Distal and non-distal ordered abelian groups

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University of Notre Dame Model Theory Seminar
December 5, 2017
Convention: $\mathbb{M}$ is a first order structure, possibly highly saturated.

**Definition**

A sequence $(a_i)_{i \in I}$ from $\mathbb{M}^n$ is **$A$-indiscernible** if for all $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ from $I$ we have $a_{i_1} \cdots a_{i_m} \equiv_A a_{j_1} \cdots a_{j_m}$.
Indiscernible sequences

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**Example**

- A strictly increasing sequence $(q_n)$ in $(\mathbb{Q}, <)$
- A sequence $(a_n)$ of algebraically independent numbers in $(\mathbb{C}; 0, 1, +, -, \cdot)$, for example, $(\exp(\sqrt{p_n}))$, where $p_n = n$th prime.
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The non-independence property (NIP)

Definition (for monster model $\mathbb{M}$)

$\mathbb{M}$ is NIP iff for every formula $\varphi(x, y)$, every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M} \models x$ and every $b \in \mathbb{M} \models y$, there is $\epsilon \in \{0, 1\}$ such that eventually $\models \varphi(a_i, b)^\epsilon$. 
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The asymptotic couple $(\Gamma_{\text{log}}, \psi)$ of the field of logarithmic transseries.
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More examples of NIP theories

forking and dividing

Map of the Universe

Nice Properties of Theories

<table>
<thead>
<tr>
<th>Property</th>
<th>$\omega$-stable</th>
<th>superstable</th>
<th>stable (NOP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>strongly minimal</td>
<td>$o$-minimal</td>
<td>dp-minimal</td>
<td>NIP</td>
</tr>
<tr>
<td>NIP</td>
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<tr>
<td>NSOP$_1$</td>
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<td>NTP$_2$</td>
<td>NSOP$_{\infty}$</td>
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<td>NSOP$_3$</td>
<td>NSOP$_4$</td>
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</tr>
</tbody>
</table>

Click a property above to highlight region and display details. Or click the map for specific region information.

NIP (dependent)

Examples

- $(\mathbb{Q}^n, <_1, \ldots, <_n)$
- $(\mathbb{R}, +, 0, 1, <)$

Contains:

- dp-minimal
- $o$-minimal
- strongly minimal
- stable
- superstable
- $\omega$-stable

Questions? Suggestions? Corrections? email me: gconant@nd.edu

References

Update Log

Features Displaying Poorly?
Stable structures

**Definition**

A sequence \((a_i)_{i \in I}\) from \(\mathbb{M}^n\) is **totally indiscernible** (over \(A\)) if for all distinct \(i_1, \ldots, i_m\) and distinct \(j_1, \ldots, j_m\) from \(I\) we have

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ACF, every abelian group, infinite set, DCF
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Examples of stable structures

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  - dp-minimal
- NIP
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- NSOP_n
- NSOP_{n+1}
- NSOP_\omega

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stable (NOP)

- infinitely refining equivalence relations
- a strictly stable superflat graph
- infinitely cross-cutting equivalence relations
- DCF_p
- free group on \( n > 1 \) generators
- SCF_p
- \( (\mathbb{Z}^d, +, 0) \)

Definition

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1. $I = I_1 + (c) + I_2$, $I_1$ nonempty without greatest element, $I_2$ nonempty without least element,
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- Let $G$ be monster model of $(\mathbb{Q}^>, \cdot, <)$, written additively.
A non-distal ordered abelian group

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- The above shows that $S_2, T_2, T_2^+$ are order-isomorphic to $(\mathbb{Q}, <)$. 

For a prime $p \neq 2$, since $G = pG$, $S_p, T_p, T_p^+$ are trivial.
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- \( G = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}_2 \varepsilon_q \) with the lexicographic order.

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The Cluckers-Halupczok language for relative QE for OAGs

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The relative QE result for OAGs

Theorem (Cluckers, Halupczok)

In the theory of ordered abelian groups, each $L_{qe}$-formula $\psi(\bar{x}, \bar{\eta})$, where $\bar{x}$ are home sort variables, and $\bar{\eta}$ are auxiliary sort variables, is equivalent to an $L_{qe}$-formula $\phi(\bar{x}, \bar{\eta})$ in family union form:

$$\phi(\bar{x}, \bar{\eta}) = \bigvee_{i=1}^{k} \exists \bar{\theta}(\xi_i(\bar{\eta}, \bar{\theta}) \land \psi_i(\bar{x}, \bar{\theta})),$$

where $\bar{\theta}$ are auxiliary sort variables, the formulas $\xi_i(\bar{\eta}, \bar{\theta})$ live purely in the auxiliary sorts, each $\psi_i(\bar{x}, \bar{\theta})$ is a conjunction of literals (i.e., of atoms and negated atoms), and for any ordered abelian group $G$ and any $\bar{\beta}$ in the auxiliary sort of $G$ corresponding to $\bar{\eta}$, the $L_{qe}(G)$-formulas

$$\{\xi_i(\bar{\beta}, \bar{\alpha}) \land \psi_i(\bar{x}, \bar{\alpha}) : 1 \leq i \leq k, \bar{\alpha} \in \text{auxiliary sorts of } G\}$$

are pairwise inconsistent.

Corollary: Definable functions in $G$ are piecewise linear.
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Corollary: Definable functions in $G$ are piecewise linear.
### Theorem

Suppose $G$ is an ordered abelian group such that $S_p$ is finite for all primes $p$. Then the following are equivalent:

1. $G$ is distal.
2. $G/pG$ is finite for all primes $p$.
3. $G$ is dp-minimal. \[(2) \iff (3) \text{ Jahnke, Simon, Walsberg (2017)}\]

**Proof.**

1. $G$ is distal.

   Thanks to the assumptions on $S_p$, we can arrange full QE for $G$. Then the argument generalizes the one for $(\mathbb{Q}^+, \cdot, <)$.

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The assumptions include the strongly dependent ordered abelian groups. To my knowledge, all known ordered abelian groups that have full QE fall under these assumptions.
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Distal and non-distal OAGs in an easy case

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