions:

**Proposition 4.** If \( f \) and \( g \) are continuous at \( x = a \), then

1. \( f + g \) is continuous at \( x = a \);
2. \( f - g \) is continuous at \( x = a \);
3. \( f \cdot g \) is continuous at \( x = a \);
4. \( f/g \) is continuous at \( x = a \), provided \( g(a) \neq 0 \).

**Example.**

1. The converses of the above properties are not true. In particular, we can find functions \( f \) and \( g \) which are discontinuous at \( x = a \), but such that \( f + g \), \( f - g \), \( fg \), or \( f/g \) is continuous. For example, define 

\[
f(x) := \begin{cases} 
0, & x < 0 \\
1, & x \geq 0
\end{cases} \quad \text{and} \quad g(x) := \begin{cases} 
1, & x < 0 \\
0, & x \geq 0
\end{cases}.
\]

Neither of these functions is continuous at \( x = 0 \), but \( h(x) := f(x) + g(x) \) is continuous on all of \( \mathbb{R} \), since \( h \equiv 1 \).

We finish with a very useful property of continuous functions: they play nicely with limits. This shouldn’t be surprising, since we defined continuous functions to be those functions which behave the way our study of limits tells us that they should.
Proposition 5. If \( f \) is continuous at \( x = g(a) \) and \( \lim_{x \to a} g(x) \) exists, then
\[
\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right).
\]

Example. The function \( f(x) = \sqrt{x} \) is continuous on its domain, so
\[
\lim_{x \to 3} \sqrt{x^2 + 7} = \sqrt{\lim_{x \to 3} (x^2 + 7)} = \sqrt{16} = 4.
\]

Other Examples

Example. All but the first of these examples were written by Steven Heilman.

1. Find constants \( b \) and \( c \) so that the function
\[
f(x) := \begin{cases} 
4x + b, & \text{if } x < -2 \\
cx^2, & \text{if } -2 \leq x \leq 2 \\
12, & \text{if } 2 < x
\end{cases}
\]
is continuous.

(Solution) Certainly \( f \) is continuous when \( x \neq -2 \) and \( x \neq 2 \); we need to make \( f \) continuous at these two points as well. First, let’s make \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \).

We have
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} cx^2 = 4c \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 12 = 12,
\]
so in order for \( \lim_{x \to 2} f(x) \) to exist and equal \( f(2) \), we need \( c = 3 \). Then
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} 3x^2 = 12.
\]

But
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x + b) = b - 8,
\]
so in order for \( \lim_{x \to 2^-} f(x) \) to exist and equal \( f(-2) \), we must have \( b = 20 \).

2. Evaluate \( \lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) \), if this limit exists.

(Solution) Notice that we cannot split this into a difference of two limits, since neither \( \lim_{t \to 0} 1/t \) nor \( \lim_{t \to 0} 1/(t^2 + t) \) exists. Instead, we first combine the fractions:
\[
\frac{1}{t} - \frac{1}{t^2 + t} = \frac{t + 1}{t(t + 1)} - \frac{1}{t(t + 1)} = \frac{t}{t(t + 1)} = \frac{1}{t + 1}.
\]

Having made this simplification, we have
\[
\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \frac{1}{t + 1} = 1.
\]
3. Evaluate the following limit, if it exists.

\[ \lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}. \]

(Solution) As with one of our above examples, this limit looks like 0/0, so we’ll multiply both the numerator and denominator by the conjugate of the denominator:

\[ \frac{x}{\sqrt{1 + 3x} - 1} = \frac{x}{\sqrt{1 + 3x} - 1} \cdot \frac{\sqrt{1 + 3x} + 1}{\sqrt{1 + 3x} + 1} = \frac{x(\sqrt{1 + 3x} + 1)}{1 + 3x} = \frac{\sqrt{1 + 3x} + 1}{3}. \]

This leads us to

\[ \lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1} = \lim_{x \to 0} \frac{\sqrt{1 + 3x} + 1}{3} = \frac{2}{3}. \]

4. Is there a real number \( a \) such that the following limit exists?

\[ \lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} \]

If so, find the value of \( a \) and the value of the limit.

(Solution) Notice that

\[ \lim_{x \to -2} (x^2 + x - 2) = 0, \]

so in order for this limit to exist, we’ll need \( x + 2 \) to be a factor of the numerator. This will allow us to cancel \( x + 2 \) from both the numerator and denominator and evaluate the limit. To this end, notice that

\[ 3x^2 + ax + a + 3 = (x + 2)(3x + (a - 6)) + 15 - a. \]

(A little polynomial division will give you this expression.) Thus, if \( a = 15 \), we’ll have \( 3x^2 + ax + a + 3 = (x + 2)(3x + (a - 6)) \), making \( x + 2 \) a factor of our numerator. Then

\[ \lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} = \lim_{x \to -2} \frac{3x^2 + 15x + 18}{(x + 2)(x - 1)} = \lim_{x \to -2} \frac{3(x + 2)(x + 3)}{(x + 2)(x - 1)} = \lim_{x \to -2} \frac{3(x + 3)}{x - 1} = \frac{3}{-3} = -1. \]