This week we’ll discuss a couple of things from homework 2, and then mention some hints about homework 3.

**Homework 3, Problem 1**

We have the following picture (without the colors), where we know that \( AC \parallel ED, BC \parallel EF, \text{ and } AB \parallel DF \):

The first thing we’re asked to show is that \( EB = BD \). There are a few different ways to verify this, and one nice way is to look at the parallelograms highlighted in the above figure. From the parallelogram \( ABDC \) we see that \( BD = AC \), and from the parallelogram \( AEBC \) we see that \( EB = AC \). Once we have \( EB = BD \), we see that \( B \) is the midpoint of \( ED \), and thus the perpendicular bisector of \( ED \) passes through \( B \).

For part (b) of problem 1 it feels like there’s not anything to say — the statement almost seems obvious. **You need to say something.** In particular, think about the definition of the altitude of \( B \) and show that the perpendicular bisector of \( ED \) satisfies this definition.

**Homework 3, Problem 2**

Next we want to think about proving that the altitudes of any triangle must intersect. Following the hint, we do this analytically, placing one vertex of our triangle at \((0, 0)\), another at \((0, a)\), and the third at \((b, c)\). We have some freedom when placing our vertices, so we place them such that the side along the \( y \)-axis — going from \((0, 0)\) to \((0, a)\) — is the longest side of our triangle, and such that \( b > 0 \). The fact that our longest side is along the \( y \)-axis tells us that

\[
a^2 \geq b^2 + c^2 > c^2,
\]

so \( a > c \). We also know that the line from \((0, a)\) to \((b, c)\) is no longer than the line from \((0, 0)\) to \((0, a)\), so

\[
b^2 + (a - c)^2 \leq a^2 \quad \Rightarrow \quad (a - c)^2 < a^2 \quad \Rightarrow \quad a - c < a,
\]

so \( c > 0 \). So we in fact know that \( 0 < c < a \). Here’s a picture of our setup:
We can now find each of the altitudes analytically. For instance, the altitude of \((0, a)\) is perpendicular to the line from \((0, 0)\) to \((b, c)\). This line has slope \(c/b\), so the slope of the altitude of \((0, a)\) must be \(-b/c\). Since this line passes through \((0, a)\), it has equation

\[ y = -\frac{b}{c}x + a. \]

(Notice that it’s important that \(c \neq 0\).) We can similarly find the altitudes of \((b, c)\) and \((0, 0)\), and verify that these three lines intersect at \((c(a - c)/b, c)\).

**Homework 3, Problem 8**

Finally we want to re-prove a result about angle bisectors using vector analysis. In the following diagram we have \(\angle BAD = \angle DAC\):

We’d like to show that

\[ \frac{BD}{AB} = \frac{DC}{AC}. \]

(1)

As suggested by the hint, we can situate \(A\) at the origin \((0, 0)\) of the plane \(\mathbb{R}^2\) and use \(\vec{v}_1, \vec{v}_2\) to represent the vectors from \(A\) to \(B\) and \(C\), respectively. This allows us to represent the line segment between \(B\) and \(C\) as

\[ \{\lambda \vec{v}_1 + (1 - \lambda) \vec{v}_2 | 0 \leq \lambda \leq 1\}. \]

In particular, we can write the vector from \(A\) to \(D\) as \(\vec{v}_\lambda := \lambda \vec{v}_1 + (1 - \lambda) \vec{v}_2\) for some particular \(0 < \lambda < 1\). If we reach way back in time to Math 32B we can recall that the angle between nonzero vectors \(\vec{u}\) and \(\vec{v}\) can be obtained as

\[ \theta = \arccos \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \right). \]
In particular, 

\[ \angle BAD = \arccos \left( \frac{\vec{v}_1 \cdot \vec{v}_\lambda}{\|\vec{v}_1\| \|\vec{v}_\lambda\|} \right) \quad \text{and} \quad \angle DAC = \arccos \left( \frac{\vec{v}_2 \cdot \vec{v}_\lambda}{\|\vec{v}_2\| \|\vec{v}_\lambda\|} \right). \]

Since we have \( \angle BAD = \angle DAC \), we must have 

\[ \vec{v}_1 \cdot \vec{v}_\lambda \|\vec{v}_2\| = \vec{v}_2 \cdot \vec{v}_\lambda \|\vec{v}_1\|. \]

Now 

\[ \vec{v}_1 \cdot \vec{v}_\lambda \|\vec{v}_2\| = (\lambda \vec{v}_1 \cdot \vec{v}_1 + (1 - \lambda)\vec{v}_1 \cdot \vec{v}_2)\|\vec{v}_2\| = \lambda \|\vec{v}_1\|^2 \|\vec{v}_2\| + (1 - \lambda)\vec{v}_1 \cdot \vec{v}_2 \|\vec{v}_2\|, \quad (2) \]

and similarly 

\[ \vec{v}_2 \cdot \vec{v}_\lambda \|\vec{v}_1\| = (\lambda \vec{v}_2 \cdot \vec{v}_1 + (1 - \lambda)\vec{v}_2 \cdot \vec{v}_2)\|\vec{v}_1\| = \lambda \vec{v}_1 \cdot \vec{v}_2 \|\vec{v}_1\| + (1 - \lambda)\|\vec{v}_1\| \|\vec{v}_2\|^2. \quad (3) \]

We can subtract (3) from (2) to obtain 

\[ 0 = \vec{v}_1 \cdot \vec{v}_\lambda \|\vec{v}_2\| - \vec{v}_2 \cdot \vec{v}_\lambda \|\vec{v}_1\| = \lambda \|\vec{v}_1\|(\|\vec{v}_1\| \|\vec{v}_2\| - \vec{v}_1 \cdot \vec{v}_2) + (1 - \lambda)\|\vec{v}_2\|(\vec{v}_1 \cdot \vec{v}_2 - \|\vec{v}_1\| \|\vec{v}_2\|). \]

This expression can then be solved for \( \lambda \). (Be sure to notice that \( \vec{v}_1 \cdot \vec{v}_2 \neq \|\vec{v}_1\| \|\vec{v}_2\| \), since \( \vec{v}_1 \) and \( \vec{v}_2 \) are not scalar multiples of one another.)

Let’s return our attention to equation (1). The vector from \( D \) to \( B \) is 

\[ \vec{v}_1 - \vec{v}_\lambda = \vec{v}_1 - [\lambda \vec{v}_1 + (1 - \lambda)\vec{v}_2] = (1 - \lambda)\vec{v}_1 - (1 - \lambda)\vec{v}_2, \]

so 

\[ \overrightarrow{BD} = \|\vec{v}_1\| -(1 - \lambda)\vec{v}_2\| = (1 - \lambda)\|\vec{v}_1 - \vec{v}_2\|. \]

Similarly, the vector from \( C \) to \( D \) is 

\[ \vec{v}_\lambda - \vec{v}_2 = \lambda \vec{v}_1 + (1 - \lambda)\vec{v}_2 - \vec{v}_2 = \lambda \vec{v}_1 - \lambda \vec{v}_2, \]

so 

\[ \overrightarrow{DC} = \|\lambda \vec{v}_1 - \lambda \vec{v}_2\| = \lambda \|\vec{v}_1 - \vec{v}_2\|. \]

Of course \( \overrightarrow{AB} = \|\vec{v}_1\| \) and \( \overrightarrow{AC} = \|\vec{v}_2\| \). Once we know what \( \lambda \) is, we can substitute into 

\[ \frac{\overrightarrow{BD}}{\overrightarrow{AB}} = \frac{(1 - \lambda)\|\vec{v}_1 - \vec{v}_2\|}{\|\vec{v}_1\|} \quad \text{and} \quad \frac{\overrightarrow{DC}}{\overrightarrow{AC}} = \frac{\lambda \|\vec{v}_1 - \vec{v}_2\|}{\|\vec{v}_2\|}, \]

to make sure that these two expressions agree.