

GENERALIZED VERSHIK'S THEOREM AND GENERIC METRIC STRUCTURES

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ABSTRACT. We compare three notions of genericity of separable metric structures. Our analysis provides a general model theoretic technique of showing that structures are generic in descriptive set theoretic (topological) sense and in measure theoretic sense. In particular, it gives a new perspective on Vershik's theorems on genericity and randomness of Urysohn's space among separable metric spaces.

1. INTRODUCTION

There are several ways to define the notion of a “generic” metric structure. In this article we compare the model theoretic and two topological approaches to this question.

This work was motivated by Anatoly Vershik's results on genericity and randomness of the Urysohn space among separable metric spaces, Theorems 1 and 2 in [Ver02]. Vershik considers the collection of all separable metric spaces as a topological space, let us call it \mathfrak{S} . Some elements of \mathfrak{S} are (isometric to) the Urysohn space. Vershik shows that this set is G_δ dense in \mathfrak{S} , which leads to the conclusion that the Urysohn space is in a sense “a generic” separable metric space. Then he shows that for any “reasonable” probability measure on \mathfrak{S} , the collection of metric spaces isometric to the Urysohn space is of measure 1. This leads to the conclusion that the Urysohn space is in a sense “the random” metric space.

In his talk at the workshop on the Urysohn space at Ben-Gurion University (May 2006), Vershik said that his results had been motivated by model theoretic properties of the (countable) random graph, and that the theorems in [Ver02] are in some sense the analogues of the appropriate facts in classical model theory, although the context is different: instead of countable structures one deals with topological spaces of cardinality the continuum. In this paper we aim to show that the analogy goes much farther.

Indeed, *countable discrete* structures are replaced in this context with *separable metric* spaces, so classical model theory is not the appropriate general framework. We would like to convince the reader that there exists a natural generalization of *discrete* first order logic to the *continuous* context, in which Vershik's results are the true analogues of classical facts, the Urysohn space is an analogue of the random graph, and discrete countable models are no more than a particular case of separable continuous structures. So from our point of view, properties of the Urysohn space discovered by Vershik are

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much more than results inspired by certain similarities between this structure and the random graph; in a sense, both of these are particular cases of the same model theoretic phenomenon, which we intend to describe here.

Continuous first order logic, recently introduced by Itai Ben-Yaacov and the author in [BU], allows one to study classes of metric spaces (maybe equipped with continuous extra-structure, e.g. a collection of uniformly continuous functions from the spaces to \mathbb{R}) from model theoretic point of view. Once working in this context, many results in classical model theory generalize to analytic structures. This paper is devoted to the connection between model theory and descriptive set theory, which is very well-developed in the classical context, i.e. studying Polish spaces of countable structures for a given countable signature, countable models of a countable universal theory, etc. We will refer the reader to the excellent expository paper by Greg Hjorth, [Hj04].

Working in the context of continuous first model theory, we adapt some basic facts and techniques from [Hj04] and show how one defines a Polish topology on the space of e.g. all separable models of a certain universal continuous theory. Having done that, we discuss three different notions of genericity of a structure. One is model theoretic, genericity of a model of a universal theory among its peers). The other two are topological, genericity of a structure as an element of the appropriate Polish space in two different ways: in the sense of Baire category theory and in the sense of measure theory. Let us state things more precisely.

Let K be a “reasonable” class of separable metric structures. In our context K will normally be the class of all separable models of a certain universal continuous first order theory. From the model theoretic point of view, a generic structure in K is a structure in which “anything that can happen” in K happens. Such structures are called “existentially closed” for K . We will give precise definitions later.

On the other hand, one can consider K as a Polish space (i.e. there is a natural topology on K with respect to which K is a complete separable metric space). One can call a structure “generic” for K in topological sense if its isomorphism class is a “big” subset of K . One natural notion of “bigness” in this context is G_δ dense. Another one comes from measure theory: one can consider natural measures on the space K and ask what are the sets of measure 1.

In this article we have several primary goals:

- (i) Introduce the general model theoretic framework and the relevant notion of genericity.
- (ii) Construct the Polish space of separable metric structures.
- (iii) Connect the notions of genericity. More precisely, we explain how a model theoretic notion of genericity gives rise to G_δ dense sets in the appropriate Polish space \mathfrak{S} and sets of measure 1 with respect to any “reasonable” probability measure on \mathfrak{S} . In particular, this provides a powerful general technique for showing that certain structures are topologically generic and random (as it allows us to use well-developed model theoretic tools for this purpose).

- (iv) Conclude with some examples. In particular, we discuss model theory of Urysohn space and show that our results generalize Vershik's theorems on its "topological" genericity.

2. PRELIMINARIES AND BASICS

2.1. Continuous logic. Continuous first order logic was introduced in [BU] and developed further by Itai Ben-Yaacov, Alexander Berenstein, C. Ward Henson and the author. We refer the reader to [BBHU] for a detailed exposition. We will now try to summarize some important basic notions, facts and notations.

Just as in classical predicate logic, one starts with a fixed *signature (vocabulary)* τ . In this paper, τ will be countable. A signature (vocabulary) is a collection of function symbols and predicate symbols as well as continuity moduli for all these symbols. There is a distinguished predicate symbol $d(x, y)$, which will correspond to the metric.

Given a vocabulary τ , one constructs the continuous *language* L which corresponds to it, which consists of continuous first order τ -*formulae*. As in classical first order logic, formulae are constructed by induction using *connectives* and *quantifiers*. Any countable collection of continuous functions from $[0, 1]^k$ to $[0, 1]$ (for any k) which is dense in the set of all such continuous functions can be taken as the set of connectives. We will assume that the following functions are among our connectives:

- (i) The *constant function* q for every $q \in [0, 1] \cap \mathbb{Q}$
- (ii) *pointwise minimum* $([0, 1]^2 \rightarrow [0, 1])$
- (iii) *pointwise maximum* $([0, 1]^2 \rightarrow [0, 1])$
- (iv) *Multiplication by q* , $[x \mapsto x \cdot q]$, for every $q \in [0, 1] \cap \mathbb{Q}$ $([0, 1] \rightarrow [0, 1])$
- (v) *negation*, $[x \mapsto 1 - x]$, $([0, 1] \rightarrow [0, 1])$
- (vi) *dotminus* or *implication*: Truncated (at 0) minus $[(x, y) \mapsto x \dot{-} y]$ $([0, 1]^2 \rightarrow [0, 1])$
- (vii) Truncated (at 1) *plus* $[(x, y) \mapsto x + y]$ $([0, 1]^2 \rightarrow [0, 1])$
- (viii) $(x, y) \mapsto |x - y|$ $([0, 1]^2 \rightarrow [0, 1])$

Of course, some of the functions above can be defined using the others, but we are not looking for "minimal" systems of connectives here.

The continuous *quantifiers* are \inf_x and \sup_x . As in classical first order logic, we only allow quantification over elements.

So the following are examples of formulae:

- $d(x, y)$
- $d(x, y) \dot{-} d(y, x)$
- $\inf_{x, y} d(x, y)$
- $\sup_{x, y, z} (d(x, z) \dot{-} (d(x, y) + d(y, z)))$

As usual, formulae with no "free variables" (i.e. each variable is in a scope of one of the quantifiers) are called *sentences*. The first two formulae above are not sentences, while the last two are.

An L -pre-structure is a set M equipped with *interpretations* for all τ -symbols such that d is interpreted as a pseudometric, each predicate symbol is interpreted as a function from (some power of) M to $[0, 1]$, each function symbol is interpreted as a function from (some power of) M to M , and all of them respect their continuity moduli with respect to d . In other words:

- $d^M: M^2 \rightarrow [0, 1]$ is a pseudometric
- For every n -ary predicate symbol P , we have $P^M: M^n \rightarrow [0, 1]$ uniformly continuous with respect to d (respecting the continuity modulus of P dictated by τ)
- For every n -ary function symbol f , we have $f^M: M^n \rightarrow M$ uniformly continuous with respect to d (respecting the continuity modulus of f dictated by τ)

A *structure* is a pre-structure in which d is a complete metric.

See [BU] or [BBHU] for more details (on e.g. continuity moduli). Formal definitions of these notions are not important for us here; but it is crucial that the interpretation of each predicate symbol and of each function symbol is uniformly continuous, and uniformly so in all L -structures (this is what we need the continuity moduli for). Uniform continuity allows us to take ultraproducts of L -structures and obtain e.g. compactness of first order continuous logic.

Note that given a structure M , one can easily define (by induction) the M -value of φ for any sentence φ . We will denote this value by φ^M (it is a real number in the interval $[0, 1]$).

Note also that there is no particular importance for the interval $[0, 1]$, but every predicate symbol must have bounded range (again, so that ultraproducts will work), and by rescaling we may assume it is in fact always $[0, 1]$.

A *condition* is a statement concerning the value of a sentence φ . For example, $\varphi \leq \varepsilon$, $\varphi = 0$, $\varphi < \varepsilon$ are conditions (where $\varepsilon \in [0, 1]$). We will call conditions of the form $\varphi \leq \varepsilon$, $\varphi = 0$, etc *closed conditions* and those of the form $\varphi < \varepsilon$, etc *open conditions*.

Note that as we will mostly work with conditions of the form $\varphi \leq \varepsilon$ and $\varphi < \varepsilon$, the continuous quantifiers \inf_x and \sup_x can be viewed as analogues of the existential and the universal quantifiers respectively.

It is clear what it means for a structure M to *satisfy* a condition α , and we write $M \models \alpha$. If every structure which satisfies α also satisfies β , we say that β *follows* from α and write $\alpha \models \beta$. If Λ is a collection of conditions and M is a structure, we say that M is a model for (of) Λ if M satisfies every condition in Λ , and write $M \models \Lambda$.

A *theory* T is a collection of *closed* conditions which is consistent (i.e. there is a structure M which satisfies all the conditions in T , $M \models T$). We will always assume that theories are closed under entailment, i.e. if $\alpha \in T$ and $\alpha \models \beta$, then $\beta \in T$. We denote by $\text{Mod}(T)$ the class of all models of T .

We encourage the reader to have a look at examples of continuous languages and theories presented in [BU] and [BBHU].

We shall not discuss ultraproduct constructions in this paper. Again, curious readers are referred to [BU] or [BBHU]. An important consequence is the *Compactness Theorem* for continuous logic, which will be useful for us:

Fact 2.1. (*Compactness Theorem*) *Let Λ be a collection of closed conditions which is finitely satisfiable (i.e. every finite subset of Λ has a model). Then Λ has a model.*

Let $M \subseteq N$ be L -structures. We say that M is an *elementary submodel* of N ($M \prec N$) if for every L -sentence φ we have $\varphi^M = \varphi^N$. We say that a theory T is *model complete* if for every $M, N \models T$, $M \subseteq N \Rightarrow M \prec N$. Most theories are not model complete; we will discuss this notion more later. T is model complete if (but not only if) it *eliminates quantifiers*; see more in [BU] or [BBHU].

Note that continuous first order logic is a natural generalization of classical first order logic. Indeed, every classical first order theory can be viewed as a continuous theory in which the metric is discrete.

2.2. Polish space of separable continuous structures. Let τ be a fixed countable continuous vocabulary. For simplicity we assume that τ is relational (i.e. no function symbols). Let L be the corresponding (countable) continuous language. We denote the space of all L -continuous separable structures M with a distinguished countable dense subset $\mathbb{N} \subseteq M$ by \mathfrak{S} . Consider the following topology on \mathfrak{S} : basic open sets are of the form $U_{\varphi(\bar{x}), \bar{a}, \varepsilon} = U_{\varphi(\bar{a}), \varepsilon} = \{M \in \mathfrak{S} : \varphi^M(\bar{a}) < \varepsilon\}$ where $\varphi(\bar{x})$ is a *quantifier free* L -formula, $\bar{a} \in \mathbb{N}$, $\varepsilon \in [0, 1] \cup \{\infty\}$.

Proposition 2.2. *\mathfrak{S} with the topology above is a Polish space.*

Proof. Let $\langle R_i : i < \omega \rangle$ be an enumeration of τ , R_0 being the metric. Let k_i be the arity of R_i (so $k_0 = 2$).

By section 2 of [Hj04] the product space

$$X = [0, 1] \sqcup_i \mathbb{N}^{k_i}$$

is Polish. We can view \mathfrak{S} as a subspace of X via the following embedding $\phi : \mathfrak{S} \rightarrow X$:

$$\phi(M) = \langle f_i : i < \omega \rangle \text{ such that } f_i \text{ is precisely } R_i^M \text{ on the dense subset } \mathbb{N} \text{ of } M.$$

Note that the fact that R_0 is a pseudometric and all the rest of the predicates respect the appropriate continuity moduli with respect to it is a collection of closed conditions. The fact that R_0 is an actual metric can be expressed as a collection of open conditions. So \mathfrak{S} can be viewed as a G_δ subset of a Polish space, and therefore, by Lemma 2.2 in [Hj04], \mathfrak{S} is a Polish space itself. QED_{2.2}

Let T be an L -theory. We denote the space of all elements of \mathfrak{S} which are models of T by \mathfrak{S}_T . So $\mathfrak{S} = \mathfrak{S}_\emptyset$.

2.3. Universal theories and existentially closed models.

- Definition 2.3.**
- (i) We call a theory *universal* if it is (the closure under entailment of) a collection of conditions of the form $[\sup_{\bar{x}} \varphi(\bar{x}) = 0]$ where φ is quantifier free.
 - (ii) Let K be a class of L -structures. We call $M \in K$ *existentially closed* for K if the following holds: for every $M \subseteq N \in K$, a quantifier free formula $\varphi(x, \bar{y})$ and a tuple $\bar{b} \in M$, we have $\inf_x^M \varphi(x, \bar{b}) = \inf_x^N \varphi(x, \bar{b})$.
 - (iii) If T is a universal theory we say that $M \models T$ is existentially closed for T if it is existentially closed for $K = \text{Mod}(T)$. When T is clear from the context we omit it and say “ M is existentially closed” or “ M is an e.c. structure” or “ M is an e.c. model”.

Remark 2.4. For $M \in K$, to be existentially closed for K means in a sense that anything which is quantifier free definable with parameters in M , that can happen in some model in K , happens already in M . In this sense, existentially closed models are “generic” among structures in K .

- Example 2.5.*
- (i) Atomless probability algebras are existentially closed among all probability algebras (see [BU] or [BBHU]).
 - (ii) Hilbert spaces equipped with a unitary operator U with full spectrum ($\text{Spec}(U) = S^1$) are e.c. among all Hilbert spaces equipped with a unitary operator, see [BUZ].
 - (iii) Atomless probability algebras with an aperiodic automorphism are e.c. among probability algebras equipped with an automorphism, see [BH].

Let T be a universal theory, $K = \text{Mod}(T)$, and K^{ec} be the class of e.c. models of T . We call K^{ec} the *continuous Robinson theory* of T . One may ask: is K^{ec} elementary (i.e. is there a continuous theory T^* such that $K^{ec} = \text{Mod}(T^*)$)? The answer is not always positive, even in the classical (discrete) context. For example, the Robinson theory of groups (i.e. T is collection of first order sentences which are true in all groups, K is the class of all groups, and K^{ec} consists of all groups which are existentially closed) is not elementary. But often the answer is yes; in this case we say that T admits a model companion and call T^* the *model companion* of T .

It is easy to see that in this case T^* is *model complete*: if $M, N \models T^*$ and $M \subseteq N$, then $M \prec N$. It does not necessarily eliminate quantifiers; if it does, we call it the *model completion* of T .

Remark 2.6. In Example 2.5 above, the classes of e.c. models are in fact elementary, and the appropriate theories are the model companions, and even the model completions of the universal theories.

Observation 2.7. Let T be a universal theory. Then \mathfrak{S}_T is a closed subset of \mathfrak{S} , and therefore a Polish space.

Proof. Clear.

QED_{2.7}

3. INDUCTIVE THEORIES

Recall that we assume that theories are closed under entailment, i.e. every closed condition which follows from T is already in T . We denote by T^o the collection of all open conditions which follow from T . Let $T^{oc} = T \cup T^o$.

Let T be an L -theory and let Δ be collection of conditions (open or/and closed). We denote by T_Δ the Δ -part of T . So $T_\Delta = T^{oc} \cap \Delta$. For an L -structure M , we denote the Δ -part of $\text{Th}(M)$ by $\text{Th}_\Delta(M)$.

As usual, we define Σ_n and Π_n formulae by induction on n :

- $\Sigma_0 = \Pi_0 =$ quantifier free formulae
- Σ_{n+1} is the collection of formulae of the form $\inf_{\bar{x}} \varphi(\bar{x}, \bar{y})$ where $\varphi(\bar{x}, \bar{y}) \in \Pi_n$
- Π_{n+1} is the collection of formulae of the form $\sup_{\bar{x}} \varphi(\bar{x}, \bar{y})$ where $\varphi(\bar{x}, \bar{y}) \in \Sigma_n$

Remark 3.1. So Σ_1 is the collection of all the existential formulae, Π_1 is the collection of all the universal formulae.

Definition 3.2. (i) For $\Lambda \subseteq L$, we denote by Λ^o the collection of all open conditions of the form $\varphi < \varepsilon$ for $\varphi \in \Lambda$, $\varepsilon > 0$.
 (ii) For $\Lambda \subseteq L$, we denote by Λ^c the collection of all closed conditions of the form $\varphi \leq \varepsilon$ for $\varphi \in \Lambda$, $\varepsilon > 0$.
 (iii) Let Δ be a collection of conditions. We call a theory T a Δ -theory if $T_\Delta \models T$.
 (iv) We call a theory T *inductive* if it is axiomatizable by open conditions of the form $\sup_{\bar{x}} \inf_{\bar{y}} \varphi(\bar{x}, \bar{y}) < \varepsilon$, where φ is quantifier-free. So T is inductive if it is a Π_2^o -theory.

Remark 3.3. (i) So a theory T is universal iff it is a Δ -theory for $\Delta = \Pi_1^c$.
 (ii) Maybe the reader would expect us to work with Π_2^c -theories instead of Π_2^o . Note that if T is Π_2^c then it is inductive, and for complete theories the notions are equivalent; but as we want Theorem 3.6 to hold for all theories, not necessarily complete, and as we want our theories to define G_δ subsets of \mathfrak{S} , the natural choice is open conditions.

Lemma 3.4. *Let $\Delta = \Sigma_n^o$ or $\Delta = \Pi_n^o$ for some n (or just Δ is a collection of open conditions closed under rescaling, i.e. multiplication by scalars and the "pointwise minimum" connective). Let T be a theory and suppose that for every two L -structures M, N such that $M \models T$ and $\text{Th}_\Delta(M) \subseteq \text{Th}_\Delta(N)$, do we have $N \models T$. Then T is a Δ -theory.*

Proof. Suppose not; so there exists $N \models T_\Delta$, $N \not\models T$. By the assumption, for no $M \models T$ do we have $\text{Th}_\Delta(M) \subseteq \text{Th}_\Delta(N)$. In other words, for every $M \models T$ there exists a formula φ_M with $[\varphi_M < \varepsilon] \in \Delta$ such that $\varphi_M^M < \varepsilon$, $\varphi_M^N \geq \varepsilon$. By rescaling we may assume $\varepsilon = \frac{1}{2}$.

So the set $T \cup \{[\varphi_M \geq \frac{1}{2}]: M \models T\}$ is inconsistent. By compactness, $[\min_{i=1}^k \varphi_i < \frac{1}{2}] \in T^o$ for some finite collection of such φ_i . But Δ is closed under taking minima, and every one of the conditions $[\varphi_i < \frac{1}{2}]$ is in Δ , so (as $N \models T_\Delta$), $(\min_{i=1}^k \varphi_i)^N < \frac{1}{2}$, so for some i $\varphi_i^N < \varepsilon$, a contradiction. QED_{3.4}

Lemma 3.5. *Let T be a complete L -theory and M an L -structure with $\text{Th}_{\Sigma_n^c}(M) \subseteq T_{\Sigma_n^c}$. Then there exists $M' \models T$ and a Σ_n -elementary embedding $f: M \rightarrow M'$.*

Proof. Let $M = \langle a_\alpha : \alpha < \lambda \rangle$, M' a λ -saturated model of T . Construct $f_\alpha: A_\alpha = \{a_\beta : \beta < \alpha\} \rightarrow M'$ an increasing continuous sequence of Σ_n -elementary embeddings. Given f_α , consider the Σ_n -type in M of a_α over A_α , call it $\pi(x)$. By the assumption that $\text{Th}_{\Sigma_n^c} \subseteq T$, $f(\pi(x))$ is a type in M' over $f(A_\alpha)$ (as Σ_n is closed under “inf”), and use the saturation of M' . QED_{3.5}

Theorem 3.6. *Let T be an L -theory such that $\text{Mod}(T)$ is preserved under unions of chains, that is if $\langle M_i : i < \omega \rangle$ is an increasing chain of models of T , then the closure of the union $M = \overline{\bigcup_i M_i}$ is also a model of T . Then T is inductive.*

Proof. Suppose $\text{Mod}(T)$ is closed under unions of chains. We would like to show that $T_{\Pi_2^c} \models T$.

Let $M \models T$ and T' a complete L -theory extending $T_{\Pi_2^c}$, $N \models T'$ with $\text{Th}_{\Pi_2^c}(M) \subseteq T'$. We will show that $N \models T$, which clearly suffices by Lemma 3.4.

Construct a chain $N_0 \subseteq M_1 \subseteq N_1 \subseteq \dots$ such that:

- (i) $N_0 = N$
- (ii) $M_i \models T$, $N_i \models T'$
- (iii) $N_i \prec N_{i+1}$

If the construction is possible, we are done:

Let $M = \overline{\bigcup_i M_i} = \overline{\bigcup_i N_i}$. $M \models T$ by the assumption on T and (ii) above. On the other hand, clearly $N_0 \prec M$ (as $N_0 \prec N_i$ for all i by (iii) above), so $N = N_0 \models T$, and we are done.

Why is the construction possible?

Let $N_0 = N$. As $\text{Th}_{\Pi_2^c}(M) \subseteq \text{Th}(N) = T'$, we have $\text{Th}_{\Sigma_2^c}(N) \subseteq \text{Th}(M)$, so by Lemma 3.5, there exists $M_0 \models \text{Th}(M)$ and a Σ_2^c -embedding of N_0 into M_0 .

Let $N_0 = \langle a_\alpha : \alpha < \lambda \rangle$. Enrich the vocabulary τ with λ -many constant symbols, call the new language L' .

Claim 3.6.1. $\text{Th}_{\Sigma_1^c}(M_0, \langle a_\alpha : \alpha < \lambda \rangle) \subseteq \text{Th}(N_0, \langle a_\alpha : \alpha < \lambda \rangle)$ as L' -theories.

Proof. Clearly (as N_0 is a Σ_2^c -elementary submodel of M_0), $\text{Th}_{\Sigma_2^c}(N_0, \langle a_\alpha : \alpha < \lambda \rangle) \subseteq \text{Th}(M_0, \langle a_\alpha : \alpha < \lambda \rangle)$ as L' -theories, and therefore $\text{Th}_{\Pi_2^c}(M_0, \langle a_\alpha : \alpha < \lambda \rangle) \subseteq \text{Th}(N_0, \langle a_\alpha : \alpha < \lambda \rangle)$ as L' -theories, in particular

$$(1) \quad \text{Th}_{\Sigma_1^c}(M_0, \langle a_\alpha : \alpha < \lambda \rangle) \subseteq \text{Th}(N_0, \langle a_\alpha : \alpha < \lambda \rangle)$$

as L' -theories.

Let $[\inf_{\bar{x}} \psi(\bar{x}, \bar{a}) \leq \varepsilon]$ be a closed existential condition satisfied by M_0 with parameters $\bar{a} \in N_0$ (i.e. $\bar{a} = a_{\alpha_1}, \dots, a_{\alpha_k}$ for some $\alpha_1, \dots, \alpha_k < \lambda$). Then $M_0 \models [\inf_{\bar{x}} \psi(\bar{x}, \bar{a}) < \varepsilon']$ for every $\varepsilon' > \varepsilon$. So this is true in N_0 (by (1) above), which completes the proof of the claim.

QED_{3.6}

By the Claim above and Lemma 3.5, there exists $N_1 \models T'$ into which M_0 is Σ_1^c -embedded in the language L' . Clearly, this means that $N_0 \prec N_1$.

The rest of the construction is similar.

QED_{3.6}

We obtain the analogue of a well-known Robinson's theorem in the continuous context:

Corollary 3.7. *If T is model complete, then it is inductive.*

Corollary 3.8. *Let T be a universal theory which has a model companion T' . Then T' is inductive.*

Proof. T' is model complete.

QED_{3.8}

4. GENERIC AND RANDOM MODELS

4.1. Model completions and topological genericity.

Observation 4.1. Let T' be an inductive theory. Then $\mathfrak{S}_{T'}$ is a G_δ subset of \mathfrak{S} .

Proof. For every quantifier free formula $\varphi(\bar{x})$, $\bar{a}, \bar{b} \in \mathbb{N}$ and $\varepsilon > 0$, the open condition $[\varphi(\bar{a}, \bar{b}) < \varepsilon]$ defines an open subset of \mathfrak{S} , which we called $U_{\varphi(\bar{a}, \bar{b}), \varepsilon}$. The open condition $[\inf_{\bar{x}} \varphi(\bar{x}, \bar{b}) < \varepsilon]$ corresponds, therefore, to an open subset of \mathfrak{S} , which equals $\bigcup_{\bar{a} \in \mathbb{N}} U_{\varphi(\bar{a}, \bar{b}), \varepsilon}$. A Π_2^o condition defines a subset of \mathfrak{S} which is a (countable) intersection (over all possible $\bar{b} \in \mathbb{N}$) of sets as above; therefore it is a G_δ set. Clearly, a countable collection of Π_2^o conditions still corresponds to a G_δ set. QED_{4.1}

Fact 4.2. *Let T be a universal theory. Then the collection of separable e.c. models is dense in \mathfrak{S}_T .*

Proof. It is a standard fact that every $M \models T$ can be extended to an e.c. model $M' \models T$ (of the same density character). Now it is easy to see that one can rename the elements of M' such that a certain finite $\bar{a} \in \mathbb{N}$ remains unchanged (and so M' is indeed in a specified open neighborhood of M in \mathfrak{S}_T). QED_{4.2}

Corollary 4.3. *Let T be a universal theory which has a model companion T' . Then $\mathfrak{S}_{T'}$ is a G_δ dense subset of \mathfrak{S}_T .*

Recall that a theory T is called \aleph_0 -categorical if any two separable models of T are isomorphic.

Definition 4.4. Let T be a universal theory. We call $M \in \mathfrak{S}_T$ *generic* if the isomorphism class of M is G_δ dense in \mathfrak{S}_T .

Corollary 4.5. *Let T be a universal theory which has a model companion T' , and assume T' is \aleph_0 -categorical. Then (any) existentially closed model of T is generic in \mathfrak{S}_T .*

Proof. By Corollary 4.3, $\mathfrak{S}_{T'}$ is G_δ dense in \mathfrak{S}_T . By \aleph_0 -categoricity of T' , $\mathfrak{S}_{T'}$ is the isomorphism class of any e.c. model of T (which is in \mathfrak{S}_T). QED_{4.5}

4.2. Random structures. Once we have shown that the class of existentially closed models in \mathfrak{S} is “big” in the sense of Baire category theory, a natural question is: is there a similar measure-theoretic result? In [Ver02] Vershik shows that the Urysohn space is in a sense the random metric space. We know that the model companion of the universal theory of graphs is the random graph. Are these facts particular cases of a model theoretic phenomenon?

Let T be a universal theory, μ a probability measure on \mathfrak{S}_T satisfying

Assumption 4.6. (i) No nonempty open set has probability 0.

(ii) μ is invariant under the action of S_∞ on \mathfrak{S}_T . In other words, for every formula $\varphi(\bar{x})$, $\varepsilon > 0$ and $\bar{a}, \bar{b} \in \mathbb{N}$, we have the equality $\mu(U_{\varphi(\bar{a}), \varepsilon}) = \mu(U_{\varphi(\bar{b}), \varepsilon})$. So $\mu(U_{\varphi(\bar{a}), \varepsilon})$ does not depend on \bar{a} .

Clearly, these are very natural assumptions on a measure on \mathfrak{S}_T , once we are interested in “random structures”: first, we assume that if a certain open event occurs in some model of T , then its probability is positive. Second, we assume that in a sense isomorphic models “occur” with equal probability.

Lemma 4.7. *Let μ be as above. Then the set of all existentially closed structures in \mathfrak{S}_T has probability 1. In other words, if we pick a structure “randomly”, it is going to be existentially closed almost surely.*

Proof. Let M be a randomly chosen structure. We aim to show that with probability 1 it is existentially closed. Let $\varphi(\bar{x})$ be a formula, and suppose that in some $M \subseteq N \models T$ we have $\inf_{\bar{x}}^N \varphi(\bar{x}) \leq \varepsilon$. Let $\varepsilon' > \varepsilon$. So there exists $\bar{a} \in N$ such that $\varphi^N(\bar{a}) < \varepsilon'$, and therefore $\mu(U_{\varphi(\bar{a}), \varepsilon'}) = \delta > 0$. By the invariance of μ , $\mu(U_{\varphi(\bar{b}), \varepsilon'}) = \delta$ for every $\bar{b} \in \mathbb{N}$, and so the probability that in a randomly chosen structure M we have $\varphi^M(\bar{b}) \geq \varepsilon'$ is bounded away from 1 for each $\bar{b} \in M$.

Now clearly with probability 1 for some $\bar{b} \in M$ we have $\varphi^M(\bar{b}) < \varepsilon'$, therefore $\inf_{\bar{x}}^M \varphi(\bar{x}) \leq \varepsilon$ almost surely, and we are done. QED_{4.7}

Remark 4.8. Note that we did not really use the invariance of μ . We only need that the probability of the event $U_{\varphi(\bar{a}), \varepsilon}$ is either 0 for all \bar{a} or bounded away from 0 for all \bar{a} .

Corollary 4.9. *Let T be a universal theory which has a model companion T' . Then $\mathfrak{S}_{T'}$ is a set of probability 1 in \mathfrak{S}_T .*

Definition 4.10. We call a separable model of a universal theory T *random* if the measure of its isomorphism class in \mathfrak{S}_T is 1 with respect to any probability measure μ as in Assumption 4.6. In other words, M is a random model of T if for every μ as above, a randomly chosen structure in \mathfrak{S}_T is almost surely isomorphic to M .

Just like in Corollary 4.5 we obtain:

Corollary 4.11. *Let T be a universal theory which has a model companion T' . Assume furthermore that T' is \aleph_0 -categorical. Then any separable model of T' is a random model of T .*

Clearly, this generalizes the “randomness” of the countable random graph; see more in the following subsection.

4.3. Concluding remarks on genericity. In this section we have shown that the model theoretic notion of genericity gives rise to both Baire category theoretical and measure theoretical notions of genericity in the space \mathfrak{S} . In other words, we have shown:

Corollary 4.12. *Let T be a universal theory which admits a model companion T' . Then $\mathfrak{S}_{T'}$ is both G_δ dense in \mathfrak{S} and of measure 1 with respect to any reasonable measure on \mathfrak{S} (i.e. any measure satisfying Assumption 4.6).*

In particular, we have the following:

Corollary 4.13. *Suppose T is a universal theory which has a model companion T' , and assume furthermore that T' is \aleph_0 -categorical. Then the (unique up to isomorphism) model of T' is both the generic and the random model of T .*

Example 4.14. The atomless separable probability algebra is both the generic and the random separable probability algebra.

Proof. The theory of atomless probability algebras is the model companion of the universal theory of probability algebras by [BU]. It is also \aleph_0 -categorical, so apply Corollary 4.13. QED_{4.14}

As we have already mentioned, every classical first order theory is a continuous first order theory with discrete metric. We can therefore apply our analysis to e.g. the theory of the random graph. Recall that the theory of the random graph is the model completion of the universal theory of graphs.

Example 4.15. The random graph is the generic countable graph.

Proof. The (classical) first order theory of the random graph is the model completion (and therefore the model companion) of the universal theory of graphs. It is also \aleph_0 -categorical. So the desired conclusion follows from Corollary 4.13. QED_{4.15}

Similarly, the unique countable model of the model completion of the universal theory of graphs is the random graph in the sense defined here in Definition 4.10. Well, no surprise here: we're just saying that the random graph is, well, random.

In the following section we will show that the continuous first order theory of the Urysohn space has similar properties, and therefore Corollary 4.13 applies to it as well. One can think of this theory as the continuous analogue of the theory of the random graph: instead of the discrete predicate $R(x, y)$ in the theory of graphs which can be either true or false, we have a metric which can take any value between 0 and 1.

5. URYSOHN SPACE

Many results on the model theory of the Urysohn space here are “folklore”, but the author is not aware of any written references. In order to follow the proofs, the reader should be familiar with basics of continuous model theory slightly beyond what is sketched in section 2 of the article.

We remind the reader that the *Urysohn space* is the universal complete separable metric space, first constructed by Pavel Urysohn. Due to the limitations of the genre, we will consider the *bounded Urysohn space*, i.e. Urysohn space of diameter 1. We denote it by \mathfrak{U} .

Denote by \mathfrak{E}_n the collection of all possible distance configurations on n points of diameter 1. It will be convenient for us to think about it in the following way: $\vartheta(x_1, \dots, x_n) \in \mathfrak{E}_n$ if $\vartheta(x_1, \dots, x_n)$ is a formula of the form

$$\bigvee_{1 \leq i, j \leq k} |d(x_i, x_j) - r_{ij}|$$

where the matrix $(r_{ij})_{1 \leq i, j \leq k}$ is a distance matrix of some finite metric space of diameter 1, and \bigvee stands for the lattice operation of pointwise maximum.

Let us introduce the following notation: for $\vartheta \in \mathfrak{E}_{n+1}$, let $\vartheta \upharpoonright n$ be the restriction of ϑ to the first n variables.

Clearly, for every $\vartheta \in \mathfrak{E}_{n+1}$, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $a_1, \dots, a_n \in \mathfrak{U}$ satisfy $\vartheta \upharpoonright n(a_1, \dots, a_n) < \delta$, then there exists $a_{n+1} \in \mathfrak{U}$ such that $\vartheta(a_1, \dots, a_n, a_{n+1}) \leq \varepsilon$.

Let $T_{\mathfrak{U}}$ be the collection of all the conditions of the form

$$\left[\sup_{x_1, \dots, x_n} \inf_y \left(\frac{\varepsilon}{1 - \delta} (1 - \vartheta \upharpoonright n(x_1, \dots, x_n)) \bigwedge \vartheta(x_1, \dots, x_n, y) \right) \leq \varepsilon \right]$$

which is just one way of stating

$$\forall x_1, \dots, x_n \exists y (\vartheta \upharpoonright n(x_1, \dots, x_n) < \delta \rightarrow \vartheta(x_1, \dots, x_n, y) \leq \varepsilon)$$

Note that \bigwedge stands for the lattice operation of pointwise minimum.

The following follows from the standard Urysohn’s argument:

Fact 5.1. *The only separable complete model of $T_{\mathfrak{U}}$ is \mathfrak{U} .*

Corollary 5.2. *$T_{\mathfrak{U}}$ is \aleph_0 -categorical, and therefore a complete continuous theory.*

Proof. By (the continuous version of) Vaught’s test. QED_{5.2}

Proposition 5.3. *$T_{\mathfrak{U}}$ eliminates quantifiers.*

Proof. By the classical back-and-forth argument (see Theorem 4.16 in [BU]) using the axioms of $T_{\mathfrak{U}}$. QED_{5.3}

Corollary 5.4. $T_{\mathfrak{U}}$ is the model completion (and therefore the model companion) of the “empty” continuous universal theory (the universal theory of a metric space with no extra-structure). \mathfrak{U} is (the only) existentially closed metric space.

A natural conclusion from our analysis is the following form of Vershik’s theorems:

Corollary 5.5. *The Urysohn space (of diameter 1) is the generic and the random metric space (of diameter 1).*

Proof. The theory of the Urysohn space is the model companion of the universal theory of metric spaces and is \aleph_0 -categorical, so the result follows immediately from Corollary 4.13. QED_{5.5}

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