

GENERIC AUTOMORPHISM OF A HILBERT SPACE

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ABSTRACT. We study the model companion of the continuous theory of a Hilbert space with an automorphism, i.e., a Hilbert space expanded by a unitary action of the group \mathbb{Z} .

1. INTRODUCTION

In this work we study the continuous first order theory of unitary representations of the group \mathbb{Z} . In other words, we study the theory of a Hilbert space expanded with a unitary action of \mathbb{Z} .

Such action, π , of \mathbb{Z} on a Hilbert space \mathcal{H} is characterized by $\pi(1)$, which must be a unitary linear operator on \mathcal{H} . So it is enough to study the theory of a Hilbert space expanded with one unitary operator U . Another way of viewing the theory of a \mathbb{Z} -action is thinking of U as an automorphism of \mathcal{H} . The model companion of it will be the theory of a Hilbert space with a generic automorphism.

Let U be a unitary linear operator on \mathcal{H} . The axioms which say “ \mathcal{H} is (a convex subset of the unit ball of) a Hilbert space with U acting unitarily on it”, in the language which is an extension of the standard language of Hilbert spaces with a function symbol U , give us an (almost) universal continuous theory, we call it T_U^{\forall} .

We show that T_U^{\forall} admits a model completion T_U . The class of existentially closed models of T_U^{\forall} turn out to be precisely the class of (\mathcal{H}, U) such that \mathcal{H} is (the unit ball of) a Hilbert space and $\text{Spec}(U) = S^1$. We find a natural axiomatization for T_U and show it is superstable, not ω -stable (in the sense of [BYU]), and even not small, but ω -stable and ω -categorical up to a perturbation of U (see [BY09a]). Types are characterized completely in terms of the “relative spectrum” and studied up to domination equivalence.

Independently from our work Alex Berenstein studied the universal theory of a Hilbert space expanded by a unitary action of an arbitrary discrete group G . Let us call this theory T_G^{\forall} . In [Ber07] Berenstein shows that for every such G , T_G^{\forall} admits a model completion. So the existence of the model completion of the universal theory of a Hilbert space with a unitary operator follows also from Berenstein’s work, but his techniques give neither the explicit axiomatization nor the analysis of the space of types in terms of spectral measures which we present here. On the other hand, while our results can be

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easily extended to discrete abelian groups, there is no general procedure for axiomatizing generic actions of non-abelian (even amenable) groups.

It also follows from the analysis in [Ber07] that if G is amenable and torsion free, then the regular unitary representation of G is an existentially closed model of T_G^{\forall} . Combining Berenstein’s and our results, we see that the theory of a Hilbert space with a generic automorphism that we give an explicit axiomatization of, is precisely the theory of the regular unitary representation of \mathbb{Z} .

The paper is organized as follows:

Section 2 provides a short explanation for what precisely we mean by the “continuous theory of a Hilbert space”.

Section 3 studies the class of Hilbert spaces with an automorphism. We analyze quantifier free types of singletons in terms of their “relative spectra”, which are complex Borel measures on the circle S^1 .

The goal of section 4 is to introduce “canonical structures”. We associate with any complex Borel measure μ on S^1 a specific vector v in a canonical structure for μ .

Section 4 is devoted to finding the model completion T_U of the universal theory of a Hilbert space with an automorphism and studying its properties.

Section 5 provides further understanding of types in T_U . In particular we study orthogonality and domination and connect these notions to natural measure-theoretic properties of the appropriate relative spectra.

We assume that the reader is familiar with the basics of continuous logic as developed in [BYU] and in [BYBHU08].

2. THE THEORY OF HILBERT SPACES

Continuous first order logic deals with *bounded* complete metric spaces, which Hilbert spaces are not. Therefore, when considering a Hilbert space (and more generally, a Banach space) in this logic, we shall always, by tacit understanding, replace it with its closed unit ball. Similarly, when referring to the theory of Hilbert spaces we mean the theory of unit balls thereof, and so on. (An alternative approach consists of considering a multi-sorted structure, in which the n th sort is the closed ball of radius n , but since all these balls are isomorphic by simple dilation the unit ball alone suffices.)

The theory of Banach spaces (i.e., of unit balls thereof) is given in [BYU], based on a characterization of convex subsets of Banach spaces contained in [?]. A more self-contained presentation of the theory, in a slightly different language, is given in [BY09b], and it is this approach that we shall follow. We shall consider Banach space as structures in the language

$$\mathcal{L}_{Bs} = \{0, -, \frac{x+y}{2}, \|\cdot\|\}.$$

Notice that the unit ball is indeed closed under $\frac{x+y}{2}$. We shall write $\frac{x-y}{2}$ for $\frac{x+(-y)}{2}$, and in order to keep the logic $[0, 1]$ -valued we shall equip the unit ball with half the usual

distance, $d(x, y) = \|\frac{x-y}{2}\|$. Notice that the norm is somewhat superfluous since it can be recovered from the distance as $\|x\| = d(x, -x)$.

It is shown in [BY09b] that the class of Banach spaces is elementary, and that its theory T_{Bs} can be axiomatised as follows:

(ID)

$$\forall x \left[\frac{x+x}{2} = x \right], \quad \text{i.e.,} \quad \sup_x \left[d\left(\frac{x+x}{2}, x\right) \right] = 0,$$

(PRM)

$$\forall xyz \left[\frac{1}{2} \left(\frac{x+y}{2} + \frac{z+t}{2} \right) = \frac{1}{2} \left(\frac{z+x}{2} + \frac{t+y}{2} \right) \right], \quad \text{etc.}$$

(HOM)

$$\forall xyz \left[d\left(\frac{x+z}{2}, \frac{y+z}{2}\right) = \frac{d(x,y)}{2} \right].$$

(SYM)

$$\forall x \left[\frac{x-x}{2} = 0 \right]$$

(NORM)

$$\forall x \left[d(x, 0) = \frac{1}{2} \|x\| \right]$$

(FULL)

$$\forall x \exists y \left[\|x\| \geq \frac{1}{2} \text{ or } \frac{y}{2} = x \right] \quad \text{i.e.,} \quad \sup_x \inf_y \left[\left(\frac{1}{2} \div \|x\| \right) \wedge d\left(\frac{y+0}{2}, x\right) \right] = 0$$

The universal part of T_{Bs} is T_{sc} (for symmetric convex), which is obtained by dropping the Fullness Axiom. Its models are precisely convex subsets of unit balls of Banach spaces which are in addition symmetric around 0. One can render T_{Bs} a universal theory by adding a unary function symbol $\hat{2}$ and interpreting it as $\hat{2}x = \min(2, \frac{1}{\|x\|})x$, i.e., by multiplication by 2 truncated to keep inside the unit ball. This identity which defines $\hat{2}$ can be expressed in continuous logic (we leave the details for the reader) and implies the Fullness Axiom (indeed, $\cdot 2$ serves as a Skolem function for this axiom).

It is a fact (see [You88]) that a real Banach space is a Hilbert space if and only if it satisfies the parallelogram identity

$$(PAR) \quad \forall xy \left[\left\| \frac{x+y}{2} \right\|^2 + \left\| \frac{x-y}{2} \right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} \right],$$

Adding this last axiom to T_{Bs} we obtain T_{Hs} , the theory of (real) Hilbert spaces. The inner product of a real Hilbert space is the recovered as

$$\langle x, y \rangle = \left\| \frac{x+y}{2} \right\|^2 - \left\| \frac{x-y}{2} \right\|^2.$$

Complex Banach and Hilbert spaces will be considered in

$$\mathcal{L}_{CBs} = \{0, -, i, \frac{x+y}{2}, \|\cdot\|\},$$

where i is an additional unary function symbol. A complex Banach space is a real Banach space in which i defines an automorphism whose square is $x \mapsto -x$. This can be expressed

as:

$$(CPLX) \quad \forall xy \left[\frac{ix+iy}{2} = i\frac{x+y}{2} \ \& \ \|ix\| = \|x\| \ \& \ i^2x = -x \right]$$

(Notice that $i(-x) = -ix$ follows, and therefore $i0 = 0$.) Adding this to T_{Bs} we obtain T_{CBs} , the theory of complex Banach spaces.

Conveniently enough, a complex Hilbert space is a complex Banach space whose reduct to a real Banach space satisfies the parallelogram identity, so the theory of complex Hilbert spaces is $T_{CHs} = T_{Hs} \cup T_{CBs}$. With the convention that $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle$, $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$, the inner product is then recovered as

$$\langle x, y \rangle = \left\| \frac{x+y}{2} \right\|^2 - \left\| \frac{x-y}{2} \right\|^2 + i \left(\left\| \frac{ix+y}{2} \right\|^2 - \left\| \frac{ix-y}{2} \right\|^2 \right) = \sum_{n=0}^3 i^n \left\| \frac{i^n x + y}{2} \right\|^2.$$

Notice that this is quantifier-free definable – indeed, the positive real, negative real, positive imaginary and negative imaginary parts of $\langle x, y \rangle$ can be expressed as uniform limits of quantifier free formulae in x and y . We may therefore use the inner product freely in formulae.

For the remainder of this article, by a *Hilbert space* we shall always mean a complex one, or, if considered as a logical structure, its unit ball. The theory of complex Hilbert spaces is universal once $\dot{2}$ is added to the language, and will consequently also be denoted T_H^\forall .

The theory of infinite dimensional Hilbert spaces T_H is obtained by adding the following axiom scheme:

$$\forall \bar{x} \exists y \left[\|y\| = 1 \ \& \ \langle y, x_0 \rangle = \dots = \langle y, x_{n-1} \rangle = 0 \right].$$

(Infinite dimension can be expressed for Banach spaces as well, but less elegantly.) It was shown in [BYU] that T_H is complete and admits quantifier elimination. It is therefore the model completion of T_H^\forall .

3. RELATIVE SPECTRA AND QUANTIFIER FREE TYPES

We begin with discussing an arbitrary action of \mathbb{Z} on a Hilbert space which amounts to studying the (almost) universal theory of a Hilbert space with a unitary operator (=automorphism) U .

We start with the almost universal theory of a Hilbert space T_H^\forall as described in the previous section, expand the language by a unary function symbol U and add to the original theory universal axioms saying that U is an automorphism (i.e. a unitary linear operator). We denote this theory by T_U^\forall .

The major tool in our analysis is the spectral decomposition theorem for a unitary linear operator U , which says (see e.g. [Rud91])

Fact 3.1. *Let U be a linear unitary operator on a Hilbert space \mathcal{H} . There is a unique projection valued measure E on the complex unit circle $S^1 \subseteq \mathbb{C}$ such that*

$$U = \int_{S^1} \sigma dE$$

In other words, there exists a unique mapping from the Borel subsets of S^1 to linear operators, $A \rightarrow E(A)$ s.t.

- (1) *For each A , $E(A)$ is a self adjoint orthogonal projection on \mathcal{H} . We will sometimes also denote it by E_A .*
- (2) *$E(S^1) = id_{\mathcal{H}}$.*
- (3) *If A and B are disjoint Borel subsets of S^1 , then $E(A)$ and $E(B)$ are mutually orthogonal (i.e. $E(A)$ and $E(B)$ are projections onto orthogonal subspaces) and $E(A \cup B) = E(A) + E(B)$.*
- (4) *For each $v \in \mathcal{H}$, E_v defined on Borel subsets of S^1 by $E_v(A) = E(A)(v)$ is a vector measure such that*

$$U(v) = \int_{S^1} \sigma dE_v$$

Note that it follows that

Observation 3.2. *If E is as in 3.1, then $E(A \cap B) = E(A)E(B)$ for all A, B Borel subsets of S^1 .*

Proof. This follows from the fact that if A and B are disjoint, then $E(A)$ and $E(B)$ are mutually orthogonal projections. ■_{3.2}

Let v be a vector. We are interested in how U looks on the space spanned by v and closed under U . More accurately, we are interested in an invariant such that if two such invariants are the same, the two spaces are isomorphic.

Lemma 3.3. *Let f, g be complex-valued functions on S^1 . Then*

$$\int f dE \circ \int g dE = \int f g dE$$

Proof. Since both sides are continuous in both f and g , it is enough to prove the lemma for step functions. Since both sides are linear, it is enough to prove the lemma for characteristic functions. For a characteristic function χ_A , $\int \chi_A dE = E(A)$. Because $\chi_A \chi_B = \chi_{A \cap B}$, we need to prove $E(A)E(B) = E(A \cap B)$ — but this is true by 3.2. ■_{3.3}

Proposition 3.4. *Let v be a vector, and let μ be defined (on Borel subsets of S^1) by*

$$\mu_v(A) = (E_A v, v)$$

Then

- (1) *μ_v is a complex-valued Borel measure on the circle*
- (2) *$v \rightarrow v'$ defines a partial isomorphism iff $\mu_v = \mu_{v'}$*

Proof. Since $A \rightarrow E_A v$ is a vector measure, $A \rightarrow (E_A v, v)$ is a complex-valued measure. This proves the first part of the claim.

Assume $v \rightarrow v'$ defines a partial isomorphism. From the uniqueness part of the spectral decomposition theorem, $\mu_v = \mu_{v'}$. This proves the only if direction of the second part.

Now, assume $\mu_v = \mu_{v'}$. Clearly, $v \rightarrow v'$ defines a partial isomorphism if and only if for every n, m

$$(U^n v, U^m v) = (U^n v', U^m v')$$

Without loss of generality assume $n > m$, $n = m + k$. Then because U is unitary, so is U^m and so it is enough to show

$$(U^k v, v) = (U^k v', v')$$

We use Lemma 3.3, inductively, to calculate U^k :

$$U^k = \int \sigma^k dE$$

Therefore

$$(U^k v, v) = \left(\int \sigma^k dE(v), v \right) = \int \sigma^k d\mu_v = \int \sigma^k d\mu_{v'} = \left(\int \sigma^k dE(v'), v' \right) = (U^k v', v')$$

■_{3.4}

We will call such μ the *relative spectrum* of v , so the claim above can be rephrased as “two vectors have the same quantifier free type iff they have the same relative spectrum”. In particular, from now on we will use the terms (quantifier free) type and relative spectrum interchangeably. As μ_v depends only on the q.f. type of v , we will also write μ_p instead of μ_v where $p = \text{qftp}(v)$.

The type of a unit eigenvector with eigenvalue σ is a Dirac probability measure concentrated on σ . We will call such a type an *eigentype*. Note that if v, v' realize different eigentypes, they are orthonormal.

If A is some set, let \bar{A}^U be the closed subspace generated by A and $U^{\pm 1}$. Then two vectors v and u have the same quantifier-free type over A if and only if $v - P_{\bar{A}^U}(v)$ and $u - P_{\bar{A}^U}(u)$ have the same quantifier-free type and $P_{\bar{A}^U}(v) = P_{\bar{A}^U}(u)$. Thus, all the conclusions above can be generalized to quantifier-types over sets:

Corollary 3.5. *Let A be a set v, u vectors. Denote $u' = u - P_{\bar{A}^U}(u), v' = v - P_{\bar{A}^U}(v)$. Then the following are equivalent:*

- (1) $\text{qftp}(v/A) = \text{qftp}(u/A)$
- (2) $\text{qftp}(v/A^U) = \text{qftp}(u/A^U)$
- (3) $P_{\bar{A}^U}(v) = P_{\bar{A}^U}(u)$ and $\text{qftp}(v') = \text{qftp}(u')$
- (4) $P_{\bar{A}^U}(v) = P_{\bar{A}^U}(u)$ and $\mu_{v'} = \mu_{u'}$
- (5) $P_{\bar{A}^U}(v) = P_{\bar{A}^U}(u)$ and $\langle U^k v', v' \rangle = \langle U^k u', u' \rangle$ for all $k < \omega$

4. CANONICAL STRUCTURES

Let (\mathcal{H}, U) be a Hilbert space with a unitary operator. We have shown that two substructures $(H_1, U_1), (H_2, U_2)$ generated by two vectors u, v respectively are isomorphic if and only if $\mu_u = \mu_v$. The following asserts that there exists a *canonical structure* with which we can identify both H_1 and H_2 .

Lemma 4.1. *Let (\mathcal{H}, U) be a Hilbert space with a unitary operator, $v \in \mathcal{H}$, $\mu = \mu_v$ the relative spectrum of v ; recall that this is a complex valued measure on $S^1 \subseteq \mathbb{C}$. Let (H, U) be the substructure generated by v . Denote $H' = L^2(\mu)$, this is a Hilbert space over \mathbb{C} . Let $U': H' \rightarrow H'$ be the operator of multiplication by x , that is, for every $f \in H'$ and $x \in S^1$ we have $U'(f)(x) = xf(x)$. Then U' is a unitary operator on H' and there is an isomorphism of structures (in particular, isometry) $\pi: (H, U) \rightarrow (H', U')$ such that $\pi(v) = 1_{H'}$.*

Proof. The first statement is standard and well-known, but let us nevertheless sketch the argument. Clearly, U' is linear and invertible. Let $f \in H'$. Since the domain of f is S^1 , that is, every element x in the domain has norm 1, we obtain:

$$\|U'(f)\|^2 = \int_{S^1} |x|^2 |f(x)|^2 d\mu = \int_{S^1} |f(x)|^2 d\mu = \|f\|^2$$

In order to show that π sending v to $1 \in H'$ induces a partial isomorphism, it is enough by Proposition 3.4 to argue that $\mu_1 = \mu$ in the structure (H', U') (where μ_1 is the relative spectrum of $1 \in H'$). This is again a very easy standard fact:

$$\mu_1(A) = (E_A(1), 1) = \int_{S^1} E_A(1) d\mu = \int_A 1 d\mu = \mu(A)$$

Since polynomials are dense in $H' = L^2(\mu)$, the image under π of the set $\{U^m(v) : m \in \mathbb{N}\}$ is dense in H' . Hence π is onto. ■_{4.1}

The Lemma above also shows how to produce, given a measure μ on S^1 , a structure (H, U) generated by a vector v with $\mu_v = \mu$. So we have established the following:

Corollary 4.2. *There is a 1-1 correspondence between types of vectors over the empty set (in some model of T_V^\forall) and complex-valued Borel measure on S^1 .*

We would like to study some basic properties of the canonical structure associated with a measure μ defined in Lemma 4.1 above, namely, $(L^2(\mu), U)$ where U is simply multiplication by x . In particular, we will see that the spectral measure has an especially clear form in this case. In addition to clarifying the spectral decomposition theorem somewhat, this analysis will simplify the proof of quantifier elimination in section 4 and help us to understand domination and orthogonality of types in section 5.

Lemma 4.3. *Let μ be a complex-valued measure on S^1 , $H = L^2(\mu)$, U the unitary operator of multiplication by x on H (as in Lemma 4.1). For every $f \in H$, define the following function E'_f on Borel subsets on S^1 :*

$$E'_f(A) = \chi_A \cdot f$$

Then E'_f is a vector-valued Borel measure on S^1 and for every $g \in H$, $x \in S^1$ we have

$$\left(\int_{S^1} g dE'_f \right) (x) = g(x)f(x)$$

Proof. It is easy to verify that E'_f is indeed a vector-valued measure. In order to obtain the second part, by continuity and linearity of both sides in g , it is enough to prove the equality for a characteristic function $g = \chi_A$. But in this case

$$\left(\int_{S^1} g dE'_f \right) (x) = \left(\int_{S^1} \chi_A dE'_f \right) (x) = E'_f(A)(x) = \chi_A(x)f(x)$$

and we are done. ■_{4.3}

Lemma 4.4. *Let μ be a complex-valued measure on S^1 , $H = L^2(\mu)$, U the unitary operator of multiplication by x on H . Then for every $f \in H$, $A \subseteq S^1$ Borel we have $E'_f(A) = \chi_A \cdot f = E'_f(A)$ (see Lemma 4.3).*

Proof. It is easy to see (using the previous Lemma) that the function defined on Borel subsets of S^1 by $A \mapsto E'(A)$, where $E'(A)$ is an orthogonal self-adjoint projection on H satisfying $E'(A)(f) = E'_f(A) = \chi_A \cdot f$ is a projection-valued measure on S^1 . Hence, by uniqueness of spectral decomposition, it is enough to show that

$$U(f) = \int_{S^1} \sigma dE'_f$$

And indeed, by Lemma 4.3 for every $x \in S^1$ we have

$$\left(\int_{S^1} \sigma dE'_f \right) (x) = xf(x) = (U(f))(x)$$

as required. ■_{4.4}

5. MODEL COMPANION

So far we have worked with a (almost) universal theory. There are many equivalent ways to axiomatize the model companion, all concerned with specifying enough eigenvectors. As the theory is nonmultidimensional (see below), we expect to obtain an “existential” axiomatization (i.e. in addition to the universal theory, we’ll have to require existence of enough elements satisfying enough “basic” types). In our case the basic types are the eigentypes, and the axioms for the model companion will ensure “approximate”

existence of eigenvectors for any eigenvalue of norm 1. In other words, the axioms will ensure that the spectrum of U is as rich as possible ($\text{Spec}(U) = S^1$).

We recall some basic classical concepts and facts concerning the spectrum of a unitary operator. See e.g. [Rud91] for more details and proofs.

Definition 5.1. Let T be a bounded linear operator on \mathcal{H} .

- (1) The *spectrum* of T , written $\text{Spec}(T)$, consists of those $\sigma \in \mathbb{C}$ for which $T - \sigma I$ is not invertible (where I is the identity operator).
- (2) $\sigma \in \mathbb{C}$ is called an *approximate eigenvalue* of T , if $T - \sigma I$ is not bounded away from 0, in other words, if $\mathcal{H} \models [\inf_x |T(x) - \sigma x| = 0]$.

Recall that if T is normal, then its spectrum consists exactly of its approximate eigenvalues. So in order to state that $\text{Spec}(U) = S^1$, it is enough to say that every $\sigma \in S^1$ is an approximate eigenvalue of U . Moreover, as the spectrum of a bounded linear operator is always compact, it is enough to say this for a dense subset of S^1 .

So fix a countable dense subset Q of the unit circle. For every $\sigma \in Q$ we have the following axiom:

$$(\clubsuit) \quad \left(\inf_x [|\langle x, x \rangle - 1| + \|Ux - \sigma x\|] = 0 \right)$$

which precisely says that σ is an approximate eigenvalue of U .

We call T_U the theory consisting of T_U^\forall along with all axioms of the form (\clubsuit) for $\sigma \in Q$.

Lemma 5.2. *Let $M \models T_U$ be \aleph_1 -saturated. Then for every $\sigma \in S^1$ there are uncountably many orthonormal σ -eigenvectors.*

Proof. By saturation, it is enough to show that the set of closed conditions

$$\Theta_n = \left\{ |U(x_i) - \sigma x_i| \leq \frac{1}{n} : i < n \right\}$$

is consistent for any $n < \omega$. Choosing σ_i close enough to σ and $v_i \in M$ such that $U(v_i) = \sigma_i v_i$ (this can clearly be done by saturation using the axioms of T_U), we get

$$|U(v_i) - \sigma v_i| \leq |\sigma_i - \sigma|$$

as required. ■_{5.2}

Recall that given a set A , we denote by A^U the substructure generated by A (the smallest closed subspace containing A closed under U and U^{-1}).

Lemma 5.3. *Let $M \models T_U$ be \aleph_1 -saturated, $A \subseteq M$ countable, μ a complex Borel measure on S^1 . Then there exists $v \in M$ such that*

- $v \perp A^U$
- $\mu_v = \mu$

Proof. We first deal with $A = \emptyset$.

Let (H', U') be the canonical structure (see section 4) for μ . Let μ_1 denote the relative spectrum of $1 \in H'$ (as in Lemma 4.1), so $\mu_1 = \mu$. By Proposition 3.4, it is enough to find $v \in M$ such that $\text{qftp}(v, M) = \text{qftp}(1, H')$. By saturation of M , it is enough to show that the following set of conditions is approximately finitely satisfiable in M :

$$\Theta(x) = \{\langle U^n x, x \rangle = \alpha_n : n \in \mathbb{N}\}$$

where $\alpha_n = \langle x^n, 1 \rangle^{H'}$, that is, $\langle (U')^n 1, 1 \rangle$ in H' . In other words,

$$\alpha_n = \int_{S^1} \sigma^n d\mu_1 = \int_{S^1} \sigma^n d\mu$$

So it is enough to find for any $\varepsilon > 0$ a vector $v \in M$ such that

$$\left| \int_{S^1} \sigma^n d\mu_v - \int_{S^1} \sigma^n d\mu \right| < \varepsilon$$

(since we are working with complex-valued measures, this has the obvious meaning: we relax both the real and the imaginary parts by ε)

Let m be big enough. Choose a partition of S^1 into m equal intervals I_0, \dots, I_m and vectors $u_{m,k}$ in M (for $k < m$) of norm 1 such that $u_{m,k}$ is an eigenvector for U whose eigenvalue lies in the (interior of the) interval I_k (this is possible as $M \models T_U$ and is \aleph_1 -saturated). Let

$$u_m = \sum_{k < m} \mu(I_k) u_{m,k}$$

Note that $\mu_{u_m}(I_k) = \mu(I_k)$ for all $k < m$ and u_m is of norm 1. So the sequence $\langle \mu_{u_m} : m < \omega \rangle$ weakly converges to μ , and therefore for every continuous function f on S^1 we have

$$\lim_{m \rightarrow \infty} \int_{S^1} f d\mu_{u_m} = \int f d\mu$$

In particular,

$$\lim_{m \rightarrow \infty} \int_{S^1} \sigma^n d\mu_{u_m} = \int \sigma^n d\mu$$

which is clearly what we want.

If $A \neq \emptyset$, we add to $\Theta(x)$ defined above requirements stating that $x \perp A^U$ and choose $u_{m,k}$ orthogonal to A^U . Since M is an \aleph_1 -saturated model of T_U , by Lemma 5.2 there is no problem. ■_{5.3}

Theorem 5.4. (1) T_U is a complete theory with quantifier elimination.

(2) T_U is not small and therefore not ω -stable.

- (3) T_U is superstable nonmultidimensional.
- (4) There are unbounded cardinals λ s.t. the number of models of density λ of T_U is λ . T_U has continuum many nonisomorphic separable models.

Proof. (1) We prove elimination of quantifiers by applying the “extending partial isomorphisms” criterion (see [BYU], Theorem 4.16). Given two \aleph_1 -saturated models M, N of T_U , a subset $A \subseteq M$, a partial isomorphism $f: A \rightarrow N$, and an element $v \in M$, we would like to find $u \in N$ such that $f \cup \{(v, u)\}$ is a partial automorphism; equivalently, $\text{qftp}(v/A)$ in M is the same as $\text{qftp}(u/f(A))$ in N . By Corollary 3.5 it is easy to see that we may assume that $A = \bar{A}^U$ and $v \perp A$, and the conclusion follows from Lemma 5.3.

Note that completeness of T follows: since the type of $\text{acl}(\emptyset) = \{0\}$ is the same in all models of T_U , it follows that every two saturated models of T_U (of the same density character) are isomorphic.

- (2) Note that if v, v' realize different eigentypes, they are orthonormal. Therefore, there is a set of types (the eigentypes) of cardinality continuum which is discrete in the metric topology on the type space. So the density of the type space (1-types over \emptyset) is the continuum.

Hence T_U is not small, in particular, not ω -stable.

- (3) Types over the empty set dominate all the types: given a 1-type of a vector v over a set A , it is dominated by the type over the empty set of the part of v which is orthogonal to A , and similarly for n -types. Therefore T is nonmultidimensional.

There are only continuum many types over the empty set. Let A be a set of cardinality $\kappa \geq 2^{\aleph_0}$. Without loss of generality, we can assume A is closed under $U^{\pm 1}$ and rational linear combinations. The type of a vector v over A depends only on the type of the orthogonal component of v to A and the projection of v on the closure of A . There are vectors in A as close as we want to the projection of v on \bar{A} , so the set of types has a dense subset, in the metric topology, of cardinality κ . Therefore the theory is superstable.

- (4) Let $\kappa = \aleph_\alpha$ be a cardinal, $\kappa \geq 2^{\aleph_0}$. A model M 's isomorphism type is completely characterized by the dimension of the vectors of every complex-valued measure. At least every combination of Dirac measures gives a different isomorphism type. Therefore there are at most, and at least, $|\alpha|^{2^{\aleph_0}}$. There are unbounded cardinals κ such that this equals κ . If $\kappa \leq 2^{\aleph_0}$, there are 2^κ models of density κ . In particular, there are 2^{\aleph_0} separable nonisomorphic models.

■_{5.4}

We obtain the following characterization for existential closed models of the universal theory of a Hilbert space with a unitary operator:

Corollary 5.5. *A model (H, U) of T_U^\forall is existentially closed if and only if in $H \text{ Spec}(U) = S^1$ (if and only if (H, U) is a model of T_U).*

Proof. As T_U eliminates quantifiers, it is the model companion of T_U^\forall . Therefore a model of T_U^\forall is existentially closed if and only if it is a model of T_U if and only if the spectrum of U in H contains Q if and only if it is S^1 (U is unitary, therefore normal and bounded, so its spectrum is compact, and consists precisely of the approximate eigenvalues of U). ■_{5.5}

In [Ber07] Berenstein shows in particular that if G is an amenable discrete torsion free group, then the regular unitary representation of G (the space $L^2(G)$ expanded with the natural action of G) is an existentially closed model of the universal theory of unitary representations of G . As we've just shown that T_U can be viewed as the theory of existentially closed representations of the group $G = \mathbb{Z}$, we can conclude:

Corollary 5.6. *T_U is the complete continuous theory of the regular representation of \mathbb{Z} .*

Definition 5.7. Let (H_1, U_1) and (H_2, U_2) be two models of T_U^\forall . We say that they are *isomorphic up to arbitrarily small perturbations of the automorphism U* if there for every $\varepsilon > 0$ there exists an isometry $f: H_1 \rightarrow H_2$ such that $\|f^{-1} \circ U_2 \circ f - U_1\| \leq \varepsilon$.

Another corollary from our characterization of existentially closed models of T_U^\forall is the following:

Corollary 5.8. *The theory T_U is \aleph_0 -categorical and \aleph_0 stable up to perturbation of the automorphism U .*

Proof. Let $M_0 = (H_0, U_0)$, $M_1 = (H_1, U_1)$ be two separable models of T_U . Let H be the free amalgamation (orthogonal sum) of H_0 and H_1 . We define two unitary operators on H : \hat{U}_i for $i = 0, 1$ extends U_i on H_i and is the identity on H_{1-i} . Let $N_i = (H, \hat{U}_i)$.

It is easy to see that \hat{U}_i is unitary. Moreover, clearly $\text{Spec}(\hat{U}_i) = S^1$, hence $N_i \models T_U$ by Corollary 5.5.

It is well known that any two unitary representations of \mathbb{Z} (on a separable Hilbert space H) with full spectrum are approximately unitarily equivalent (see, for example, [Con91] or Theorem II.4.4 in [Dav96]). This means that there exists a sequence of unitary operators T_n on H such that $\|T_n^{-1} \hat{U}_2 T_n - \hat{U}_1\| \leq \frac{1}{n}$. Clearly, these T_n induce isometries from H_1 onto H_2 which are as required.

\aleph_0 -stability is proven in a similar fashion: one can show that for any separable $M \models T_U$, there is a separable $N \models T_U$ which realizes all types over M up to perturbation of the automorphism.

In a sense, up to perturbation of the automorphism, over a countable set there is a countable dense subset of types (types which are dominated by the rational eigentypes, i.e. eigentypes of rational points on the circle). Similarly, up to perturbation of the automorphism, every separable model is “isomorphic” to the direct sum of infinitely many copies of H_q for q rational points on the circle, where H_q is the space generated by one eigenvector with eigenvalue q .

■_{5.8}

Discussion 5.9. So although there are continuum many separable models of T_U up to *isometry*, there is a unique such model up to arbitrarily small perturbations of the automorphism.

Let us remark that this fact explains somewhat the name of the paper. Recall that an automorphism of a countable structure M is often called *generic* if its conjugacy class in the Polish group $\text{Aut}(M)$ is dense and co-meagre. In [Usv] the second author observed that if a universal continuous theory T has an \aleph_0 -categorical model companion, then its unique model is “generic” in descriptive set theoretic sense. It follows easily that if T is a theory of a separable structure M_0 with an automorphism σ , then the model theoretically generic automorphism (the automorphism in the unique model M of the model companion) will be a generic element of $\text{Aut}(M)$ (in the sense defined above).

It is quite easy to see that this notion of genericity is too strong for most isometry groups of separable metric structures (e.g., the isometry group of the Urysohn space or the Unitary group - the group of isometries of a separable Hilbert space). On the other hand, analogously to the definition of λ -stability in continuous logic, the “correct” definition for genericity in the context of metric structures seems to be different. Instead of requiring that the conjugacy class of the automorphism is co-meagre, one should make the requirement on the closure of the conjugacy class in the metric of uniform convergence. Under this definition, the (model-theoretically) generic automorphism of a Hilbert space becomes indeed generic. Moreover, it is quite easy to see that the argument in [Usv] extends to showing that if a universal theory of a structure with an automorphism σ has a model companion which is \aleph_0 categorical up to perturbations of σ , then the model theoretically “generic” automorphism is also generic in the weak descriptive theoretic sense mentioned above.

Let \mathcal{H} be the monster model of T_U . Any model mentioned from now on will be an elementary submodel of \mathcal{H} , any set - a subset of \mathcal{H} .

We conclude this section by characterizing independence in T_U .

Observation 5.10. (1) For any set A , $\text{acl}(A) = \text{dcl}(A) =$ the completion of the minimal invariant subspace containing A .

$$(2) A \downarrow_C B \iff \text{dcl}(AC) \cap \text{dcl}(C)^\perp \perp^{\mathcal{H}} \text{dcl}(BC) \cap \text{dcl}(C)^\perp$$

Proof. (1) Clear.

(2) Just observe that $A \downarrow_C B \iff \text{dcl}(A)$ is independent from $\text{dcl}(B)$ over $\text{dcl}(C)$ in the theory of pure Hilbert space. Alternatively, note that the relation defined above satisfies the axioms of independence relation in stable theories and use the uniqueness of the independence notion.

6. ORTHOGONALITY AND DOMINATION

In this section we characterize orthogonality and domination of types in T_U . Note that T_U is stable, so these notions have the usual meaning. We will connect them to natural relations between the relative spectra.

We will mention three different notions of orthogonality. The first one is orthogonality of types, which we denote by $p \perp q$ as usual. The second one is singularity of measures, we denote it also by $\mu_p \perp \mu_q$ (which is again the standard notation in measure theory), hoping no confusion will arise. Remembering that a 1-type over the empty set is completely determined by the appropriate measure and vice versa, theorem 6.2 will justify these identical notations. Lastly, we will denote by $\perp^{\mathcal{H}}$ orthogonality in the sense of \mathcal{H} as a Hilbert space.

Recall: two measures μ_1, μ_2 are called *singular* (equivalently, μ_1 is called singular with respect to μ_2) if they are supported on disjoint subsets. As mentioned above, we denote this situation by $\mu_1 \perp \mu_2$. Also, a measure μ_1 is said to be *absolutely continuous* with respect to a measure μ_2 if $[\mu_2(A) = 0] \implies [\mu_1(A) = 0]$ for every set A . We denote this by $\mu_1 \ll \mu_2$.

We will give full analysis of orthogonality and domination of types over the empty set. As every type is completely determined by a sum of an algebraic type and a type over the empty set, this yields understanding of the general case.

Let us first note

Observation 6.1. *Let v, u be two vectors in \mathcal{H} , then $\mu_v \perp \mu_u$ if and only if $E_v \perp E_u$.*

Proof. The “if” direction is clear. The “only if” is true since $E(A)$ is a self-adjoint projection for every A , so if $\langle E(A)(v), v \rangle = 0$, then $E_v(A) = E(A)(v) = 0$. ■_{6.1}

Theorem 6.2. *Let $p, q \in S_1(\emptyset)$ be two pure 1-types.*

- (1) $p \perp q \iff \mu_p \perp \mu_q$
- (2) $p \triangleleft q \iff \mu_p \ll \mu_q$

Proof. We prove the two clauses simultaneously.

First of all, it is easy to see that p, q are weakly orthogonal if and only if they are orthogonal: suppose p and q are weakly orthogonal, let M be an \aleph_1 -saturated model and $p', q' \in S(M)$ nonforking extensions of p, q respectively. Let $a \models p', b \models q'$. Note that as $a \downarrow_{\emptyset} M$, $\text{dcl}(a) \perp^{\mathcal{H}} M$, same for b . Now, as p and q are weakly orthogonal, we get $\text{dcl}(a) \perp^{\mathcal{H}} \text{dcl}(b)$, so clearly $a \downarrow_M b$ also.

Recall that $p \perp_w q$ if and only if every two realizations of them are independent, i.e. if and only if for every $v \models p, u \models q$, $\text{dcl}(v) \perp^{\mathcal{H}} \text{dcl}(u)$, which means that for every $m, n \in \mathbb{Z}$, $\langle U^m v, U^n u \rangle = 0$.

Now assume $\mu_p \perp \mu_q$. Choose $v \models p, u \models q$. By 6.1, the appropriate E -measures E_v and E_u are orthogonal. So there exist $A_v, A_u \subseteq S^1$ disjoint such that E_v, E_u concentrate on A_v, A_u respectively. Recall that as $A_v \cap A_u = \emptyset$, $E(A_v)$ and $E(A_u)$ are projections

onto orthogonal subspaces. E_v concentrates on A_v , therefore

$$v = E(S^1)(v) = E_v(S^1) = E_v(A_v) = E(A_v)(v)$$

So $v \in \mathcal{H}_{A_v} = \text{Im}(E(A_v))$ and similarly for all moments of v (\mathcal{H}_{A_v} is an invariant subspace). So we showed that $\text{dcl}(v) \subseteq \mathcal{H}_{A_v}$. The same argument shows that $\text{dcl}(u) \subseteq \mathcal{H}_{A_u}$, but $\mathcal{H}_{A_v} \perp^{\mathcal{H}} \mathcal{H}_{A_u}$, which concludes the proof of the ‘‘if’’ direction of the first clause.

Assume next that $\mu_p \ll \mu_q$. Let $u \models q$ ($\mu_u = \mu_q$), M any model containing u , $H = \text{dcl}(u)$ the substructure generated by u . Recall that by Lemma 4.1, we can identify $(H, U \upharpoonright H)$ with $L^2(\mu_q)$ equipped with the unitary operator which is simply multiplication by x . Under this identification we have $u = 1_H$ (the constant function 1 in H).

Let $\frac{d\mu_p}{d\mu_q}$ be the Radon-Nikodym derivative of μ_p with respect to μ_q . It exists by the assumption $\mu_p \ll \mu_q$ and belongs to $L^1(\mu_q)$. So $v = \sqrt{\frac{d\mu_p}{d\mu_q}}$ is in H . Let $A \subseteq S^1$ Borel. Recalling that by Lemma 4.4 in H we have $E_v(A) = \chi_A \cdot v$, we get

$$\begin{aligned} \mu_v(A) &= (E_A(v), v) = \int_{S^1} E_A \left(\sqrt{\frac{d\mu_p}{d\mu_q}} \right) \sqrt{\frac{d\mu_p}{d\mu_q}} d\mu_q = \int_{S^1} \chi_A \sqrt{\frac{d\mu_p}{d\mu_q}} \sqrt{\frac{d\mu_p}{d\mu_q}} d\mu_q = \\ &= \int_A \frac{d\mu_p}{d\mu_q} d\mu_q = \mu_p(A) \end{aligned}$$

so $\mu_v = \mu_p$, hence $v \models p$. Consequently, every model realizing q realizes p (in fact, this is even true for every *substructure*, that is, every model of T_U^{\forall} ; moreover, in the definable closure of every realization of q there is a realization of p), so certainly $p \triangleleft q$.

The remaining two directions follow easily from the analysis above. Assume $\mu_p \not\ll \mu_q$. Then by Lebesgue decomposition, $\mu_p = \mu'_p + \nu_p$, $\nu_p \perp \mu_q$, $\mu'_p \ll \mu_q$. Let $u \models q$, and again assume $u = 1_H$ where H is as above, and construct $v' \in H$ whose spectral measure is μ'_p (using the appropriate Radon-Nikodym derivative). Let w be such that $\mu_w = \nu_p$. Now $v = v' + w$ satisfies p , and clearly v and u are not independent, as v' and u are not orthogonal (polynomials are dense in H , so no vector in H is orthogonal to $u = 1_H$). Alternatively, p and q can not be orthogonal since both dominate the nonalgebraic type p' , $\mu_{p'} = \mu'_p$.

Lastly, assume μ_p is not absolutely continuous with respects to μ_q . Then p dominates a nonalgebraic type which is orthogonal to q (similarly to the previous argument), and therefore p can not be dominated by q . ■_{6.2}

The following corollary can be extracted from the proof of Theorem 6.2.

Corollary 6.3. *Assume p, q are two nonalgebraic nonorthogonal types. Then there exists a nonalgebraic type r such that $r \triangleleft p$ and $r \triangleleft q$.*

Proof. The type whose measure is the part of μ_p which is absolutely continuous with respect to μ_q is as required. ■_{6.3}

The corollary above is trivial for discrete superstable theories (as any type is a product of regular types). Continuous superstable theories are not as nice in this respect, we do not have density of regular types, and no general decomposition theorem for types is known. By a theorem of Hernandez (see [Her92] or [She]) the corollary above is true for any stable first order theory, once the notion of domination is replaced with a weaker notion of domination between infinite powers of types. The same proof works in the continuous context; alternatively, use Shelah’s generalization ([She]) to “almost good frames” (and apply Shelah’s results treating the monster model as a homogeneous monster and working with \aleph_1 -saturated elementary submodels). We have just seen that a stronger result holds.

Question 6.4. Is T_U an example of some general phenomenon? Does Corollary 6.3 hold for every continuous superstable theory? Is there a reasonable decomposition theorem for types in a continuous superstable theory?

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