The Evolute

Before providing a solution to the challenge problem, we want to give an explicit parametrization for the evolute of a regular curve whose signed curvature never vanishes.

If \( \tilde{\gamma}: (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^2 \) is a unit-speed curve whose signed curvature \( \tilde{k}(s) \) is never zero, then the evolute of \( \tilde{\gamma} \) is defined by

\[
\tilde{\epsilon}(t) := \tilde{\gamma}(t) + \tilde{R}(t)\hat{n}(t), \quad \text{for all } t \in (\tilde{\alpha}, \tilde{\beta}),
\]

where \( \tilde{R}(t) = 1/\tilde{k}(t) \) is the signed radius of curvature of \( \tilde{\gamma} \), and \( \hat{n}(t) \) is the standard unit normal vector. If \( \gamma: (\alpha, \beta) \to \mathbb{R}^2 \) is regular (but not necessarily unit-speed), we can define its evolute to be the evolute of a unit-speed reparametrization of \( \gamma \). Suppose \( \tilde{\gamma}(s) = \gamma(\phi(s)), s \in (\tilde{\alpha}, \tilde{\beta}) \), is a unit-speed reparametrization of \( \gamma \). The unit normal vectors of \( \gamma \) and \( \tilde{\gamma} \) are clearly related by

\[
\hat{n}(t) = \tilde{\hat{n}}(\phi^{-1}(t)),
\]

and we have previously seen a similar relation for their signed curvatures:

\[
k(t) = \tilde{k}(\phi^{-1}(t)).
\]

This of course means that the signed radius of curvature \( R(t) \) of \( \gamma \) is given by

\[
\tilde{R}(\phi^{-1}(t)) = R(t)\hat{n}(t),
\]

which is a reparametrization of the evolute \( \tilde{\epsilon} \) of \( \tilde{\gamma} \). That is,

\[
\epsilon(t) := \gamma(t) + R(t)\hat{n}(t), \quad \text{for all } t \in (\alpha, \beta)
\]

(1)

gives a parametrization for the evolute of \( \gamma: (\alpha, \beta) \to \mathbb{R}^2 \), provided \( \gamma \) is a regular curve whose signed curvature \( k(t) \) never vanishes. In particular, \( \gamma \) need not be unit-speed for (1) to hold.

Equation (1) will be useful in parametrizing the evolute of a regular curve \( \gamma(t) \); by taking advantage of the equation

\[
\frac{d}{dt}\hat{\epsilon}(t) = k(t)\|\tilde{\gamma}(t)\|\tilde{n}(t)
\]

(2)

that defines signed curvature, we can give an even more explicit formula. In particular, we can take the inner product of each side of (2) with the vector \( \hat{n}(t) \) to obtain

\[
\langle \hat{\epsilon}(t), \hat{n}(t) \rangle = k(t)\|\tilde{\gamma}(t)\|\langle \hat{n}(t), \hat{n}(t) \rangle = k(t)\|\tilde{\gamma}(t)\|.
\]

The last equality above uses the fact that \( \hat{n}(t) \) is a unit-length vector. Because \( \gamma \) is assumed to be regular, \( \|\tilde{\gamma}(t)\| \) is never zero, so we have

\[
k(t) = \frac{\langle \hat{\epsilon}, \hat{n}(t) \rangle}{\|\tilde{\gamma}(t)\|},
\]
and thus
\[ R(t) = \frac{\|\dot{\gamma}(t)\|}{\langle \dot{t}(t), \dot{n}(t) \rangle}. \]

We can unpack the expression \( \langle \dot{t}, \dot{n}(t) \rangle \) a bit. In particular, differentiating the equation \( \dot{t}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \) gives
\[ \dot{t}(t) = \frac{d}{dt} \left( \frac{1}{\|\dot{\gamma}(t)\|} \right) \dot{\gamma}(t) = \frac{1}{\|\dot{\gamma}(t)\|^2} \gamma(t) \dot{\gamma}(t). \]

Because \( \dot{n}(t) \) is perpendicular to \( \dot{\gamma}(t) \), the first term on the right will not contribute to our inner product, and we find that
\[ \langle \dot{t}(t), \dot{n}(t) \rangle = \frac{1}{\|\dot{\gamma}(t)\|} \langle \dot{\gamma}(t), \dot{n}(t) \rangle. \]

So in fact we have
\[ R(t) = \frac{\|\dot{\gamma}(t)\|^2}{\langle \dot{\gamma}(t), \dot{n}(t) \rangle}. \]  

(3)

Before moving on to the problem, we’ll derive one more equation. Let’s write \( \gamma \) as \( \gamma(t) = (x(t), y(t)) \) for some smooth functions \( x, y : (\alpha, \beta) \to \mathbb{R} \). We immediately have the unit tangent vector, and thus unit normal vector, for \( \gamma \):
\[ \dot{t}(t) = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}} (\dot{x}(t), \dot{y}(t)) \quad \text{and} \quad \dot{n}(t) = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}} (-\dot{y}(t), \dot{x}(t)). \]

We can then compute the denominator we see in (3):
\[ \langle \dot{\gamma}(t), \dot{n}(t) \rangle = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}} (\dot{x}(t)\dot{y}(t) - \dot{y}(t)\dot{x}(t)). \]

Because \( \|\dot{\gamma}(t)\|^2 = \dot{x}(t)^2 + \dot{y}(t)^2 \), we then have
\[ R(t) = \frac{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}{x(t)\dot{y}(t) - \dot{y}(t)x(t)}. \]  

(4)

Finally, the parametrization given by (1) becomes
\[ \epsilon(t) = (x(t), y(t)) + \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{x(t)\dot{y}(t) - \dot{y}(t)x(t)} (\dot{y}(t), \dot{x}(t)), \]  

(5)

for all \( t \in (\alpha, \beta) \).

The Problem

Let \( \gamma : (\alpha, \beta) \to \mathbb{R}^2 \) be a unit-speed parametrized curve whose signed curvature \( k(t) \) is negative for all \( t \), and let \( \epsilon(t) \) be the signed curvature of \( \gamma \), defined as in (1). Assume that \( \dot{k}(t) > 0 \) for all \( t \).

(a) Prove that the arc-length function \( s_\epsilon(t) \) of \( \epsilon \) is given by \( s_\epsilon(t) = -R(t) + C \) for some constant \( C \).

(b) Prove that the signed curvature \( k_\epsilon(t) \) of \( \epsilon \) is given by \( k_\epsilon(t) = \frac{-1}{R(t)R(t)} \).
(c) For \( t \in (\alpha, \beta) \), let \( N_t \) be the line in \( \mathbb{R}^2 \) in the direction of \( \hat{n}(t) \) containing \( \gamma(t) \). Prove that for all \( t \), the tangent vector \( \hat{t}_t(t) \) is rooted at a point of \( N_t \) and points in the direction of \( N_t \).

(d) Let \( \gamma : (0, \pi) \to \mathbb{R}^2 \) be defined by
\[
\gamma(t) = a(t - \sin(t), 1 - \cos(t)),
\]
where \( a > 0 \) is constant\(^1\). Prove that the evolute of \( \gamma \) is
\[
\epsilon(t) = a(t + \sin(t), -1 + \cos(t)).
\]

(e) Prove that \( \gamma \) is a translation of a reparametrization of \( \epsilon \).

(Solution) (a) Because \( \gamma \) is unit-speed we can show that its signed curvature satisfies
\[
\hat{n}(t) = -k(t)\gamma(t).
\]
(This is Exercise 2.2.1 of the book.) From this we see that
\[
\dot{\epsilon}(t) = \dot{\gamma}(t) + \dot{R}(t)\hat{n}(t) + R(t)\dot{\hat{n}}(t) = (1 - R(t)k(t))\dot{\gamma}(t) + \dot{R}(t)\hat{n}(t) = \dot{R}(t)\hat{n}(t).
\]
So \( \|\dot{\epsilon}(t)\| = |\dot{R}(t)||\hat{n}(t)| = |\dot{R}(t)| \). Now \( \dot{R}(t) = -k(t)/k(t)^2 \), so \( \dot{R}(t) < 0 \), since \( k(t) > 0 \). So \( |\dot{R}(t)| = -\dot{R}(t) \), and then
\[
s_\epsilon(t) = \int_{t_0}^t \|\dot{\epsilon}(u)\|\,du = \int_{t_0}^t -\dot{R}(u)\,du = -R(t) + C
\]
for some constant \( C \), as desired.

(b) Let \( \hat{t}_t(t) \) and \( \hat{n}_t(t) \) be the unit tangent and unit normal vectors for \( \epsilon(t) \), respectively. Because \( \epsilon(t) = \dot{R}(t)\hat{n}(t) \) we have
\[
\hat{t}_t(t) = \frac{\dot{\epsilon}(t)}{\|\dot{\epsilon}(t)\|} = \frac{\dot{R}(t)}{|\dot{R}(t)|}\hat{n}(t) = -\hat{n}(t).
\]
Then, writing \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \),
\[
\hat{n}_t(t) = J\hat{t}_t(t) = -J\hat{n}(t) = -J(J\hat{t}(t)) = -J^2\hat{t}(t) = \hat{t}(t),
\]
because \( J^2 = -I \). Now because the inner product \( \langle \hat{t}(t), \hat{n}(t) \rangle \) is identically zero, we have
\[
0 = \frac{d}{dt}(\langle \hat{t}(t), \hat{n}(t) \rangle) = \langle \dot{\hat{t}}(t), \hat{n}(t) \rangle + \langle \hat{t}(t), \dot{\hat{n}}(t) \rangle.
\]
Combining this with the fact that \( \gamma \) is unit-speed, we see that the signed curvature of \( \gamma \) is given by
\[
k(t) = \frac{\langle \dot{\hat{t}}(t), \hat{n}(t) \rangle}{\|\dot{\gamma}(t)\|} = -\langle \dot{\hat{t}}(t), \hat{n}(t) \rangle.
\]
Now the signed curvature of \( \epsilon \) is given by
\[
k_\epsilon(t) = \frac{\langle \dot{\hat{t}}_t(t), \hat{n}_t(t) \rangle}{\|\dot{\epsilon}(t)\|} = \frac{\langle \dot{\hat{t}}(t), \hat{n}(t) \rangle}{-\dot{R}(t)} = \frac{k(t)}{-\dot{R}(t)} = \frac{-1}{R(t)\dot{R}(t)},
\]
as we had hoped.

\(^1\)Note that \( \gamma \) is not unit-speed.
(c) Points on the line $N_t$ have the form $\gamma(t) + s\hat{n}(t)$, since $N_t$ passes through $\gamma(t)$ and is directed by $\hat{n}(t)$; certainly the point $\epsilon(t) = \gamma(t) + R(t)\hat{n}(t)$ is of this form. Moreover, because we showed that $\dot{\epsilon}(t) = -\hat{n}(t)$, the vector $\dot{\epsilon}(t)$ must point along $N_t$.

(d) We want to take advantage of equation (5). We have $x(t) = a(t - \sin t)$ and $y(t) = a(1 - \cos t)$, so

$$\dot{x}(t) = a(1 - \cos t), \quad \dot{y}(t) = a \sin t, \quad \ddot{x}(t) = a \sin t, \quad \ddot{y}(t) = a \cos t.$$ 

Notice that

$$\dot{x}(t)^2 + \dot{y}(t)^2 = a^2(1 - 2 \cos t + \cos^2 t) + a^2 \sin^2 t = a^2(2 - 2 \cos t),$$

and

$$\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t) = a^2(\cos t - \cos^2 t) - a^2 \sin t = a^2(\cos t - 1).$$

So we have

$$\frac{\dot{x}(t)^2 + \dot{y}(t)^2}{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)} = \frac{a^2(2 - 2 \cos t)}{a^2(\cos t - 1)} = -2.$$ 

According to (5) we then have

$$\epsilon(t) = (x(t), y(t)) - 2(\dot{y}(t), \dot{x}(t))$$

$$= a(t - \sin t, 1 - \cos t) - 2a(-\sin t, 1 - \cos t)$$

$$= a(t - \sin t + 2 \sin t, 1 + \cos t - 2(1 - \cos t))$$

$$= a(t + \sin t, -1 + \cos t),$$

just as expected.

(e) Notice that we have

$$\epsilon(t - \pi) = a((t - \pi) + \sin(t - \pi), -1 + \cos(t - \pi))$$

$$= a(t - \sin(t - \pi), -1 - \cos(t))$$

$$= a(t - \sin(t), 1 - \cos(t)) - a(\pi, 2)$$

$$= \gamma(t) - a(\pi, 2).$$

So $\gamma(t) = \epsilon(t - \pi) + a(\pi, 2)$, and we see that $\gamma$ is a translation of a reparametrization of $\epsilon$. 

\[\Diamond\]