2 Vector Products

2.1 The Dot Product

The first product we’d like to define between vectors is one that measures the extent to which the vectors are pointing in the same direction. That is, if \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors pointing in the same direction, their product should be positive, while if they point in opposite directions, their product should be negative. This product should also have a feature we don’t see in products of numbers: it should be possible for a product of two vectors to be 0 even though neither of the vectors is zero. In particular, if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal, or perpendicular, then their product should be 0, since they’re pointing neither in the same nor the opposite direction from each other.

It’s possible to come up with a product strictly from these geometric properties, if a bit tedious. Instead we’ll define the dot product algebraically, and then show that it has the properties we hoped for.

**The Dot Product.** Suppose \( \mathbf{u} \) and \( \mathbf{v} \) are vectors with \( n \) components:

\[
\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle, \quad \mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle.
\]

Then the dot product of \( \mathbf{u} \) with \( \mathbf{v} \) is

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.
\]

Notice that the dot product of two vectors is a scalar, and also that \( \mathbf{u} \) and \( \mathbf{v} \) must have the same number of components in order for \( \mathbf{u} \cdot \mathbf{v} \) to be defined.

For example, if \( \mathbf{u} = \langle 1, 2, 4, -2 \rangle \) and \( \mathbf{v} = 2, 1, 0, 3 \rangle \), then

\[
\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + 2 \cdot 1 + 4 \cdot 0 + (-2) \cdot 3 = -2.
\]

It’s interesting to note that the dot product is a product of two vectors, but the result is not a vector. In particular, if \( \mathbf{u}, \mathbf{v} \), and \( \mathbf{w} \) are vectors, then \( \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) \) doesn’t make any sense, because \( \mathbf{v} \cdot \mathbf{w} \) is a scalar. Before we verify that the dot product has the geometric properties we’d hoped for, we’ll point out some algebraic properties of the dot product.

**Theorem 2.1.** Suppose \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors with the same number of components as each other, \( \lambda \) is a scalar, and let \( \mathbf{0} \) denote the zero vector. Then

(i) \( \mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0 \).

(ii) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \).

(iii) \( (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v}) = \lambda (\mathbf{u} \cdot \mathbf{v}) \).

(iv) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \) and \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \).
Theorem 2.1 is telling us that the dot product shares several nice properties with the usual product of numbers. We’ve already mentioned one that it does not share, however. Since, for example,

$$\langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 1 \cdot 0 + 0 \cdot 1 = 0,$$

we see that the dot product admits zero divisors, meaning that $u \cdot v$ can be 0 even when neither $u$ nor $v$ is 0. If we’re comfortable with the algebra of the dot product, we can make our first connection with geometry. Recall that the length of a vector $u$ is computed using the distance formula:

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Then we can notice that

$$u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 = \|u\|^2.$$

So the dot product of a vector with itself is the square of the vector’s length. This fits with our expectation that the product of two vectors pointing in the same direction be a positive number, since $\|u\|^2 > 0$ whenever $u \neq 0$. In fact, we can use the observation that $u \cdot u = \|u\|^2$ to compute the angle $\theta$ between any two vectors $u$ and $v$.

Consider the vectors in Figure 5. We’ve labeled the acute angle $\theta$ between them, but there’s another angle $2\pi - \theta$ that also lies between them. Whenever we say “let $\theta$ be the angle between two vectors,” we assume that $0 \leq \theta \leq \pi$. That is, we always take $\theta$ to be the smaller of the two angles between the two vectors.

From Figure 5 we can compute the length of the vector $u - v$ using the law of cosines:

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta.$$

But using what we know about how the dot product is related to the length of a vector, we can also compute the length of $u - v$ using the dot product:

$$\|u - v\|^2 = (u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v = \|u\|^2 + \|v\|^2 - 2u \cdot v.$$

So

$$\|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta = \|u\|^2 + \|v\|^2 - 2u \cdot v.$$
meaning that
\[ \mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta. \]

So we can compute the angle \( \theta \) between \( \mathbf{u} \) and \( \mathbf{v} \) using the dot product:
\[ \theta = \arccos \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right). \]

Notice also that the dot product has the geometric properties we set out hoping for. If the vectors are pointing in the same direction, then \( \theta = 0 \), so
\[ \mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos(0) = \| \mathbf{u} \| \| \mathbf{v} \| > 0. \]

If the vectors are orthogonal, then \( \theta = \frac{\pi}{2} \), so
\[ \mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos(\pi/2) = 0. \]

In fact, whenever the dot product between vectors \( \mathbf{u} \) and \( \mathbf{v} \) is positive, the angle between \( \mathbf{u} \) and \( \mathbf{v} \) is acute, meaning that \( \mathbf{u} \) and \( \mathbf{v} \) are pointing in the same general direction. If \( \mathbf{u} \cdot \mathbf{v} < 0 \), then the angle between \( \mathbf{u} \) and \( \mathbf{v} \) is obtuse.

**Example 2.1.** The vectors in Figure 5 are \( \mathbf{u} = \langle 1, 2 \rangle \) and \( \mathbf{v} = \langle 3, 1 \rangle \). Compute \( \theta \).

*(Solution)* First we compute the lengths of \( \mathbf{u} \) and \( \mathbf{v} \):
\[ \| \mathbf{u} \| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 2^2} = \sqrt{5}, \quad \| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{3^2 + 1^2} = \sqrt{10}. \]

Since \( \mathbf{u} \cdot \mathbf{v} = 1 \cdot 3 + 2 \cdot 1 = 5 \),
\[ \theta = \arccos \left( \frac{5}{\sqrt{5} \sqrt{10}} \right) = \arccos \left( \sqrt{\frac{5}{10}} \right) = \arccos \left( \frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}. \]

So the angle \( \theta \) between \( \mathbf{u} \) and \( \mathbf{v} \) is \( \pi/4 \).

**Example 2.2.** Find the angle \( \theta \) in the figure below.

![Diagram of a triangle with vectors labeled (-1+sqrt3,1+sqrt3) and (1+sqrt3,-1+sqrt3)](image)
(Solution) Let \( P \) be the point \((1 + \sqrt{3}, -1 + \sqrt{3})\) and let \( Q \) be the point \((-1 + \sqrt{3}, 1 + \sqrt{3})\). We’ll let \( u \) be the vector whose baspoint is \( P \) and whose terminal point is the origin, and we’ll let \( v \) have basepoint \( P \) and terminal point \( Q \), so that
\[
u = \langle -1 - \sqrt{3}, 1 - \sqrt{3} \rangle, \quad \text{and} \quad v = \langle -2, 2 \rangle.
\]
The angle between \( u \) and \( v \) is \( \theta \), since these vectors agree with the sides of our triangle which are adjacent to \( \theta \). This means that
\[
\cos \theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{(-1 - \sqrt{3}) \cdot (-2) + (1 - \sqrt{3}) \cdot 2}{\sqrt{(-1 - \sqrt{3})^2 + (1 - \sqrt{3})^2} \sqrt{(-2)^2 + 2^2}} = \frac{4}{\sqrt{8}\sqrt{8}} = \frac{1}{2}.
\]
Since the only angle \( \theta \) satisfying \( 0 \leq \theta \leq \pi \) and \( \cos \theta = 1/2 \) is \( \pi/3 \), we must have \( \theta = \pi/3 \) in the figure.

The dot product not only allows us to compute the angle between two vectors, it also allows us to compute the projection of one vector onto another. Consider the vectors \( u \) and \( v \) in Figure 6, which are the same as those in Figure 5. Since \( u = (1, 2) \), \( u \) has \( x \)-component 1 and \( y \)-component 2, but what is its \( v \)-component? That is, how much is \( u \) pointing in the \( v \) direction? One way to assess this is to project \( u \) onto \( v \), producing a vector \( \text{proj}_v u \) which points in the same direction as \( v \) and moves as much in the \( v \)-direction as does \( u \). (Notice that if \( \theta \) is obtuse, then \( \text{proj}_v u \) actually points in the \textit{opposite} direction from \( v \).)

So how do we go about finding \( \text{proj}_v u \)? If \( 0 \leq \theta \leq \pi/2 \), then \( \text{proj}_v u \) points in the same direction as \( v \), so
\[
\text{proj}_v u = \| \text{proj}_v u \| e_v,
\]
where \( e_v \) is the unit vector pointing in the direction of \( v \). Since the triangle formed by \( u \), \( \text{proj}_v u \), and the dashed line segment in Figure 6 is a right triangle,
\[
\| \text{proj}_v u \| = \| u \| \cos \theta.
\]
Substituting this into our expression for \( \text{proj}_v u \), and using what we know about how the dot product relates to the angle between two vectors, we have
\[
\text{proj}_v u = \| u \| \cos \theta \frac{v}{\|v\|} = \| u \| \frac{u \cdot v}{\|u\|\|v\|} \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|^2} v.
\]
Notice that \( \text{proj}_v u \) can also be written in terms of the unit vector:
\[
\text{proj}_v u = \frac{u \cdot v}{\|v\|} e_v.
\]
We sometimes call \( \frac{u \cdot v}{\|v\|} \) the \textit{component of} \( u \) \textit{along} \( v \).

Example 2.3. Compute \( \text{proj}_v u \), with \( u \) and \( v \) as in Figures 5 and 6.
Figure 6

(Solution) Recall that \( u = (1, 2) \) and \( v = (3, 1) \), so

\[
\text{proj}_v u = \frac{1 \cdot 3 + 2 \cdot 1}{3^2 + 1^2} (3, 1) = \frac{5}{10} (3, 1) = (\frac{3}{2}, \frac{1}{2})
\]

is the projection of \( u \) onto \( v \).

Example 2.4. (§13.3, Exercise 81 of [1]) Calculate the force (in newtons) required to push a 40-kg wagon up a 10°, frictionless incline.

(Solution) Let \( v \) be a vector pointing down the incline, and let \( F_g \) be the force on the wagon due to gravity. The magnitude of this force is \( \|F_g\| = 40g \) newtons, where \( g = 9.8 \) represents the acceleration due to gravity. We can decompose this force into two vectors, one of which is parallel to the incline and one of which is perpendicular, as seen in the following figure:

Figure 7

Here \( F_{\parallel v} \) is the projection of \( F_g \) onto \( v \), and \( F_{\perp v} = F_g - F_{\parallel v} \) is the normal force perpendicular to the incline. The force that results from pushing the wagon up the incline needs only to counteract the force \( F_{\parallel v} \). Because the angle between \( F_{\parallel v} \) and \( F_g \) is 90° - 10° = 80°, \( F_{\parallel v} \) has has magnitude

\[
\|F_{\parallel v}\| = \|\text{proj}_v F_g\| = \|F_g\| \cos(80°) \approx 40 \cdot 9.8 \cdot 0.1736 \approx 68.07 \text{N}.
\]

So a force of 68.07 newtons must be applied to push the wagon up the incline.

2.2 The Cross Product

The next vector product we introduce is the **cross product**, which is only defined between vectors in three dimensions. Like the dot product, the cross product has some incredible geometric properties, but we’ll define the cross product algebraically before observing these
Before defining the cross product we must first define determinants for $2 \times 2$ and $3 \times 3$ matrices. The determinant of a $2 \times 2$ matrix is the difference of its diagonal products:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$ 

We then define the determinant of a $3 \times 3$ matrix using the $2 \times 2$ determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$ 

There isn’t much intuition behind these definitions, but they aren’t difficult to apply. For instance:

$$\begin{vmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ -4 & 2 & 6 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 \\ 2 & 6 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ -4 & 6 \end{vmatrix} + 4 \begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 1(3 \cdot 6 - 1 \cdot 2) - 0(1 \cdot 6 - 1 \cdot (-4)) + 4(1 \cdot 2 - 3 \cdot (-4)) = 72.$$ 

We can now define the cross product as a determinant of a $3 \times 3$ matrix whose top-row entries are vectors.

**The Cross Product.** The cross product of vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is the vector

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = u_2 v_3 - u_3 v_2 \ i - (u_1 v_3 - u_3 v_1) \ j + (u_1 v_2 - u_2 v_1) \ k.$$ 

**Example 2.5.** Compute $u \times v$, where $u = \langle 1, 0, 4 \rangle$ and $v = \langle 1, 3, 1 \rangle$.

(Solution) We have

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 4 \\ 1 & 3 & 1 \end{vmatrix} = 0 \begin{vmatrix} 4 \end{vmatrix} + 1 \begin{vmatrix} 1 \end{vmatrix} + 1 \begin{vmatrix} 0 \end{vmatrix} = -12i + 3j + 3k,$$

so $u \times v = \langle -12, 3, 3 \rangle$. 

Notice that unlike the dot product, the cross product of two vectors is another vector. This means that expressions such as $(u \times v) \times w$ make sense, where $u, v, w$ are all
three-dimensional vectors. We’re now ready to make some geometric observations about the cross product of two vectors. The first is that the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. This can be checked in general from the definition of the cross product, but we’ll just verify it for the vectors in Example 2.5. We have

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \langle 1, 0, 4 \rangle \cdot \langle -12, 3, 3 \rangle = 1 \cdot (-12) + 4 \cdot 3 = 0$$

and

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \langle 1, 3, 1 \rangle \cdot \langle -12, 3, 3 \rangle = 1 \cdot (-12) + 3 \cdot 3 + 1 \cdot 3 = 0,$$

so indeed $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. This orthogonality condition tells us that $\mathbf{u} \times \mathbf{v}$ must lie on the line that’s orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, but we still don’t know which direction $\mathbf{u} \times \mathbf{v}$ is pointing or its magnitude. The question of which direction $\mathbf{u} \times \mathbf{v}$ points has an interesting answer: it points in the direction that makes $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ a right-handed system. To see what is meant by this, look at Figure 7. Here we’ve plotted $\mathbf{u} = \langle 1, 0, 4 \rangle$, $\mathbf{v} = \langle 1, 3, 1 \rangle$, and the cross product $\mathbf{u} \times \mathbf{v}$, as well as $-\mathbf{u} \times \mathbf{v}$. Since $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, it must lie on the line containing the green and red vectors in this figure. If you take your right hand and curl your fingers so that they point from $\mathbf{u}$ to $\mathbf{v}$, your thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$, the green vector in this figure. This is the sense in which $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ forms a right-handed system, and it holds for any three-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$.

We now know the direction in which $\mathbf{u} \times \mathbf{v}$ points, but we still don’t know its magnitude. This also has an interesting geometric description. Notice in Figure 7 that the vectors $\mathbf{u}$ and $\mathbf{v}$ form a parallelogram $\mathcal{P}$ in the plane that’s orthogonal to $\mathbf{u} \times \mathbf{v}$. We call this the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, and its area is equal to the magnitude of $\mathbf{u} \times \mathbf{v}$:

$$\text{Area}(\mathcal{P}) = \|\mathbf{u} \times \mathbf{v}\|.$$  

We can describe $\|\mathbf{u} \times \mathbf{v}\|$ more concretely using the angle between $\mathbf{u}$ and $\mathbf{v}$:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta,$$

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where $\theta$ is the angle between $u$ and $v$. Notice that this means that $u \times v$ and $v \times u$ have the same magnitude; since we already know that these vectors point in opposite directions, we can now conclude that

$$u \times v = -v \times u.$$ 

For this reason, we say that the cross product is *anticommutative*.

Our last application of vector products in this section is to compute volumes of parallelepipeds. Given three nonzero vectors $u, v, w$, we have a parallelepiped formed from these vectors, an example of which can be seen in Figure 8. A parallelepiped can be thought of as a three-dimensional analogue to the parallelogram. Notice that if one of the vectors lies in the plane spanned by the other two vectors, then the parallelepiped lies entirely within this plane, and thus has no volume. An interesting application of the dot and cross products is that the volume of the parallelepiped spanned by $u, v$, and $w$ can be computed by the following **triple scalar product**:

$$V = |u \cdot (v \times w)|.$$ 

**Example 2.6.** The vectors $\langle \ell, 0, 0 \rangle$, $\langle 0, w, 0 \rangle$, and $\langle 0, 0, h \rangle$ span a rectangular prism with length $\ell$, width $w$, and height $h$. Use the triple scalar product to verify that this rectangular prism has volume $|\ell wh|$.

*(Solution)* Write the vectors as

$$l = \langle \ell, 0, 0 \rangle, \quad w = \langle 0, w, 0 \rangle, \quad h = \langle 0, 0, h \rangle.$$ 

Then the volume of the prism spanned by these vectors is

$$V = |l \cdot (w \times h)|.$$ 

We first compute $w \times h$:

$$w \times h = \begin{vmatrix} i & j & k \\ 0 & w & 0 \\ 0 & 0 & h \end{vmatrix} = \langle wh, 0, 0 \rangle.$$
Then

\[ V = |\langle \ell, 0, 0 \rangle \cdot \langle wh, 0, 0 \rangle| = |\ell wh|, \]

as we suspected.

\[ \diamond \]

References