Some trigonometry

Today we want to approach trigonometry in the same way we’ve approached geometry so far this quarter: we’re relatively familiar with the subject, but we want to write down just a few definitions and then rigorously derive some of the other characterizations (identities) with which we’re familiar.

Definitions

Given an angle $0 < \theta < 90^\circ$, we’ll define the values of six trigonometric functions at $\theta$ according to the following figures:

![Figure 1: Definitions of trigonometric functions.](image)

Specifically, the circles in the above figures are of unit radius. In the first figure, the blue segment is tangent to the circle at the point $(1,0)$. We define $\tan \theta$ to be the distance from $(1,0)$ to the point where the blue tangent line intersects the ray making an angle $\theta$ with the $x$-axis. We define $\sec \theta$ to be the distance from the origin $(0,0)$ to the point where the ray intersects the blue tangent line. The red segment meets the ray at the point where the ray intersects the unit circle, and is perpendicular to the $x$ axis. We define $\sin \theta$ to be the length of this segment — that is, the distance from the point where the ray intersects the unit circle to the $x$-axis. Finally, the figure on the right is dual to the figure on the left; we draw the blue line tangent to the circle at $(0,1)$ and define cotangent as we previously defined tangent. We similarly define cosecant and cosine. Notice that we do not define the trigonometric functions according to things like “opposite over hypotenuse;” instead, we’ll derive these sorts of identities from the definitions given. We should emphasize that these are not the standard definitions, but are equivalent.

Some identities

The first identity we’ll derive is a familiar characterization of the tangent function.
Proposition 1. Given a right triangle \( \triangle ABC \), with \( \angle ABC = 90^\circ \), \( \tan(\angle BAC) = BC/AB \). Colloquially, the tangent is the opposite length over the adjacent length.

(Proof.) With \( \triangle ABC \) satisfying the hypotheses of the proposition, let \( D \) be the point on the line \( AB \) which is on the same side of \( A \) as \( B \), and for which \( AD = 1 \). (Of course we can find such a point by intersecting \( AB \) with a unit circle.) Notice that \( D \) may not lie between \( A \) and \( B \), as is the case in Figure 2. Now let \( \ell \) be the line passing through \( D \) which is perpendicular to \( AB \), and let \( E \) be the point where \( \ell \) intersects \( AC \). Since the triangles \( \triangle ABC \) and \( \triangle ADE \) share \( \angle A \) and \( \angle ABC = \angle ADE \), these two triangles are similar. So we have

\[
\frac{BC}{ED} = \frac{AB}{AD}.
\]

(1)

Since \( D \) was chosen to make \( AD = 1 \), \( ED = \tan(\angle BAC) \), by the definition of the tangent. So (1) tells us that

\[
\frac{BC}{\tan(\angle BAC)} = \frac{AB}{1} \quad \Rightarrow \quad \tan(\angle BAC) = \frac{BC}{AB},
\]

as desired. \( \diamond \)

A very similar argument will allow you to prove the following familiar descriptions of the sine and cosine functions.

Proposition 2. Given a right triangle \( \triangle ABC \), with \( \angle ABC = 90^\circ \), \( \sin(\angle BAC) = BC/AC \) and \( \cos(\angle BAC) = AB/AC \).

Next we’ll prove another characterization of the tangent function which is sometimes given as the definition.

Proposition 3. For any angle \( 0 < \theta < 90^\circ \), \( \tan \theta = \sin \theta / \cos \theta \).

(Proof.) Consider the triangles \( \triangle OAC \) and \( \triangle OBD \) in Figure 3, where the circle is a unit circle. Since these triangles share the angle \( \angle A \) and we have \( \angle OCA = 90^\circ = \angle ODB \), \( \triangle OAC \) is similar to \( \triangle OBD \), so

\[
\frac{BD}{AC} = \frac{OD}{OC}.
\]

(2)

Since \( OD = 1 \), we know that \( BD = \tan \theta \). Because \( OA = 1 \), we have \( AC = \sin \theta \) and \( OC = \cos \theta \). Substituting into (2) gives us

\[
\frac{\tan \theta}{\sin \theta} = \frac{1}{\cos \theta} \quad \Rightarrow \quad \tan \theta = \frac{\sin \theta}{\cos \theta},
\]

as desired. \( \diamond \)
An argument very similar to that of Proposition 3 tells us that

**Proposition 4.** For any angle $0 < \theta < 90^\circ$, $\cot \theta = \cos \theta / \sin \theta$.

This immediately gives the more familiar description of the cotangent:

**Corollary 5.** For any angle $0 < \theta < 90^\circ$, $\cot \theta = 1 / \tan \theta$.

We’re also familiar with reciprocal relationships for sine/cosecant and cosine/secant. Let’s prove (one of) these more directly.

**Proposition 6.** For any angle $0 < \theta < 90^\circ$, $\sec \theta = 1 / \cos \theta$.

*(Proof)*. Let’s refer again to Figure 3. We showed in the proof of Proposition 3 that $\triangle OAC$ is similar to $\triangle OBD$, meaning that

$$\frac{OB}{OA} = \frac{OD}{OC}. \quad (3)$$

But according to the definitions of the trigonometric functions we have $OB = \sec \theta$ and $OC = \cos \theta$. Certainly $OD = OA = 1$, meaning that

$$\frac{\sec \theta}{1} = \frac{1}{\cos \theta},$$

as we claimed. $\diamondsuit$

Of course we can very similarly show that

**Proposition 7.** For any angle $0 < \theta < 90^\circ$, $\csc \theta = 1 / \sin \theta$.

Our last proposition today is an angle addition formula, and will require the most complicated proof.

**Proposition 8.** Given angles $0 < \theta_1, \theta_2 < 90^\circ$ with $\theta_1 + \theta_2 < 90^\circ$, we have

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.$$
(Proof.) Consider the triangles in Figure 4. We have chosen $A$ so that $OA = 1$, which immediately gives $AB = \tan \theta_1, OB = \sec \theta_1$, and $AD = \tan(\theta_1 + \theta_2)$. The first observation we make is that 

$$\angle EBC = 90^\circ - \angle OBA = 90^\circ - (90^\circ - \theta_1) = \theta_1.$$ 

So the triangles $\triangle BCE$ and $\triangle OBA$ share the angles $\angle EBC = \angle AOB$ and $\angle CEB = \angle BAO$, and thus are similar. So

$$\frac{BC}{OB} = \frac{CE}{AB}. \quad (4)$$

Next, Proposition 1 (of these notes, not Elements) tells us that 

$$\tan \theta_2 = \frac{BC}{OB},$$

so $BC = \sec \theta_1 \tan \theta_2$. Substituting into (4), we have 

$$\frac{\sec \theta_1 \tan \theta_2}{\sec \theta_1} = \frac{CE}{\tan \theta_1} \Rightarrow CE = \tan \theta_1 \tan \theta_2.$$

Knowing $CE$ allows us to compute $OF$: 

$$OF = OA - FA = OA - CE = 1 - \tan \theta_1 \tan \theta_2.$$ 

This is promising, since it gives us the denominator of the identity we’re trying to prove. Our knowledge of $CE$ also helps us to compute $BE$, which in turn gives us $CF$. The similarity of $\triangle BCE$ and $\triangle OBA$ gives us

$$\frac{BE}{OA} = \frac{CE}{AB} \Rightarrow BE = \frac{\tan \theta_1 \tan \theta_2}{\tan \theta_1} = \tan \theta_2.$$ 

From this we find that 

$$CF = AB + BE = \tan \theta_1 + \tan \theta_2.$$ 

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Finally, we notice that triangles $\triangle OAD$ and $\triangle OFC$ share $\angle A$ and each have a right angle, and thus are similar. This gives us

$$\frac{AD}{CF} = \frac{OA}{OF} \Rightarrow \frac{\tan(\theta_1 + \theta_2)}{\tan \theta_1 + \tan \theta_2} = \frac{1}{1 - \tan \theta_1 \tan \theta_2},$$

and a simple rearrangement gives us the desired identity. $\diamondsuit$