This week we’re going to discuss Möbius transformations, which are the bijective, conformal maps from the Riemann sphere to itself, and which are the subject of the fourth homework problem from the current set.

Möbius Transformations

We say that a map \( T : \mathbb{C} \to \mathbb{C} \) is a Möbius transformation if we may write \( T \) as

\[
T(z) = \frac{az + b}{cz + d}
\]

for some complex numbers \( a, b, c, d \in \mathbb{C} \) with the property that \( ad - bc \neq 0 \). Notice that \(-d/c\) is not in the domain of \( T \), even when \( c \neq 0 \). We can fix this by extending \( T \) to be defined on the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). We define

\[
T(-d/c) = \infty, \quad \text{and} \quad T(\infty) = \frac{a}{c},
\]

where \( a/c = \infty \) if \( c = 0 \). The extended complex plane \( \overline{\mathbb{C}} \) is also called the Riemann sphere, and we have defined \( T \) to be a map from the Riemann sphere to itself.

We exclude the situation where \( ad - bc = 0 \) to make sure that \( T \) is bijective. In case \( ad = bc \), we have

\[
T(z) = \frac{az + b}{cz + d} = \frac{a^2cz + abc}{ac^2z + acd} = \frac{a(acz + bc)}{c(acz + ad)} = \frac{a}{c},
\]

for all \( z \in \overline{\mathbb{C}} \), making \( T \) as far from bijective as possible.

It was stated above that the Möbius transformations are the bijective, conformal maps from the Riemann sphere to itself. This can be taken as an analytic statement about angle-preserving maps from \( \overline{\mathbb{C}} \) to itself, but a much more interesting approach to take is to use stereographic projection to realize \( \overline{\mathbb{C}} \) as a sphere in 3-space, and then to see the Möbius transformations as the rigid motions of the sphere. This is done in the fantastic video *Möbius Transformations Revealed*.

Generators

When proving that the Möbius transformations have certain properties, it is often easier to prove that the property holds for transformations of special forms, and then to conclude that the property holds for all Möbius transformations. Specifically, we want to show that all Möbius transformations can be written as compositions of translations, dilations, rotations, and the inversion transformation. To fit with Dr. Greene’s suggestion in problem 4, we’ll treat the rotations, dilations, and translations together. So we’re claiming that the maps of the form

\[
T(z) = az + b \quad \text{and} \quad T(z) = 1/z
\]

(1)
generate all of the Möbius transformations. To see that this is the case, suppose we have \( T(z) = \frac{az + b}{cz + d} \). If \( c = 0 \), then
\[
T(z) = \frac{a}{d} z + \frac{b}{d}
\]
is composed of a dilation, a rotation, and a translation, and thus fits into the first class of maps. If \( c \neq 0 \) then we can define
\[
T_1(z) = z + \frac{d}{c}, \quad T_2(z) = \frac{1}{z}, \quad \text{and} \quad T_3(z) = \frac{bc - ad}{c^2} z + \frac{a}{c}.
\]
Notice that \( T_1 \) and \( T_3 \) fall into the first class of maps, and \( T_2 \) is an inversion. Most importantly, we have
\[
T_3 \circ T_2 \circ T_1(z) = T_3 \left( \frac{1}{z + d/c} \right) = \frac{bc - ad}{c^2 z + cd} + \frac{a}{c}
\]
\[
= \frac{bc - ad}{c^2 z + cd} + \frac{a(cz + d)}{c^2 z + cd} = \frac{ac z + bc}{cz + d}.
\]
So we may write any Möbius transformation as a composition of maps of the forms listed in (1). This means that if we can prove that \( z \mapsto 1/z \) and \( z \mapsto az + b \) share a certain property, and that this property is preserved under composition, then all Möbius transformations have this property.

**Generalized Circles**

Among the many important properties that Möbius transformations enjoy, the one we’ll focus on is this: they map generalized circles to generalized circles. By a generalized circle in \( \mathbb{C} \) we mean either a usual circle in \( \mathbb{C} \) or a line in \( \mathbb{C} \), along with the point at infinity. We’ll prove that Möbius transformations have this property by showing that all maps of the forms listed in (1) have this property. That this property is preserved under composition is automatic: certainly if \( f \) and \( g \) map generalized circles to generalized circles, then \( f \circ g \) will do so as well.

**Proposition 1.** For any \( a, b \in \mathbb{C} \), with \( a \neq 0 \), the map \( T(z) = az + b \) takes generalized circles to generalized circles.

(*Proof.*) First let’s suppose we have a circle \( \mathcal{C} \subset \mathbb{C} \), so that we can write
\[
\mathcal{C} = \{ z \in \mathbb{C} | |z - z_0| = r \}
\]
for some \( z_0 \in \mathbb{C} \) and some \( r > 0 \). We want to show that \( T(\mathcal{C}) \) is a (generalized) circle. Indeed, for any \( z \in \mathcal{C} \) we have
\[
|T(z) - T(z_0)| = |(az + b) - (az_0 + b)| = |az - az_0| = |a| |z - z_0| = |a|r,
\]
so \( T(\mathcal{C}) \) is contained in the circle of radius \( |a|r \) centered at \( T(z_0) \). Conversely, suppose \( w \in \mathbb{C} \) lies on the circle of radius \( |a|r \) centered at \( T(z_0) \). Then \( T(a^{-1}(w - b)) = w \), and
\[
|a^{-1}(w - b) - z_0| = |a^{-1}(w - b) - a^{-1}(T(z_0) - b)| = |a^{-1}(w - T(z_0))| = |a|^{-1} |w - T(z_0)| = r,
\]

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so \( a^{-1}(w - b) \in \mathbb{C} \). This means that \( w \in T(\mathbb{C}) \), so we see that \( T(\mathbb{C}) \) is precisely the circle of radius \(|a|r\) centered at \( z_0 \).

Proving that \( T \) takes lines to lines is left for you to do. Suggestion: write your line \( L \) as

\[
L = \{z_0 + \lambda z_1 | \lambda \in \mathbb{R}\}
\]

for some \( z_0, z_1 \in \mathbb{C} \).

**Proposition 2.** The inversion map \( T(z) = 1/z \) takes generalized circles to generalized circles.

*Proof.* This time we’ll do the case where our generalized circle is a line and leave the circle case for you to do. Given \( 0 \neq z \in \mathbb{C} \), we write \( z = x + iy \in \mathbb{C} \). We can then express a line \( L \) as

\[
L = \{x + iy | Ax + By = C\},
\]

where \( A \) and \( B \) are not both 0. Let’s write the image \( T(z) \) as \( u + iv \). Notice that

\[
T(x + iy) = \frac{1}{x + iy} = \frac{1}{x + iy} \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},
\]

so

\[
u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}. \tag{2}
\]

Notice that

\[
u^2 + v^2 = \frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2}. \tag{3}
\]

Since \( z \neq 0 \), we can rewrite \( Ax + By = C \) as

\[
A \frac{x}{x^2 + y^2} + B \frac{y}{x^2 + y^2} = \frac{C}{x^2 + y^2}.
\]

Substituting (2) and (3), we have

\[
Au - Bv = C(u^2 + v^2).
\]

If \( C \neq 0 \), this is the equation of a circle in the variables \( u, v \), meaning that \( T \) takes our line (a generalized circle) to a circle in \( \mathbb{C} \). Notice also that 0 is on this circle, and is the image of \( \infty \) under \( T \). If \( C = 0 \) then \( T(L) \) is given by the equation \( Au = Bv \), giving us a line passing through 0, which is also a generalized circle. Notice that we can only have \( C = 0 \) if our original line passes through the point 0.

The case of a circle is left to you. This case can be handled very similarly to our approach for the line. A circle may be written as

\[
C = \{x + iy | Ax + By + C(x^2 + y^2) = D\},
\]

where \( C \neq 0 \), and the rest of the analysis will proceed similarly to what was done above. \( \diamond \)

Once we finish the proofs of Propositions 1 and 2, we’ll have the desired result. Starting with any Möbius transformation \( T(z) \), we can write

\[
T(z) = T_3 \circ T_2 \circ T_1(z),
\]

where \( T_2(z) \) is the inversion map and \( T_1(z), T_3(z) \) are of the form \( z \mapsto az + b \). Then any generalized circle will be mapped by \( T_1 \) to a generalized circle which will be mapped by \( T_2 \) to another generalized circle, and finally this will be mapped by \( T_3 \) to a generalized circle.