Towards a model theory of logarithmic transseries

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The (Ordered) Valued Field $\mathbb{T}_{\log}$

**Definition (The valued field $\mathbb{T}_{\log}$ of logarithmic transseries)**

$$\mathbb{T}_{\log} := \bigcup_{n} \mathbb{R}[[\mathcal{L}_n]] \quad \text{union of spherically complete Hahn fields}$$

where $\mathcal{L}_n$ is the **ordered group of logarithmic transmonomials**:

$$\mathcal{L}_n := \ell_0 \cdot \ldots \cdot \ell_n = \{\ell_0^r \cdot \ldots \cdot \ell_n^r : r_i \in \mathbb{R}\}, \quad \ell_0 = x, \ell_{m+1} = \log \ell_m$$

ordered such that $\ell_i \succ \ell_{i+1}^m > 1$ for all $m \geq 1$, $i = 0, \ldots, n - 1$.

Typical elements of $\mathbb{T}_{\log}$ look like:

- $-2x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots$

- $\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$
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Note: $\mathbb{T}_{\log}$ is a real closed field and thus has a definable ordering.
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Note: $\mathbb{T}_\text{log}$ is a real closed field and thus has a definable ordering. Also: Residue field is $\mathbb{R}$ and value group $\Gamma_\text{log}$ is additive copy of $\bigcup_n \mathcal{L}_n$ with reverse ordering.
The derivation on $\mathbb{T}_{\log}$

$\mathbb{T}_{\log}$ comes equipped with the usual termwise derivative and logarithmic derivative:

$$f \mapsto f'$$

$$f \mapsto f^\dagger := \frac{f'}{f}, \quad (f \neq 0)$$

subject to the usual rules: $\ell_0' = 1, \ell_1' = \ell_0^{-1}$, etc.

For example:

- $(x^3 \log x + \sqrt{x} + 2 + \cdots)' = 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots$
- $\ell_n^\dagger = \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$
- $(\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots)' = -\frac{1}{x \log x (\log \log x)^2} - \frac{2}{x \log x (\log \log x)^3} + \cdots$
- $(\ell_0^r \cdots \ell_n^r)^\dagger = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1}$

This derivative makes $\mathbb{T}_{\log}$ into a differential field with field of constants $\mathbb{R}$.
**Definition**

A field $K$ is an ordered valued differential field. We call $K$ an **$H$-field** if

1. For all $f \in K$, if $f > C$, then $f' > 0$;
2. $O = C + \mathfrak{o}$ where $O = \{ g \in K : |g| \leq c \text{ for some } c \in C \}$ is the (convex) valuation ring of $K$ and $\mathfrak{o}$ is the maximal ideal of $O$.

**Example**

$T_{\log}$ is an $H$-field, also any Hardy field containing $\mathbb{R}$ is an $H$-field.

**Example**

$T$, the differential field of logarithmic-exponential transseries is naturally an $H$-field, and contains $T_{\log}$. It is closed under exp. Typical element:

$$-3e^x + e^{\frac{e^x}{\log x}} + e^{\frac{e^x}{\log^2 x}} + e^{\frac{e^x}{\log^3 x}} + \cdots - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \cdots + e^{-x} + 2e^{-x^2} + \cdots$$
The asymptotic couple \((\Gamma, \psi)\) of an \(H\)-field \(K\)

**Fact**

For \(f \in K^\times\) such that \(v(f) \neq 0\), the values \(v(f')\) and \(v(f^\dagger)\) depend only on \(v(f)\).

\[
\begin{array}{ccc}
K & \overset{'}{\rightarrow} & K \\
\downarrow v & & \downarrow v \\
\Gamma & \overset{'}{\rightarrow} & \Gamma
\end{array} \quad \begin{array}{ccc}
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**Definition (Rosenlicht)**

The pair \((\Gamma, \psi)\) is the *asymptotic couple of \(K\).*
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**Definition (Rosenlicht)**

The pair \((\Gamma, \psi)\) is the asymptotic couple of \(K\).

**Theorem (G)**

\(\text{Th}(\Gamma_{\log}, \psi)\), the asymptotic couple of \(\mathbb{T}_{\log}\), has quantifier elimination in a natural language and is model complete and has NIP.
Both $\mathbb{T}$ and $\mathbb{T}_{\log}$ enjoy two additional (first-order) properties:

- **$\omega$-free**: this is a very strong and robust property which prevents certain deviant behavior
  \[
  \forall f \neq 0 \exists g \neq 1[g' \simeq f] \quad \& \quad \forall f \exists g \succ 1[f + 2g^{\dagger \dagger'} + 2(g^{\dagger \dagger})^2 \succeq g^{\dagger}]
  \]

- **newtonian**: this is a variant of “differential-henselian”; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations ($\partial \mapsto \phi \partial$).
$H$-fields: two technical properties

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$\mathbb{T}_{\log}$ satisfies both of these properties because it has integration and is a union of spherically complete $H$-fields, each with a smallest “comparability class”:

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^\mathbb{R} \cdots \ell_n^\mathbb{R}]]$$
Another nice property:

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We call a real closed $H$-field $K$ **Liouville closed** if

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$T$ is Liouville closed, however...

$T_{\log}$ is NOT Liouville closed:

$$(T_{\log})' = T_{\log} \quad \text{but} \quad (T_{\log}^\times)^\dagger \neq T_{\log}$$

E.g., an element $f$ such that $f^\dagger = 1$ would have to behave like $e^x$. ..
Let $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preceq\}$

The following result is the starting point for the model theory of $\mathbb{T}_{\log}$:

**Theorem (Aschenbrenner, van den Dries, van der Hoeven, 2015)**

$\text{Th}_{\mathcal{L}}(\mathbb{T})$ is axiomatized by:
The field $\mathbb{T}$: a success story

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- real closed, $\omega$-free, newtonian, $H$-field such that $\forall \epsilon \prec 1, \partial(\epsilon) \prec 1$;
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Furthermore, $\mathbb{T}$ is model complete as an $\mathcal{L}$-structure.

Recall: a structure $M$ is *model complete* if every definable subset of $M^n$ is existentially definable (for every $n$).
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Furthermore, $\mathbb{T}$ is model complete as an $\mathcal{L}$-structure.

Recall: a structure $M$ is *model complete* if every definable subset of $M^n$ is existentially definable (for every $n$). A starting point for model completeness of $\mathbb{T}_{\log}$ is to try to make both $(\mathbb{T}_{\log} \times)^\dagger$ and its complement existentially definable.
Investigating \((\mathbb{T}_{\log}^\times)^\dagger\)

\[ f \in (\mathbb{T}_{\log}^\times)^\dagger \iff \text{there exists } g \in \mathbb{T}_{\log}^\times \text{ such that } g^\dagger = f \]
Investigating $(\mathbb{T}^\times_{\log})^\dagger$

$f \in (\mathbb{T}^\times_{\log})^\dagger \iff \text{there exists } g \in \mathbb{T}^\times_{\log} \text{ such that } g^\dagger = f$

Given $f \in \mathbb{T}^\times_{\log}$, we can write it uniquely as

$$f = c \ell^r_0 \cdots \ell^r_n (1 + \epsilon) \text{ for some infinitesimal } \epsilon \prec 1 \text{ and some } c \in \mathbb{R}^\times$$
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Then we compute the logarithmic derivative:

\[ (c\ell_0^r \cdots \ell_n^r (1 + \epsilon))^\dagger = r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1} \cdots \ell_n^{-1} + \frac{\epsilon'}{1 + \epsilon} \]

"small"
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Given \(f \in \mathbb{T}_{\log}^\times\), we can write it uniquely as

\[ f = c\ell_0^{r_0} \cdots \ell_n^{r_n}(1 + \epsilon) \quad \text{for some infinitesimal } \epsilon < 1 \text{ and some } c \in \mathbb{R}^\times \]

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\[ (c\ell_0^{r_0} \cdots \ell_n^{r_n}(1 + \epsilon))^\dagger = r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1} \cdots \ell_n^{-1} + \frac{\epsilon'}{1 + \epsilon} \]

"small"

Note: \(\nu(\ell_0^{-1} \cdots \ell_n^{-1}) \in \Psi := \psi(\Gamma_{\log}) \) and \(\nu(\epsilon'/(1 + \epsilon)) > \Psi\).
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Fact

\( f \not\in (\mathbb{T}_{\log}^\times)^\dagger \iff \text{there exists } g \in \mathbb{T}_{\log}^\times \text{ such that } \nu(f - g^\dagger) \in \Psi^\downarrow \setminus \Psi \)
Introducing LD-H-fields

From now on all $H$-fields will have asymptotic integration ($\Gamma = (\Gamma \neq)'$).

Let $K$ be an $H$-field and $LD \subseteq K$.

We call the pair $(K, LD)$ an LD-$H$-field if:

**LD1** LD is a $C_K$-vector subspace of $K$;

**LD2** $(K^\times)^\dagger \subseteq LD$;

**LD3** $I(K) := \{ y \in K : y \preceq f' \text{ for some } f \in \mathcal{O} \} \subseteq LD$; and

**LD4** $v(LD) \subseteq \Psi \cup (\Gamma >)' \cup \{ \infty \}$.

Example $(T \log, (T \times \log)\dagger)$ and $(T, T)$ are both \(\Psi\)-closed LD-$H$-fields.
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**LD4** $v(LD) \subseteq \Psi \cup (\Gamma^>')' \cup \{\infty\}$.

We say an LD-$H$-field $(K, LD)$ is **$\Psi$-closed** if:

**E1** For every $a \in K \setminus LD$, there is $b \in LD$ such that $v(a - b) \in \Psi \downarrow \setminus \Psi$; and

**E2** $LD = (K^\times)\dagger$. 

Example ($\mathrm{Tlog}$, $(\mathrm{Tlog} \times \mathrm{Tlog})\dagger$) and ($\mathrm{Tlog}$, $\mathrm{Tlog}$) are both $\Psi$-closed LD-$H$-fields.
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**Example**

$(\mathbb{T}_{log}, (\mathbb{T}^\times)^\dagger_{log})$ and $(\mathbb{T}, \mathbb{T})$ are both $\Psi$-closed LD-$H$-fields.

Allen Gehret (UIUC)  Logarithmic transseries  Thesis Defense
Model completeness conjecture for $T_{\text{log}}$

Let $\mathcal{L}_{\text{LD}} := \mathcal{L} \cup \{\text{LD}\}$ where LD is a unary relation symbol.
Model completeness conjecture for $T_{\log}$

Let $\mathcal{L}_{LD} := \mathcal{L} \cup \{LD\}$ where LD is a unary relation symbol. Let $T_{\log}$ be the $\mathcal{L}_{LD}$-theory whose models are precisely the LD-$H$-fields $(K, LD)$ such that:

1. $K$ is real closed, $\omega$-free, and newtonian;
2. $(K, LD)$ is $\Psi$-closed; and
3. $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$, where $(\Gamma, \psi)$ is the asymptotic couple of $K$. 

Conjecture

The theory $T_{\log}$ is model complete.

Embedding version of conjecture

Let $(K, LD)$ and $(L, LD_1)$ be models of $T_{\log}$ and suppose $(E, LD_0)$ is an $\omega$-free LD-$H$-subfield of $(K, LD)$ with $E_1$ such that $(Q \Gamma E, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$. Let $i : (E, LD_0) \to (L, LD_1)$ be an embedding of LD-$H$-fields. Assume $(L, LD_1)$ is $|K|^+$ saturated. Then $i$ extends to an embedding $(K, LD) \to (L, LD_1)$ of LD-$H$-fields.
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Given LD-H-fields \((K, LD)\) and \((L, LD^*)\) such that \(K \subseteq L\), we say that \((L, LD^*)\) is an extension of \((K, LD)\) (notation \((K, LD) \subseteq (L, LD^*))\) is \(LD^* \cap K = LD\).
Given LD-H-fields \((K, \text{LD})\) and \((L, \text{LD}^*)\) such that \(K \subseteq L\), we say that \((L, \text{LD}^*)\) is an extension of \((K, \text{LD})\) (notation \((K, \text{LD}) \subseteq (L, \text{LD}^*)\)) is \(\text{LD}^* \cap K = \text{LD}\).

**Proposition**

Suppose \(L\) is an algebraic extension of \(K\), \((K, \text{LD})\) has \(\text{E1}\), and \((\Gamma, \psi) \models \text{Th}(\Gamma_{\text{log}}, \psi)\). Then there is a **unique** LD-set \(\text{LD}^* \subseteq L\) such that \((K, \text{LD}) \subseteq (L, \text{LD}^*)\); equipped with this LD-set, \((L, \text{LD}^*)\) **also has E1**.

**Important case:** \(L\) is a real closure of \(K\).
Suppose $K \subseteq L$ is an extension of $H$-fields such that $L = K(C_L)$, so $L$ is a constant field extension of $K$.

**Proposition**

Suppose $K$ is henselian, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$, and $(K, LD)$ has E1. Then there is a **unique** LD-set $LD^* \subseteq L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ **also has E1**.

Thus adding new constants will never be an issue!
The $\Psi$-closure of an LD-$H$-field

**Definition**

We say an LD-$H$-field extension $(K^\Psi, LD^\Psi)$ of $(K, LD)$ is a **$\Psi$-closure of $(K, LD)$** if $K^\Psi$ is real closed, $(K^\Psi, LD^\Psi)$ is $\Psi$-closed, and for any LD-$H$-field extension $(L, LD^*)$ of $(K, LD)$ such that $L$ is real closed and $(L, LD^*)$ is $\Psi$-closed, there is an embedding $(K^\Psi, LD^\Psi) \rightarrow (L, LD^*)$ of LD-$H$-fields over $(K, LD)$.

**Proposition**

Suppose $(K, LD)$ has E1, is $\lambda$-free, and $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$. Then $(K, LD)$ has a $\Psi$-closure. Furthermore, every $\Psi$-closure will be differentially-algebraic over $K$, and its asymptotic couple will model $\text{Th}(\Gamma_{\log}, \psi)$. 
Newtonization: a reduction to the linear case

Suppose $K$ is $\omega$-free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let $K^{nt}$ be the *newtonization* of $K$ (a newtonian extension of $K$ with a universal property).

What we would like to prove: Suppose $(K, \text{LD})$ has E1. Then $\text{LD}^{nt} := \text{LD} + I(K^{nt})$ is the unique LD-set on $K^{nt}$ such that $(K, \text{LD}) \subseteq (K^{nt}, \text{LD}^{nt})$; equipped with this LD-set, $(K^{nt}, \text{LD}^{nt})$ also satisfies E1.

This can be reduced to the linear case: Conjecture 1 There is a linearly newtonian $H$-field $L$ such that $K \subseteq L \subseteq K^{nt}$ and $\text{LD}^{\ast} := \text{LD} + I(L)$ is the unique LD-set on $L$ such that $(K, \text{LD}) \subseteq (L, \text{LD}^{\ast})$; equipped with this LD-set, $(L, \text{LD}^{\ast})$ also satisfies E1.

Linearly newtonian is the fragment of newtonian that only involves degree 1 differential polynomials (differential operators).
Newtonization: a reduction to the linear case

Suppose $K$ is $\omega$-free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let $K^{nt}$ be the newtonization of $K$ (a newtonian extension of $K$ with a universal property).

What we would like to prove:

Suppose $(K, LD)$ has E1. Then $LD^{nt} := LD + I(K^{nt})$ is the unique LD-set on $K^{nt}$ such that $(K, LD) \subseteq (K^{nt}, LD^{nt})$; equipped with this LD-set, $(K^{nt}, LD^{nt})$ also satisfies E1.

This can be reduced to the linear case:
Newtonization: a reduction to the linear case

Suppose $K$ is $\omega$-free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let $K^{nt}$ be the newtonization of $K$ (a newtonian extension of $K$ with a universal property).

What we would like to prove:

Suppose $(K, LD)$ has E1. Then $LD^{nt} := LD + I(K^{nt})$ is the unique LD-set on $K^{nt}$ such that $(K, LD) \subseteq (K^{nt}, LD^{nt})$; equipped with this LD-set, $(K^{nt}, LD^{nt})$ also satisfies E1.

This can be reduced to the linear case:

Conjecture 1

There is a linearly newtonian $H$-field $L$ such that $K \subseteq L \subseteq K^{nt}$ and $LD^* := LD + I(L)$ is the unique LD-set on $L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ also satisfies E1.

Linearly newtonian is the fragment of newtonian that only involves degree 1 differential polynomials (differential operators).
Newtonization: a reduction to the linear case

Suppose $K$ is $\omega$-free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let $K^{nt}$ be the newtonization of $K$ (a newtonian extension of $K$ with a universal property).

What we would like to prove:

Suppose $(K, LD)$ has E1. Then $LD^{nt} := LD + I(K^{nt})$ is the unique LD-set on $K^{nt}$ such that $(K, LD) \subseteq (K^{nt}, LD^{nt})$; equipped with this LD-set, $(K^{nt}, LD^{nt})$ also satisfies E1.

This can be reduced to the linear case:

Conjecture 1

There is a linearly newtonian $H$-field $L$ such that $K \subseteq L \subseteq K^{nt}$ and $LD^* := LD + I(L)$ is the unique LD-set on $L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ also satisfies E1.

Linearly newtonian is the fragment of newtonian that only involves degree 1 differential polynomials (differential operators).
Two more cases we need to handle

Conjecture 2 (Differentially-transcendental immediate extension case)

Suppose \((L, \text{LD}^*)\) is an LD-\(H\)-field extension of \((K, \text{LD})\) such that \((K, \text{LD}), (L, \text{LD}^*) \models T_{\log}\), and suppose there is \(y \in L \setminus K\) such that \(K\langle y \rangle\) is an immediate extension of \(K\) (so \(y\) is necessarily differentially transcendental over \(K\) since \(K\) is asymptotically d-algebraically maximal). Then \(\text{LD}_y := \text{LD} + I(K\langle y \rangle)\) is the unique LD-set on \(K\langle y \rangle\) such that \((K, \text{LD}) \subseteq (K\langle y \rangle, \text{LD}_y)\); equipped with this LD-set, \((K\langle y \rangle, \text{LD}_y)\) also satisfies E1.

Conjecture 3 (Copy of \(\mathbb{Z}\) case)

Similar statement, but for adjoining “copies of \(\mathbb{Z}\)” to the \(\Psi\)-set of \(K\).

Model completeness follows from resolving Conjectures 1, 2, and 3.