Today we want to briefly recall the definition of the **Weingarten map** associated to an oriented surface $S \subset \mathbb{R}^3$ and use this map to define the **Gaussian curvature** of $S$. We will then compute the Gaussian curvature using a surface patch $\sigma: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

Unfortunately we will not have time to really interpret the Gaussian curvature in a meaningful way today, but you’re certainly encouraged to read about this quantity in the textbook or online, or to come chat about it in office hours. Our real goal today is somewhat ulterior — we’re using the Gaussian curvature as an excuse to reinforce the relationship between the Weingarten map and the second fundamental form.

### The Weingarten map and Gaussian curvature

Let $S \subset \mathbb{R}^3$ be an oriented surface, by which we mean a surface $S$ along with a continuous choice of unit normal $\hat{N}_p$ for each $p \in S$. As you have seen in lecture, this choice of unit normal defines a map

$$\nu: S \to S^2,$$

with $\nu(p) := \hat{N}_p$, which we call the **Gauss map** of $S$. For instance, Figure 1 illustrates the Gauss map of a hyperbolic paraboloid, with $\nu$ mapping curves in $S$ to curves in $S^2$. For each point $p \in S$ we have the derivative map

$$D_p \nu: T_p S \to T_{\nu(p)} S^2.$$

The “sneaky trick” that Andy mentioned in lecture is this: the vector space $T_{\nu(p)} S^2$ is the unique plane in $\mathbb{R}^3$ passing through the origin and perpendicular to $\nu(p)$. But this is precisely how we describe $T_p S$, so we see that $T_{\nu(p)} S^2 = T_p S$. We may then define the **Weingarten map**

$$W_p := -D_p \nu: T_p S \to T_p S,$$

and endomorphism of $T_p S$. You might recall from linear algebra that we can usually do quite a bit more with an endomorphism on a vector space than we can with a general linear map between vector spaces. For instance, we immediately know how to define eigenvalues and determinants for endomorphisms, and if our vector space has an inner product we can readily use an endomorphism to define a bilinear form on the space. Indeed, we’ll do these sorts of things with the Weingarten map.

**Definition.** Let $S \subset \mathbb{R}^3$ be an oriented surface and let $W_p$ be its Weingarten map at a point $p \in S$. The **Gaussian curvature** of $S$ is defined by

$$K(p) := \det(W_p)$$

for each $p \in S$.

**Remark.** We might sometimes care about the **mean curvature** of $S$, defined to be half the trace of $W_p$, but today we’ll restrict our attention to Gaussian curvature. Surfaces which locally minimize
area — such as soap films — have zero mean curvature, and are called \textit{minimal surfaces}.

As promised, we will omit a good interpretation of the Gaussian curvature today and focus instead on how we might compute this quantity. In particular, suppose we have a surface patch \( \sigma: U \subseteq \mathbb{R}^2 \to S \) whose orientation matches that of \( S \). We would like to use \( \sigma \) to compute \( K(p) \) for a fixed \( p \in S \). Notice that if we can find a matrix representation of \( \mathcal{W}_p \) with respect to some basis of \( T_p S \) then we can easily compute \( K(p) = \det(\mathcal{W}_p) \). Also, the surface patch \( \sigma \) gives us an obvious basis \( \beta_{\sigma,p} = (\sigma_u, \sigma_v) \) of \( T_p S \). The question now is whether or not we can determine \[ [\mathcal{W}_p]_{\beta_{\sigma,p}}, \] the matrix representation of \( \mathcal{W}_p \) with respect to the basis \( \beta_{\sigma,p} \). We will accomplish this by taking advantage of the relationship between the Weingarten map and the second fundamental form. Yesterday’s lecture defined the \textit{second fundamental form} \( \Pi_p: T_p S \times T_p S \to \mathbb{R} \) by

\[
\Pi_p(w_1, w_2) = I_p(\mathcal{W}_p(w_1), w_2),
\]

and showed that the second fundamental form can also be written as

\[
\Pi_p = Ld\sigma^2 + 2Mdudv + Nd\sigma^2,
\]

where

\[
L = \Pi_p(\sigma_u, \sigma_u), \quad M = \Pi_p(\sigma_u, \sigma_v) = \Pi_p(\sigma_v, \sigma_u), \quad \text{and} \quad N = \Pi_p(\sigma_v, \sigma_v).
\]

Let’s write the desired matrix representation of \( \mathcal{W}_p \) as

\[
[\mathcal{W}_p]_{\beta_{\sigma,p}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

and use this relationship with the second fundamental form to determine the values of \( a, b, c, \) and \( d \). We immediately see that

\[
\mathcal{W}_p(\sigma_u) = a\sigma_u + c\sigma_v \quad \text{and} \quad \mathcal{W}_p(\sigma_v) = b\sigma_u + d\sigma_v.
\]
This allows us to see that
\[ L = II_p(\sigma_u, \sigma_u) = I_p(\mathcal{W}_p(\sigma_u), \sigma_u) = I_p(a \sigma_u + c \sigma_v, \sigma_u) = aE + cF, \]
where \(E = I_p(\sigma_u, \sigma_u)\) and \(F = I_p(\sigma_u, \sigma_v)\) are the familiar coefficients of the first fundamental form. Similarly,
\[ N = II_p(\sigma_v, \sigma_u) = I_p(\mathcal{W}_p(\sigma_v), \sigma_v) = I_p(b \sigma_u + d \sigma_v, \sigma_v) = bF + dG. \]
We may write \(M\) either as
\[ M = II_p(\sigma_v, \sigma_u) = I_p(a \sigma_u + c \sigma_v, \sigma_v) = aF + cG \]
or as
\[ M = II_p(\sigma_v, \sigma_u) = I_p(b \sigma_u + d \sigma_v, \sigma_u) = bE + dF. \]
We may then write these four equations as a single matrix equation:
\[
\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} aE + cF & bE + dF \\ bF + dG & aF + cG \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
That is,
\[
[W_p]_{\beta_{\sigma, p}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix},
\]
(1)
We can now write the Gaussian curvature of \(S\) in terms of the coefficients of the first and second fundamental forms:
\[
\det(\mathcal{W}_p) = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \det \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{LN - M^2}{EG - F^2}.
\]
So we have
\[
K = \frac{LN - M^2}{EG - F^2}.
\]
(2)
**Example.** Let \(S \subset \mathbb{R}^3\) be the surface determined by the equation \(z = x^2 + \alpha y^2\) for some \(\alpha \in \mathbb{R}\), and let \(p \in S\) be the origin in \(\mathbb{R}^3\). Determine the Gaussian curvature of \(S\) at \(p\).

*(Solution)* This surface admits an obvious parametrization, given by
\[ \sigma(u, v) := (u, v, u^2 + \alpha v^2), \quad \text{for all } (u, v) \in \mathbb{R}^2. \]
To compute the Gaussian curvature using (2) we’ll need to find \(E, F, G, L, M,\) and \(N\). We start with
\[ \sigma_u = (1, 0, 2u), \quad \text{and} \quad \sigma_v = (0, 1, 2\alpha v), \]
so
\[ E = 1 + 4u^2, \quad F = 4uv, \quad \text{and} \quad G = 1 + 4\alpha^2 v^2. \]
Because \(p = \sigma(0, 0)\), we see that \(E(p) = G(p) = 1\) and \(F(p) = 0\). Next we compute
\[ \sigma_{uu} = (0, 0, 2), \quad \sigma_{uv} = (0, 0, 0), \quad \text{and} \quad \sigma_{vv} = (0, 0, 2\alpha). \]
Because 
\[ \sigma_u \times \sigma_v = (-2u, -2\alpha v, 1), \]
we see that \( \hat{N}_p = (0, 0, 1) \). Finally we have 
\[ L(p) = \sigma_{uu} \cdot \hat{N}_p = 2, \quad M(p) = \sigma_{uv} \cdot \hat{N}_p = 0, \quad \text{and} \quad N(p) = \sigma_{vv} \cdot \hat{N}_p = 2\alpha. \]

Now that we have all of these coefficients, we may as well write down the matrix representation of \( W_p \) in the basis \( (\sigma_u, \sigma_v) \). According to (1) we have 
\[
[W_p]_{\beta, \sigma, p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2\alpha \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2\alpha \end{pmatrix},
\]
so \( \sigma_u \) and \( \sigma_v \) are both eigenvectors of the Weingarten map, with corresponding eigenvalues 2 and \( 2\alpha \), respectively. We call the eigenvalues of \( W_p \) the \textit{principal curvatures} of \( S \) at \( p \). Finally we conclude that 
\[ K(p) = \det(W_p) = 4\alpha. \]

Notice that the sign of the Gaussian curvature agrees with the sign of \( \alpha \). We know how to plot \( S \); do our computations here agree with our physical intuition?