This short note is meant to clear up some confusion about derivatives of smooth maps — which are linear transformations — and matrix representations of derivatives, including the special case of Jacobians.

Given a smooth map \( f: \mathbb{R}^n \to \mathbb{R}^m \) we can write
\[
f(x) = (f^1(x), \ldots, f^m(x))
\]
for each \( x \in \mathbb{R}^n \). We then have the Jacobian matrix of \( f \) at \( x \), given by
\[
J_f(x) := \begin{pmatrix}
\frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n}
\end{pmatrix}.
\]
(1)

We sometimes call the Jacobian the derivative of \( f \), but we’ve also defined a linear map
\[
D_x f: T_x \mathbb{R}^n = \mathbb{R}^n \to T_{f(x)} \mathbb{R}^m = \mathbb{R}^m
\]
and called this map the derivative of \( f \). This can be clarified as follows: the Jacobian \( J_f(x) \) is the matrix representation of \( D_x f \) with respect to the standard bases of \( \mathbb{R}^n \) and \( \mathbb{R}^m \). If we chose bases \( \beta, \beta' \) of \( \mathbb{R}^n, \mathbb{R}^m \), respectively, other than the standard bases, we would expect the matrix representation \( [D_x f]_{\beta'}^\beta \) to be something other than \( J_f(x) \).

Now if we have a smooth map \( f: S_1 \to S_2 \) between smooth surfaces we know that we obtain the derivative map
\[
D_x f: T_x S_1 \to T_{f(x)} S_2
\]
for each \( x \in S_1 \). Perhaps we’re inclined to say something about the Jacobian of \( f \); the problem with this is that the vector spaces \( T_x S_1 \) and \( T_{f(x)} S_2 \) that serve as domain and codomain of \( D_x f \) don’t have standard bases. Without a preferred set of coordinates, “the Jacobian of \( f \)” isn’t very meaningful\(^1\).

Though \( T_x S_1 \) and \( T_{f(x)} S_2 \) don’t have standard bases, we do know how to obtain bases for these vector spaces. In particular, let
\[
\sigma_1: U_1 \subset \mathbb{R}^2 \to S_1 \quad \text{and} \quad \sigma_2: U_2 \subset \mathbb{R}^2 \to S_2
\]
be allowable surface patches, with \( x \in \text{im}(\sigma_1) \) and \( f(x) \in \text{im}(\sigma_2) \). Then we have bases \( \beta_{\sigma_1,x} = ((\sigma_1)_u, (\sigma_1)_v) \) and \( \beta_{\sigma_2,f(x)} = ((\sigma_2)_{\tilde{u}}, (\sigma_2)_{\tilde{v}}) \) for \( T_x S_1 \) and \( T_{f(x)} S_2 \), respectively. So we can find a matrix
\[
[D_x f]_{\beta_{\sigma_2,f(x)}}^\beta_{\sigma_1,x}
\]
\(^1\)If we fix coordinates on \( S_1 \) and \( S_2 \) we can write a matrix of derivatives as in (1), and this will be the Jacobian described in the proposition. But without coordinates, we don’t have a Jacobian.
representing $D_xf$ with respect to these bases. The following proposition tells us how to realize this matrix as the Jacobian of a particular map between subsets of $\mathbb{R}^2$.

**Proposition.** Let $f$, $\sigma_1$, $\sigma_2$, and $x \in S_1$ be as above. Then

$$[D_xf]_{\beta_{\sigma_2,f(x)}}^{\beta_{\sigma_1,x}} = J_{\sigma_2^{-1} \circ f \circ \sigma_1}(\sigma_1^{-1}(x)).$$

To prove this proposition, we’ll need the following lemma:

**Lemma.** Let $f$, $\sigma_1$, $\sigma_2$, and $x \in S_1$ be as above. Then

1. $(D_x\sigma_1^{-1})(\vec{v}) = [\vec{v}]_{\beta_{\sigma_1,x}}$ for every $\vec{v} \in T_xS_1$;
2. $(D_{f(x)}\sigma_2^{-1})(\vec{v}) = [\vec{v}]_{\beta_{\sigma_2,f(x)}}$ for every $\vec{v} \in T_{f(x)}S_2$.

**Proof.** We can write $\sigma_1^{-1}: \text{im}(\sigma) \to U_1$ as $\sigma_1^{-1}(x) = (u(x), v(x))$, where $u, v: \text{im}(\sigma) \to \mathbb{R}$ are the familiar coordinate functions. Then, because $D_xu = du$ and $D_xv = dv$, we have

$$(D_x\sigma_1^{-1})(\vec{v}) = \begin{pmatrix} du(\vec{v}) \\ dv(\vec{v}) \end{pmatrix}$$

for each $\vec{v} \in T_xS_1$. Now if $\vec{v} = a(\sigma_1)_u + b(\sigma_1)_v$, then $du(\vec{v}) = a$ and $dv(\vec{v}) = b$, so

$$(D_x\sigma_1^{-1})(\vec{v}) = \begin{pmatrix} a \\ b \end{pmatrix} = [\vec{v}]_{\beta_{\sigma_1,x}},$$

as claimed. This establishes the first claim; the second is proved similarly. 

**Proof of proposition.** The matrix representation of interest is the unique matrix such that

$$[D_xf(\vec{v})]_{\beta_{\sigma_2,f(x)}} = [D_xf]_{\beta_{\sigma_2,f(x)}}^{\beta_{\sigma_1,x}} [\vec{v}]_{\beta_{\sigma_1,x}}$$

for every $\vec{v} \in T_xS_1$. Now we can write $f$ in a silly way:

$$f = \sigma_2 \circ (\sigma_2^{-1} \circ f \circ \sigma_1) \circ \sigma_1^{-1}.$$

By the chain rule we have

$$D_xf = D_{\sigma_2^{-1}(f(x))} \sigma_2 \circ D_{\sigma_1^{-1}(x)}(\sigma_2^{-1} \circ f \circ \sigma_1) \circ D_x\sigma_1^{-1},$$

so for every $\vec{v} \in T_xS_1$ we have

$$D_xf(\vec{v}) = D_{\sigma_2^{-1}(f(x))} \sigma_2 \circ D_{\sigma_1^{-1}(x)}(\sigma_2^{-1} \circ f \circ \sigma_1) \circ D_x\sigma_1^{-1}(\vec{v})$$

$$= D_{\sigma_2^{-1}(f(x))} \sigma_2 \circ D_{\sigma_1^{-1}(x)}(\sigma_2^{-1} \circ f \circ \sigma_1)([\vec{v}]_{\beta_{\sigma_1,x}})$$

$$= (D_{\sigma_2^{-1}(f(x))} \sigma_2)(J_{\sigma_2^{-1} \circ f \circ \sigma_1}(\sigma_1^{-1}(x))[\vec{v}]_{\beta_{\sigma_1,x}}).$$

The second equality above comes from the first claim in our lemma, and the third equality uses the fact that $\sigma_2^{-1} \circ f \circ \sigma_1$ is a map between subsets of $\mathbb{R}^2$, so it has a Jacobian matrix. Now we can
apply $D_f(x)\sigma_2^{-1}$ to each side of our equality. According to the second claim of our lemma, the left hand side will become

$$(D_f(x)\sigma_2^{-1})(D_x f)(\vec{v}) = [D_x f(\vec{v})]_{\beta_{\sigma_2^{-1},f(x)}}.$$ 

On the right hand side we have

$$(D_f(x)\sigma_2^{-1} \circ D_{\sigma_2^{-1}(f(x))}\sigma_2)(J_{\sigma_2^{-1}(f(x))\sigma_1^{-1}(x)})(\vec{v})_{\beta_{\sigma_1^{-1},x}} = J_{\sigma_2^{-1}(f(x))\sigma_1^{-1}(x)}(\vec{v})_{\beta_{\sigma_1^{-1},x}}.$$ 

Altogether,

$$[D_x f(\vec{v})]_{\beta_{\sigma_2^{-1},f(x)}} = J_{\sigma_2^{-1}(f(x))\sigma_1^{-1}(x)}(\vec{v})_{\beta_{\sigma_1^{-1},x}}.$$ 

Comparing with (2), we see that

$$[D_x f]_{\beta_{\sigma_1^{-1},x}} = J_{\sigma_2^{-1}(f(x))\sigma_1^{-1}(x)}(\vec{v})_{\beta_{\sigma_1^{-1},x}},$$ 

exactly as expected. $\diamondsuit$