Distal and non-distal ordered abelian groups

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Indiscernible sequences

Convention: $\mathbb{M}$ is a first order structure, possibly highly saturated.

**Definition**

A sequence $(a_i)_{i \in I}$ from $\mathbb{M}^n$ is $A$-indiscernible if for all $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ from $I$ we have $a_{i_1} \cdots a_{i_m} \equiv_A a_{j_1} \cdots a_{j_m}$
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**Example**

- A strictly increasing sequence $(q_n)$ in $(\mathbb{Q}, <)$
- A sequence $(a_n)$ of algebraically independent numbers in $(\mathbb{C}; 0, 1, +, -, \cdot)$, for example, $(\exp(\sqrt{p_n}))$, where $p_n = n$th prime.
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The non-independence property (NIP)

**Definition**

\( \mathcal{M} \) is NIP iff for every formula \( \varphi(x, y) \), every indiscernible sequence \( (a_i)_{i \in I} \) from \( \mathcal{M}^{|x|} \) and every \( b \in \mathcal{M}^{|y|} \), there is \( \epsilon \in \{0, 1\} \) such that eventually

\[ \models \varphi(a_i, b)^\epsilon. \]
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The asymptotic couple $(\Gamma_{\log}, \psi)$ of the field of logarithmic transseries.
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More examples of NIP theories

**NIP (dependent)**
- Examples
  - $\langle Q^n, <_1, \ldots, <_n \rangle$
  - $\langle T, +, 0, 1, \leq, \prec \rangle$
- Contains:
  - dp-minimal
  - o-minimal
  - strongly minimal
  - stable
  - superstable
  - $\omega$-stable

**Nice Properties of Theories**

<table>
<thead>
<tr>
<th>Property</th>
<th>$\omega$-stable</th>
<th>superstable</th>
<th>stable (NOP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>strongly minimal</td>
<td>o-minimal</td>
<td>dp-minimal</td>
<td></td>
</tr>
<tr>
<td>NIP</td>
<td>supersimple</td>
<td>simple (NTP)</td>
<td></td>
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<tr>
<td>NSOP$_1$</td>
<td>NTP$_1$</td>
<td>NTP$_2$</td>
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<tr>
<td>NSOP$_3$</td>
<td>NSOP$_4$</td>
<td>NSOP$_{3+1}$</td>
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<tr>
<td>NSOP$_{\infty}$</td>
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</tbody>
</table>

Click a property above to highlight region and display details. Or click the map for specific region information.

**Map of the Universe**

Questions? Suggestions? Corrections? email me: gconant@nd.edu

References

Update Log

Features Displaying Poorly?
Stable structures

Definition

A sequence \((a_i)_{i \in I}\) from \(\mathbb{M}^n\) is **totally indiscernible (over \(A\))** if for all distinct \(i_1, \ldots, i_m\) and distinct \(j_1, \ldots, j_m\) from \(I\) we have

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ACF, every abelian group, infinite set, DCF
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Examples of stable structures

Map of the Universe

Nice Properties of Theories

- \(\omega\)-stable
- superstable
- stable (NOP)
- strongly minimal
- o-minimal
- dp-minimal
- NIP
- supersimple
- simple (NTP)
- NSOP
- NSOP_1
- NTP
- NTP_1
- NTP_2
- NSOP
- NSOP_{n+1}
- NSOP_\infty

Examples of stable (NOP)
- infinitely refining equivalence relations
- a strictly stable superflat graph
- infinitely cross-cutting equivalence relations
- DCF_p
- free group on \(n > 1\) generators
- SCF_p
- \((\mathbb{Z}^n, +, 0)\)

Definition

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We say that $\mathbb{M}$ is distal if for every tuple $d$, for every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}^p$ such that

1. $I = I_1 + (c) + I_2$, $I_1$ nonempty without greatest element, $I_2$ nonempty without least element,
2. $(a_i)_{i \in I_1 + I_2}$ is $d$-indiscernible,

then $(a_i)_{i \in I}$ is $d$-indiscernible.

Slogan: everything in sight is secretly governed by linear order(s) existing somewhere.
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- Any o-minimal theory, e.g. DLO, ODAG, RCF
- $(\mathbb{R}, <, +, \cdot, 0, 1, 2^\mathbb{Z})$ is distal, whereas $(\mathbb{R}, <, +, \cdot, 0, 1, 2^\mathbb{Q})$ is not distal (Hieronymi, Nell, 2017)
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- Any o-minimal theory, e.g. DLO, ODAG, RCF
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Theorem

The ordered abelian group \((\mathbb{Q}^>, \cdot, <)\) is not distal.
A non-distal ordered abelian group

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The ordered abelian group $(\mathbb{Q}^\times, \cdot, <)$ is not distal.

- Let $G$ be monster model of $(\mathbb{Q}^\times, \cdot, <)$, written additively.
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The ordered abelian group \((\mathbb{Q}^>^0, \cdot, <)\) is not distal.

- Let \(G\) be monster model of \((\mathbb{Q}^>^0, \cdot, <)\), written additively.
- \(G/pG\) is infinite for all (thus at least one) primes \(p\), fix such a \(p\).
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The ordered abelian group \((\mathbb{Q}^>, \cdot, <)\) is not distal.

- Let \(G\) be monster model of \((\mathbb{Q}^>, \cdot, <)\), written additively.
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  where \(x \equiv_m y\) is interpreted as \(x - y \in mG\).
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- Using Ramsey, we construct indiscernible sequence \((a_i)_{i \in (-1, 1)}\) which is rapidly increasing and \(a_i \not\equiv_p a_j\) for \(i \neq j\).
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For $a \in G \setminus nG$, let $H_a \subseteq G$ be the largest convex subgroup such that $a \notin H_a + nG$; set $H_a = \{0\}$ if $a \in nG$. Define $S_n := G/\sim$, with $a \sim a'$ iff $H_a = H_{a'}$, and let $s_n : G \to S_n$ be the canonical map. (Yes, this actually is all definable)
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- Given \( a = \sum_{q \in \mathbb{Q}} r_q \epsilon_q \in G \), let \( q_0 = \min\{q : r_q \not\equiv 0 \} \), then:

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H_a = G_{s_2(a)} := \bigoplus_{q > q_0} \mathbb{Z}_2 \epsilon_q \subseteq G
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- \( G = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}(2)q \) with the lexicographic order.
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- Let \( q_0 = \min\{q : r_q \neq 0\} \), then \( H'_a = G_{t_2(a)} = \sum_{q > q_0} \mathbb{Z}(2)q \subseteq G \).
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- Given $a = \sum_{q \in \mathbb{Q}} r_q \varepsilon_q \in G$, let $q_0 = \min\{q : r_q \neq 2 \text{ 0}\}$, then:

$$H_a = G_{s_2(a)} := \sum_{q > q_0} \mathbb{Z}(2) \varepsilon_q \subseteq G$$

- Let $q_0 = \min\{q : r_q \neq 0\}$, then $H'_a = G_{t_2(a)} = \sum_{q > q_0} \mathbb{Z}(2) \varepsilon_q \subseteq G$
- Let $q_0 = \min\{q : r_q \neq 0\}$, then $G_{t_2(a)+} = \sum_{q \geq q_0} \mathbb{Z}(2) \varepsilon_q \subseteq G$
An example

- $G = \bigoplus_{q \in Q} \mathbb{Z}_2^{(2)} \epsilon_q$ with the lexicographic order.

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- The above shows that $S_2, T_2, T_2^+$ are order-isomorphic to $(\mathbb{Q}, <)$. 
An example

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The Cluckers-Halupczok language for relative QE for OAGs

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In the theory of ordered abelian groups, each $L_{qe}$-formula $\psi(\bar{x}, \bar{\eta})$, where $\bar{x}$ are home sort variables, and $\bar{\eta}$ are auxiliary sort variables, is equivalent to an $L_{qe}$-formula $\phi(\bar{x}, \bar{\eta})$ in family union form:

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\phi(\bar{x}, \bar{\eta}) = \bigvee_{i=1}^{k} \exists \bar{\theta}(\xi_i(\bar{\eta}, \bar{\theta}) \land \psi_i(\bar{x}, \bar{\theta})),
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where $\bar{\theta}$ are auxiliary sort variables, the formulas $\xi_i(\bar{\eta}, \bar{\theta})$ live purely in the auxiliary sorts, each $\psi_i(\bar{x}, \bar{\theta})$ is a conjunction of literals (i.e., of atoms and negated atoms), and for any ordered abelian group $G$ and any $\bar{\beta}$ in the auxiliary sort of $G$ corresponding to $\bar{\eta}$, the $L_{qe}(G)$-formulas $\{\xi_i(\bar{\beta}, \bar{\alpha}) \land \psi_i(\bar{x}, \bar{\alpha}) : 1 \leq i \leq k, \bar{\alpha} \in \text{auxiliary sorts of } G\}$ are pairwise inconsistent.
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Theorem

Suppose $G$ is an ordered abelian group such that $S_p$ is finite for all primes $p$. Then the following are equivalent:

1. $G$ is distal.
2. $G/pG$ is finite for all primes $p$.
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Proof. 

(1) $\Rightarrow$ (2) Thanks to the assumptions on $S_p$, we can arrange full QE for $G$. Then the argument generalizes the one for $(\mathbb{Q}, >, \cdot, <)$.

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Thank You and Have a Happy Thanksgiving!!!