A Tale of Two Liouville Closures...

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Overview

1. Hardy fields

2. $H$-fields
Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
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Some nice properties of Hardy fields

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Liouville extensions and closures

- **A differential field** is a characteristic zero field $K$ equipped with a derivation $\partial : K \rightarrow K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the constant field of $K$.
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A Liouville extension of a differential field $K$ is a differential field extension $L$ of $K$ such that $C_L$ is algebraic over $C_K$ and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,

- $t_i$ is algebraic over $K(t_1, \ldots, t_{i-1})$, or
- $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
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A differential field $K$ is Liouville closed if it is real closed and for every $f, g \in K$ there is $y \in K \times$ such that $y' + fy = g$. A Liouville closure of $K$ is a Liouville closed differential field extension $L$ of $K$ that is a Liouville extension. (provisional definition)
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**Theorem (Robinson, 1972)**

Define $K^{rc} = \{ g \in G : g$ is continuous and algebraic over $K \} \subseteq G$. Then $K^{rc}$ is a Hardy field and a real closure of $K$. In the category of Hardy fields, it is **THE** real closure of $K$. 

**Corollary**

If $f \in K$, then $K\left( \int f \right)$, $K\left( e^{f} \right)$, $K\left( \log(|f|) \right)$, $K\left( \exp(\int f) \right)$ are all Hardy fields.
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If $P(Y) \in K(Y)$ and $g \in \mathcal{G}$ is differentiable such that satisfies $g' = P(g)$, then $K(g)$ is a Hardy field.
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**H-fields**

- **An H-field** is an ordered differential field $K$ such that:
  - (H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;
  - (H2) $O = C_K + \wp$ where
    
    $$O = \{ g \in K : |g| \leq c \text{ for some } c \in C_K \}$$

    and $\wp$ is the maximal ideal of the convex subring $O$ of $K$.  

Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with $O = \{ f \in K : \lim_{x \to +\infty} f \in \mathbb{R} \}$, the bounded elements, and $O = \{ f \in K : \lim_{x \to +\infty} f = 0 \}$, the infinitesimal elements.

Other examples: various fields of transseries such as $T$ and $T\log$.

Conway’s No, the ordered field of surreal numbers is a proper class-sized H-field when equipped with the Berarducci-Mantova derivation $\partial_{BM}$. 

Allen Gehret (UIUC) 

Liouville closures 

Kolchin Seminar 8 / 15
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$H$-fields

- **An $H$-field** is an ordered differential field $K$ such that:
  - (H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;
  - (H2) $\mathcal{O} = C_K + \mathcal{O}$ where

    $$\mathcal{O} = \{ g \in K : |g| \leq c \text{ for some } c \in C_K \}$$

    and $\mathcal{O}$ is the maximal ideal of the convex subring $\mathcal{O}$ of $K$.

- **Example**: A Hardy field $K \supseteq \mathbb{R}$ is an $H$-field, with
  - $\mathcal{O} = \{ f \in K : \lim_{x \to +\infty} f \in \mathbb{R} \}$, the *bounded* elements, and
  - $\mathcal{O} = \{ f \in K : \lim_{x \to +\infty} f = 0 \}$, the *infinitesimal* elements

- **Other examples**: various fields of transseries such as $\mathbb{T}$ and $\mathbb{T}_{\log}$

- **Conway’s No**, the ordered field of surreal numbers is a proper class-sized $H$-field when equipped with the Berarducci-Mantova derivation $\partial_{BM}$
If $K$ is an $H$-field, by **Liouville closure of $K$** we now mean “$H$-field extension of $K$ that is also a Liouville closure of $K$ in the previous sense”

**Theorem (Aschenbrenner, van den Dries, 2002)**

Let $K$ be an $H$-field. Then one of the following occurs:

(I) $K$ has exactly one Liouville closure up to isomorphism over $K$,

(II) $K$ has exactly two Liouville closures up to isomorphism over $K$.

What causes one or two Liouville closures?
Liouville closures of $H$-fields

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What causes one or two Liouville closures?
Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \lessdot g :\iff \exists c \in C_K^0 : |f| \leq c|g|$

The valuation of an $H$-field
Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \preceq g \iff \exists c \in C_K^0 : |f| \leq c|g|$
- Also define equivalence relation $\simeq$ on $K^\times$:
  $f \simeq g \iff f \preceq g$ and $g \preceq f$
The valuation of an $H$-field

Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \preceq g : \iff \exists c \in C_K^> : |f| \leq c|g|

- Also define equivalence relation $\sim$ on $K^\times$: $f \sim g : \iff f \preceq g$ and $g \preceq f$

- The equivalence classes $vf$ are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \iff f \preceq g.$$
Let $K$ be an $H$-field and let $f, g \in K$

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- The map $f \mapsto vf : K^\times \to \Gamma$ is a valuation.
The valuation of an $H$-field

Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \preceq g : \iff \exists c \in \mathbb{C}_K : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g : \iff f \preceq g$ and $g \preceq f$
- The equivalence classes $vf$ are elements of an ordered abelian group $\Gamma_K := \nu(K^\times)$:
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- Example: in $K = \mathbb{R}(x, \arctan(x)), x \preceq x^2$ and $\arctan(x) \asymp \pi$
Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \lesssim g : \iff \exists c \in C^>_{K} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  
  \[ f \asymp g : \iff f \preceq g \text{ and } g \preceq f \]

- The equivalence classes $vf$ are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:
  
  \[ vf + vg = v(fg), \quad vf \geq vg \iff f \preceq g. \]

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The valuation of an $H$-field

Let $K$ be an $H$-field and let $f, g \in K$

- Define $f \preceq g :\iff \exists c \in C_K^> : |f| \leq c|g|$
- Also define equivalence relation $\equiv$ on $K^\times$
  
  $f \equiv g :\iff f \preceq g$ and $g \preceq f$

- The equivalence classes $vf$ are elements of an ordered abelian group $\Gamma_K := \nu(K^\times)$:
  
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The Asymptotic Couple of an $H$-field

The derivation $\partial$ induces a map

$$\gamma = vf \mapsto \gamma' = v(f') : \Gamma^\neq := \Gamma \setminus \{0\} \to \Gamma.$$ 

We set $\Psi := \{\gamma' - \gamma : \gamma \in \Gamma^\neq\}$. Then $\Psi < (\Gamma^0>)^\prime$. 

\[ \Gamma \uparrow \quad \gamma' \quad \rightarrow \Gamma \]

$\gamma^\dagger = \gamma' - \gamma$
Trichotomy for $H$-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^0)'$ for a (necessarily unique) $\beta$. We call such a $\beta$ a **gap** in $K$. Example: $K = \mathbb{R}$.

(2) $\Psi$ has a largest element. In this case we say that $K$ is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exist; equivalently: $\Gamma = (\Gamma \neq \Gamma)'$. In this case we say that $K$ has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log \log(x), \log \log \log(x), \ldots)$.

In case (1) there are two Liouville closures. Why? You can integrate $\beta$ in such a way that its antiderivative is either infinite or infinitesimal, your choice!

In case (2) there is one Liouville closure. What about case (3)?
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Exactly one of the following statements holds:

1. $\Psi < \beta < (\Gamma > 0)'$ for a (necessarily unique) $\beta$. We call such a $\beta$ a \textbf{gap} in $K$. Example: $K = \mathbb{R}$.

2. $\Psi$ has a largest element. In this case we say that $K$ is \textbf{grounded}. Example: $K = \mathbb{R}(x, \log x)$.

3. $\sup \Psi$ does not exist; equivalently: $\Gamma = (\Gamma \neq)'$. In this case we say that $K$ has \textbf{asymptotic integration}. Example: $K = \mathbb{R}(x, \log(x), \log \log(x), \log \log \log(x), \ldots)$.

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How do gaps occur in Liouville extensions?

Let $K$ be a real closed $H$-field

- If $K$ is Liouville closed, then $K$ does not have a gap.
How do gaps occur in Liouville extensions?

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- If $K$ is Liouville closed, then $K$ does not have a gap.
- If $L = K(y)$ with $y' = f \in K$ ($y = \int f$), then $L$ has a gap if and only if $K$ has a gap.

In fact, one can detect in $K$ already whether some $g \in K$ creates a gap over $K$, i.e., $z = \exp(\int g)$ is a gap in $K(z)$...
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Proposition

The following are equivalent, for a real closed $H$-field $K$:

1. $\forall f \exists g \left[ |g| > C_K \text{ and } f - g^{\dagger\dagger} \succcurlyeq g^{\dagger} \right]$, where $g^{\dagger} := \partial(g)/g$.

2. $K$ has asymptotic integration, and no element of $K$ creates a gap.

We say that $K$ is $\lambda$-free if it satisfies condition (1) in the proposition.

Theorem (G.)

Let $K$ be an $H$-field. Then

1. $K$ has exactly one Liouville closure up to isomorphism over $K$ iff (a) $K$ is grounded, or (b) $K$ has asymptotic integration and is $\lambda$-free.

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Liouville closures of $H$-fields

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Proof sketch

Assume $K$ has asymptotic integration and is $\lambda$-free. Want to show that $K$ has one Liouville closure up to isomorphism over $K$. 

Proof by procrastination!

The primary source of Two Liouville Closures is the occurrence of gaps. $\lambda$-freeness is a gap prevention property in the sense that you can’t create a gap in “the next step.”

I proved that $\lambda$-freeness is preserved under adjoining integrals and adjoining exponential integrals. Thus we “kick the can down the road”: in constructing a Liouville closure, we are forever $\lambda$-free, so we are always at least two steps away from creating a gap, so a gap never gets created!
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