4 Images, Kernels, and Subspaces

In our study of linear transformations we’ve examined some of the conditions under which a transformation is invertible. Now we’re ready to investigate some ideas similar to invertibility. Namely, we would like to measure the ways in which a transformation that is not invertible fails to have an inverse.

4.1 The Image and Kernel of a Linear Transformation

Definition. The image of a function consists of all the values the function assumes. If \( f : X \rightarrow Y \) is a function from \( X \) to \( Y \), then
\[
\text{im}(f) = \{ f(x) : x \in X \}.
\]
Notice that \( \text{im}(f) \) is a subset of \( Y \).

Definition. The kernel of a function whose range is \( \mathbb{R}^n \) consists of all the values in its domain at which the function assumes the value 0. If \( f : X \rightarrow \mathbb{R}^n \) is a function from \( X \) to \( \mathbb{R}^n \), then
\[
\text{ker}(f) = \{ x \in X : f(x) = 0 \}.
\]
Notice that \( \text{ker}(f) \) is a subset of \( X \). Also, if \( T(x) = Ax \) is a linear transformation from \( \mathbb{R}^m \) to \( \mathbb{R}^n \), then \( \text{ker}(T) \) (also denoted \( \text{ker}(A) \)) is the set of solutions to the equation \( Ax = 0 \).

The kernel gives us some new ways to characterize invertible matrices.

Theorem 1. Let \( A \) be an \( n \times n \) matrix. Then the following statements are equivalent.

1. \( A \) is invertible.
2. The linear system \( Ax = b \) has a unique solution \( x \) for every \( b \in \mathbb{R}^n \).
3. \( \text{rref}(A) = I_n \).
4. \( \text{rank}(A) = n \).
5. \( \text{im}(A) = \mathbb{R}^n \).
6. \( \text{ker}(A) = \{ 0 \} \).

Example 13. (§3.1, Exercise 39 of [1]) Consider an \( n \times p \) matrix \( A \) and a \( p \times m \) matrix \( B \).

(a) What is the relationship between \( \text{ker}(AB) \) and \( \text{ker}(B) \)? Are they always equal? Is one of them always contained in the other?

(b) What is the relationship between \( \text{im}(A) \) and \( \text{im}(AB) \)?

(Solution)
(a) Recall that ker(AB) is the set of vectors \( x \in \mathbb{R}^m \) for which \( ABx = 0 \), and similarly that ker(B) is the set of vectors \( x \in \mathbb{R}^m \) for which \( Bx = 0 \). Now if \( x \) is in ker(B), then \( Bx = 0 \), so \( ABx = 0 \). This means that \( x \) is in ker(AB), so we see that ker(B) must always be contained in ker(AB). On the other hand, ker(AB) might not be a subset of ker(B). For instance, suppose that

\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .
\]

Then \( B \) is the identity matrix, so ker(B) = \{0\}. But every vector has image zero under \( AB \), so ker(AB) = \( \mathbb{R}^2 \). Certainly ker(B) does not contain ker(AB) in this case.

(b) Suppose \( y \) is in the image of \( AB \). Then \( y = ABx \) for some \( x \in \mathbb{R}^m \). That is,

\[
y = ABx = A(Bx),
\]

so \( y \) is the image of \( Bx \) under multiplication by \( A \), and is thus in the image of \( A \). So im(\( A \)) contains im(AB). On the other hand, consider

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .
\]

Then

\[
AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ,
\]

so im(AB) = \{0\}, but the image of \( A \) is the span of the vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). So im(AB) does not necessarily contain im(A).

\[\Box\]

**Example 14.** (§3.1, Exercise 48 of [1]) Consider a 2 × 2 matrix \( A \) with \( A^2 = A \).

(a) If \( w \) is in the image of \( A \), what is the relationship between \( w \) and \( Aw \)?

(b) What can you say about \( A \) if rank(\( A \)) = 2? What if rank(\( A \)) = 0?

(c) If rank(\( A \)) = 1, show that the linear transformation \( T(x) = Ax \) is the projection onto im(\( A \)) along ker(\( A \)).

**(Solution)**

(a) If \( w \) is in the image of \( A \), then \( w = Av \) for some \( v \in \mathbb{R}^2 \). Then

\[
Aw = A(Av) = A^2v = Av = w,
\]

since \( A^2 = A \). So \( Aw = w \).
(b) If \( \text{rank}(A) = 2 \), then \( A \) is invertible. Since \( A^2 = A \), we see that

\[
A = I_2A = (A^{-1}A)A = A^{-1}A^2 = A^{-1}A = I_2.
\]

So the only rank 2 \( 2 \times 2 \) matrix with the property that \( A^2 = A \) is the identity matrix. On the other hand, if \( \text{rank}(A) = 0 \) then \( A \) must be the zero matrix.

(c) If \( \text{rank}(A) = 1 \), then \( A \) is not invertible, so \( \ker(A) \neq \{0\} \). But we also know that \( A \) is not the zero matrix, so \( \ker(A) \neq \mathbb{R}^2 \). We conclude that \( \ker(A) \) must be a line in \( \mathbb{R}^2 \). Next, suppose we have \( w \in \ker(A) \cap \text{im}(A) \). Then \( Aw = 0 \) and, according to part (a), \( A^2 = w \). So \( w \) is the zero vector, meaning that \( \ker(A) \cap \text{im}(A) = \{0\} \). Since \( \text{im}(A) \) is neither 0 nor all of \( \mathbb{R}^2 \), it also must be a line in \( \mathbb{R}^2 \). So \( \ker(A) \) and \( \text{im}(A) \) are non-parallel lines in \( \mathbb{R}^2 \). Now choose \( x \in \mathbb{R}^2 \) and let \( w = x - Ax \). Notice that \( Aw = Ax - A^2x = 0 \), so \( w \in \ker(A) \). Then we may write \( x \) as the sum of an element of \( \text{im}(A) \) and an element of \( \ker(A) \):

\[
x = Ax + w.
\]

According to Exercise 2.2.33, the map \( T(x) = Ax \) is then the projection onto \( \text{im}(A) \) along \( \ker(A) \).

\[\Box\]

**Example 15.** (§3.1, Exercise 50 of [1]) Consider a square matrix \( A \) with \( \ker(A^2) = \ker(A^3) \). Is \( \ker(A^3) = \ker(A^4) \)? Justify your answer.

*(Solution)* Suppose \( x \in \ker(A^3) \). Then \( A^3x = 0 \), so

\[
A^4x = A(A^3x) = A0 = 0,
\]

meaning that \( x \in \ker(A^4) \). So \( \ker(A^3) \) is contained in \( \ker(A^4) \). On the other hand, suppose \( x \in \ker(A^4) \). Then \( A^4x = 0 \), so \( A^3(Ax) = 0 \). This means that \( Ax \) is in the kernel of \( A^3 \), and thus in \( \ker(A^2) \). So

\[
A^3x = A^2(Ax) = 0,
\]

meaning that \( x \in \ker(A^3) \). So \( \ker(A^4) \) is contained in \( \ker(A^3) \). Since each set contains the other, the two are equal: \( \ker(A^3) = \ker(A^4) \).

\[\Box\]

### 4.2 Subspaces

**Definition.** A subset \( W \) of the vector space \( \mathbb{R}^n \) is called a **subspace** of \( \mathbb{R}^n \) if it

(i) contains the zero vector;

(ii) is closed under vector addition;

(iii) is closed under scalar multiplication.
One important observation we can immediately make is that for any \( n \times m \) matrix \( A \), \( \ker(A) \) is a subspace of \( \mathbb{R}^m \) and \( \text{im}(A) \) is a subspace of \( \mathbb{R}^n \).

**Definition.** Suppose we have vectors \( v_1, \ldots, v_m \) in \( \mathbb{R}^n \). We say that a vector \( v_i \) is **redundant** if \( v_i \) is a linear combination of the preceding vectors \( v_1, \ldots, v_{i-1} \). We say that the set of vectors \( v_1, \ldots, v_m \) is **linearly independent** if none of them is redundant, and **linearly dependent** otherwise. If the vectors \( v_1, \ldots, v_m \) are linearly independent and span a subspace \( V \) of \( \mathbb{R}^n \), we say that \( v_1, \ldots, v_m \) form a **basis** of \( V \).

**Example 16.** (§3.2, Exercise 26 of [1]) Find a redundant column vector of the following matrix and write it as a linear combination of the preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of \( A \).

\[
A = \begin{bmatrix}
1 & 3 & 6 \\
1 & 2 & 5 \\
1 & 1 & 4
\end{bmatrix}.
\]

(Solution) First we notice that

\[
3 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} + \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix} = \begin{bmatrix}
6 \\
5 \\
4
\end{bmatrix},
\]

meaning that the third vector of \( A \) is redundant. This allows us to write a nontrivial relation among the vectors. Finally, these coefficients give us a nonzero element of \( \ker(A) \), since

\[
\begin{bmatrix}
1 & 3 & 6 \\
1 & 2 & 5 \\
1 & 1 & 4
\end{bmatrix} \begin{bmatrix}
3 \\
1 \\
-1
\end{bmatrix} = 3 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} + 1 \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix} - 1 \begin{bmatrix}
6 \\
5 \\
4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

\[\Box\]

**Example 17.** (§3.2, Exercise 53 of [1]) Consider a subspace \( V \) of \( \mathbb{R}^n \). We define the **orthogonal complement** \( V^\perp \) of \( V \) as the set of those vectors \( w \) in \( \mathbb{R}^n \) that are perpendicular to all vectors in \( V \); that is, \( w \cdot v = 0 \), for all \( v \in V \). Show that \( V^\perp \) is a subspace of \( \mathbb{R}^n \).

(Solution) We have three properties to check: that \( V^\perp \) contains the zero vector, that it is closed under addition, and that it is closed under scalar multiplication. Certainly \( 0 \cdot v = 0 \) for every \( v \in V \), so \( 0 \in V^\perp \). Next, suppose we have vectors \( w_1 \) and \( w_2 \) in \( V^\perp \). Then

\[
(w_1 + w_2) \cdot v = w_1 \cdot v + w_2 \cdot v = 0 + 0,
\]
since \( w_1 \cdot v = 0 \) and \( w_2 \cdot v = 0 \). So \( w_1 + w_2 \) is in \( V^\perp \), meaning that \( V^\perp \) is closed under addition. Finally, suppose we have \( w \) in \( V^\perp \) and a scalar \( k \). Then

\[
(kw) \cdot v = k(w \cdot v) = 0,
\]

so \( kw \in V^\perp \). So \( V^\perp \) is closed under scalar addition, and is thus a subspace of \( \mathbb{R}^n \). \( \diamond \)

**Example 18.** (§3.2, Exercise 54 of [1]) Consider the line \( L \) spanned by

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

in \( \mathbb{R}^3 \). Find a basis of \( L^\perp \). See Exercise 53.

*(Solution)* Suppose \( v \), with components \( v_1, v_2, \) and \( v_3 \), is in \( L^\perp \). Then

\[
0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = v_1 + 2v_2 + 3v_3.
\]

This is a linear equation in three variables. Its solution set has two free variables – \( v_2 \) and \( v_3 \) – and the remaining variable can be given in terms of these:

\[
v_1 = -2v_2 - 3v_3.
\]

Consider the vectors

\[
u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.
\]

We can check that \( u_1 \) and \( u_2 \) are both in \( L^\perp \), and since neither is a scalar multiple of the other, these two vectors are linearly independent. Finally, choose any vector

\[
v = \begin{bmatrix}
-2v_2 - 3v_3 \\
v_2 \\
v_3
\end{bmatrix}
\]

in \( L^\perp \) and notice that

\[
v_2u_1 + v_3u_2 = \begin{bmatrix} -2v_2 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3v_3 \\ 0 \\ v_3 \end{bmatrix} = v.
\]

So the linearly independent vectors \( u_1 \) and \( u_2 \) span \( L^\perp \), meaning that they provide a basis for this space. \( \diamond \)
4.3 The Dimension of a Subspace

**Definition.** The dimension of a subspace $V$ of $\mathbb{R}^n$ is the number of vectors in a basis for $V$, and is denoted $\dim(V)$.

We now have a new (and better!) definition for the rank of a matrix which can be verified to match our previous definition.

**Definition.** For any matrix $A$, $\text{rank}(A) = \dim(\text{im}(A))$.

**Example 19.** (§3.3, Exercise 78 of [1]) An $n \times n$ matrix $A$ is called nilpotent if $A^m = 0$ for some positive integer $m$. Consider a nilpotent $n \times n$ matrix $A$, and choose the smallest number $m$ such that $A^m = 0$. Pick a vector $v$ in $\mathbb{R}^n$ such that $A^{m-1}v \neq 0$. Show that the vectors $v, Av, A^2v, \ldots, A^{m-1}v$ are linearly independent.

*(Solution)* Suppose we have coefficients $c_0, c_1, \ldots, c_{m-1}$ so that

$$c_0v + c_1Av + c_2A^2v + \cdots + c_{m-1}A^{m-1}v = 0. \quad (7)$$

Multiplying both sides of this equation by $A^{m-1}$ gives

$$c_0A^{m-1}v + c_1A^m v + c_2AA^m v + \cdots + c_{m-1}A^{m-2}A^m v = 0,$$

meaning that $c_0A^{m-1}v = 0$. Since $A^{m-1}v \neq 0$, this means that $c_0 = 0$. So we may rewrite Equation 7 as

$$c_1Av + c_2A^2v + \cdots + c_{m-1}A^{m-1}v = 0.$$

We may then multiply both sides of this equation by $A^{m-2}$ to obtain

$$c_1A^{m-1}v + c_2A^m v + c_3AA^m v + \cdots + c_{m-1}A^{m-3}A^m v = 0.$$

Similar to before, this simplifies to $c_1A^{m-1}v = 0$. This tells us that $c_1 = 0$, so Equation 7 simplifies again to

$$c_2A^2v + c_3A^3v + \cdots + c_{m-1}A^{m-1}v = 0.$$

We may carry on this argument to show that each coefficient $c_i$ is zero. This means that the vectors $v, Av, A^2v, \ldots, A^{m-1}v$ admit only the trivial relation, and are thus linearly independent. ♦

**Example 20.** (§3.3, Exercise 79 of [1]) Consider a nilpotent $n \times n$ matrix $A$. Use the result demonstrated in Exercise 78 to show that $A^n = 0$.

*(Solution)* Let $m$ be the smallest integer so that $A^m = 0$, as in Exercise 78. According to that exercise, we may choose $v$ so that the vectors $v, Av, A^2v, \ldots, A^{m-1}v$ are linearly independent. We know that any collection of more than $n$ vectors in $\mathbb{R}^n$ is linearly dependent, so this collection may have at most $n$ vectors. That is, $m \leq n$, so

$$A^n = A^{n-m}A^m = A^{n-m}0 = 0,$$

as desired. ♦
Example 21. (§3.3, Exercise 82 of [1]) If a $3 \times 3$ matrix $A$ represents the projection onto a plane in $\mathbb{R}^3$, what is $\text{rank}(A)$?

(Solution) The rank of $A$ is given by the dimension of $\text{im}(A)$. Because $A$ represents the projection onto a plane, the plane onto which we’re projecting is precisely $\text{im}(A)$. That is, $\text{im}(A)$ has dimension 2, so $\text{rank}(A) = 2$. 

\hfill\Box
References