

# UNSTABLE CLASSES OF METRIC STRUCTURES

SAHARON SHELAH AND ALEXANDER USVYATSOV

ABSTRACT. We prove a strong nonstructure theorem for a class of metric structures with an unstable formula. As a consequence, we show that weak categoricity (that is, categoricity up to isomorphisms and not isometries) implies several weak versions of stability. This is the first step in the direction of the investigation of weak categoricity of metric classes.

## 1. INTRODUCTION AND PRELIMINARIES

**1.1. Introduction.** Categoricity and stability are two of the most basic and important notions in contemporary model theory. It is therefore not a surprise that the study of pure model theory of classes of metric structures has begun with the investigations of these concepts and connections between them. One should point out, though, that while these notions have a very precise meaning in classical model theory, there are more than one natural choices when topology is introduced as an inherent part of the structure. For example, while classical categoricity means having a unique model up to isomorphism, when working with metric structures, one has to specify which isomorphisms are being considered.

Categoricity of metric classes with respect to *isometries* has been studied independently in slightly different frameworks in [Ben05] and [ShUs837]. For a model theorist this is perhaps the most natural choice of isomorphism, since it preserves *the whole structure*. This is precisely why in many cases this choice would be considered too strict (too rigid). For example, the “correct” notion of isomorphism between Banach spaces is widely accepted to be a  $c$ -isomorphism, that is, a linear isomorphism  $T$  satisfying  $\frac{1}{c} \leq \|T\| \leq c$  for some  $c \geq 1$ . It is certainly the case that dealing with isomorphisms that are allowed to perturb the structure somewhat is significantly more challenging when it comes to model theoretic questions. In this article we make the first steps in this direction.

Stability had been known to have had applications in functional analysis even before it was systematically introduced in this context by model theorists: investigation of quantifier-free stability for Banach spaces led to the spectacular result of Krivine and

---

*Date:* March 4, 2008.

The authors thank C. Ward Henson for stimulating discussions.

Publication no. 928 on Shelah’s list of publications. Shelah thanks the Israel Science Foundation for partial support of this research, grant no. ???.

The second author thanks the Fundação para a Ciência e a Tecnologia for partial support of this research, grant no. SFRH / BPD / 34893 / 2007.

Maurey [KM] on the existence of an almost isometric copy of  $\ell_p$  in any stable Banach space. Stable definable predicates introduced and studied by the second author and Itai Ben-Yaacov in [BU] seem to capture the “correct” notion of stability in the metric context. It generalizes Krivine-Maurey stability as well as stability of positive-bounded theories (stability in Henson’s logic) studied by José Iovino in [Io]. Still, if a theory (or a formula/predicate) lacks the full strength of stability, there are natural weaker properties (which would not make sense in classical model theory since their definitions rely heavily on the existence of a metric) that might hold. We introduce and study several such notions. Some of them have already been investigated in the context of Banach spaces. For instance, “Junge-Rosenthal”-stability, which we define in section 4, is motivated by the notion of an “asymptotically symmetric” Banach space, originally due to Junge and Rosenthal, studied e.g. in [JKO].

All the results in this paper can be summarized as “different notions of instability lead to non-structure”, i.e. to the existence of many non-isomorphic models. We naturally conclude that categorical classes of metric structures (where categoricity is viewed with respect to much weaker notions of isomorphism than an isometry) are (in some sense) stable. As an example of a consequence of our analysis, let us include here the following result in model theory of Banach spaces, which (we believe) should have interesting applications, such as throwing light on model theoretic properties of particular Banach spaces (it is proven in section 4):

**Theorem 1.1.** *Let  $\mathcal{B}$  be a Banach space whose continuous theory is categorical (with respect to isomorphisms of Banach spaces) in some uncountable density. Then it is asymptotically symmetric (as defined in [JKO]).*

**1.2. Preliminaries.** The proofs in this article rely mostly on Ehrenfeucht-Mostowski constructions, hence do not require any compactness assumptions and can be carried out in very general settings, such as abstract metric classes (defined in [ShUs837]) with amalgamation and arbitrary large models. Since some additional ground work is needed in order to develop the basics of this framework, we decided not to include the details here and work in the well-developed context of a monster metric space (it is the central concept in [ShUs837]), which is already quite general.

Since the definition of a monster metric space is rather long, we will not include it here. Instead, we would like to begin with defining a simplified version, which we call a homogeneous metric monster. We refer the reader to [ShUs837] for details and proofs.

Fix a “big enough” cardinal  $\lambda^* = (\lambda^*)^{\aleph_0}$ .

**Definition 1.2.** A *homogeneous metric monster* is a metric structure of cardinality  $\lambda^*$  which is strongly homogeneous in the following sense: every partial function  $f: A \rightarrow \mathfrak{C}$  (with  $A \subseteq \mathfrak{C}$  of cardinality less than  $\lambda^*$ ) which preserves positive existential formulae (see [ShUs837], Definition 2.13), can be extended to an automorphism of  $\mathfrak{C}$ .

*Remark 1.3.* If  $\mathfrak{C}$  is a homogeneous metric monster then it is a monster metric space as defined in [ShUs837], Definition 2.17 for  $\Delta(\mathfrak{C}) = \text{positive existential formulas}$ .

From now on,  $\mathfrak{C}$  will denote monster metric space (unless stated otherwise). The reader can assume without almost any loss of generality that it is a homogeneous metric monster. All elements, tuples, sets, will be taken inside  $\mathfrak{C}$ . The finite diagram of  $\mathfrak{C}$  will be denoted by  $D$ . By “types” we will always mean  $D$ -types, which in case of a homogeneous monster is just the collection of all positive existential types. By “formula”, we always mean a  $\Delta(\mathfrak{C})$ -formula, which in case of a homogeneous monster is simply a positive existential formula.

Let  $K$  be the class of all almost elementary submodels of the monster  $\mathfrak{C}$ ,  $K^c$  the class of complete such (as defined in [ShUs837], Definition 2.19). We will call elements of  $K$  *premodels* and elements of  $K^c$  *models*.

We recall the definition of approximations of formulae and types (see more in [ShUs837], section 3):

**Definition 1.4.** (i) For a formula  $\varphi(\bar{x})$  possibly with parameters, we define  $\varphi^{[\varepsilon]}(\bar{x}) = \exists \bar{x}'(\varphi(\bar{x}') \wedge \mathbf{d}(\bar{x}, \bar{x}') \leq \varepsilon)$ . So  $\varphi^{[\varepsilon]}(\bar{x}, \bar{a}) = (\exists \bar{x}')[\mathbf{d}(\bar{x}, \bar{x}') \leq \varepsilon \wedge \varphi(\bar{x}', \bar{a})]$ .  
(ii) For a type  $p$ , define  $p^{[\varepsilon]} = \{(\bigwedge_{\ell < n} \varphi_\ell)^{[\varepsilon]} : n < \omega \text{ and } \varphi_0, \dots, \varphi_{n-1} \in p\}$ .

Recall that a pair  $(\varphi(\bar{x}), \psi(\bar{x}))$  is called  $\varepsilon$ -*contradictory* (where  $\varepsilon > 0$  as usual) if  $\mathbf{d}(\varphi^{\varepsilon}, \psi^{\varepsilon}) > \varepsilon$ . A pair of formulae (maybe with parameters)  $(\varphi(\bar{x}), \psi(\bar{x}))$  is called *contradictory* if it is  $\varepsilon$ -contradictory for some  $\varepsilon > 0$ .

We refer the reader to section 5 of [ShUs837] for the treatment of Ehrenfeucht-Mostowski models in the framework of a monster metric space. Recall that an Ehrenfeucht-Mostowski blueprint (EM-blueprint) is called *proper* for  $\mathfrak{C}$  if for any order type  $J$ , the structure  $\text{EM}(J, \Phi)$  is in  $K$ .

## 2. INSTABILITY IMPLIES STRONG NONSTRUCTURE

The following is a simplification of Definition III.2.5(1) in [Sh:e]:

**Definition 2.1.** (i) Let  $I$  be a linear order,  $\Phi$  an EM-blueprint proper for  $\mathfrak{C}$ ,  $M = \text{EM}(I, \Phi)$ ,  $\langle \bar{a}_i : i \in I \rangle$  is the skeleton of  $M$ ,  $\bar{c}, \bar{d} \in M$ . We say that  $\bar{c}$  and  $\bar{d}$  are *similar*,  $\bar{c} \sim \bar{d}$  if

- $\bar{c} = \bar{\sigma}(\bar{a}_{\bar{i}}), \bar{d} = \bar{\sigma}(\bar{a}_{\bar{j}})$
- $\bar{i}$  and  $\bar{j}$  have the same quantifier free type in  $I$  in the language of order  $\{<\}$ .

(ii) Let  $\theta(\bar{x}, \bar{y})$  be a formula in the language of order  $\{<\}$ , and let  $I, J$  be linear orders. We say that  $I$  is  $\theta(\bar{x}, \bar{y})$ -unembeddable into  $J$  if for any EM-blueprint  $\Phi$  proper for  $\mathfrak{C}$  and a function  $f: I \rightarrow \text{EM}(J, \Phi)$  there exist  $\bar{i}_1, \bar{i}_2 \in I$  such that  $I \models \theta(\bar{i}_1, \bar{i}_2)$  and  $f(\bar{i}_1) \sim f(\bar{i}_2)$ .

**Fact 2.2.** *Let  $\theta(x_0x_1, y_0y_1)$  be the following formula in the language of order:  $\theta = x_0 < x_1 \wedge y_1 < y_0$ . Let  $\lambda > |L|$ . Then there exists a family  $\langle I_\alpha : \alpha < 2^\lambda \rangle$  of linear orders of cardinality  $\lambda$  which are pairwise  $\theta(\bar{x}, \bar{y})$ -unembeddable into each other.*

*Proof.* Combine [Sh:e] VI.3.1(2) and III.2.21 (see also Definition III.2.5(2) there). QED<sub>2.2</sub>

Recall:

**Definition 2.3.** (i) A contradictory pair  $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$  is called *unstable* if there are arbitrary long indiscernible sequences  $\langle \bar{a}_i : i < \lambda \rangle, \langle \bar{b}_i : i < \lambda \rangle$  such that

$$i < j < \lambda \Rightarrow \varphi(\bar{a}_i, \bar{b}_j), \psi(\bar{a}_j, \bar{b}_i)$$

In this case we also say that the pair above has the *order property*.

- (ii) A pair as above is called *c-additively unstable* or *c-add-unstable* (for  $c \geq 0$ ) if in addition  $\mathbf{d}(\varphi, \psi) \geq c$ .
- (iii)  $\mathfrak{C}$  is called *unstable* if there exists an unstable pair.
- (iv)  $\mathfrak{C}$  is called *c-additively unstable* or *c-add-unstable* if there exists a *c-add-unstable* pair.

We now define weak embeddings between metric structures.

**Definition 2.4.** (i) Let  $\Delta$  be a set of formulae,  $c \geq 0$ , and let  $M, N \in K$ . We say that  $f: M \rightarrow N$  is a  $(\Delta, c)$ -*additive-embedding* or  $(\Delta, c)$ -*add-embedding* if it is 1-1 and for every  $\varphi(\bar{x}) \in \Delta, \bar{a} \in M$  we have  $M \models \varphi(\bar{a}) \Rightarrow N \models \varphi^{[c]}(f(\bar{a}))$ .

(ii) We say that  $f: M \rightarrow N$  is a  $(\Delta, c)$ -*additive isomorphism* or  $(\Delta, c)$ -*add-isomorphism* if it is an onto  $(\Delta, c)$ -embedding.

**Proposition 2.5.** Denote  $\theta(x_0x_1, y_0y_1) = x_0 < x_1 \wedge y_1 < y_0$  and let  $I, J$  be linear orders such that  $I$  is  $\theta$ -unembeddable into  $J$ . Assume that  $\mathfrak{C}$  is unstable with respect to some *c-contradictory pair*  $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ . Then there exists an *EM-blueprint*  $\Phi$  such that there is no  $(\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}, \frac{c}{3})$ -*add-embedding*  $f: \overline{\text{EM}(I, \Phi)} \rightarrow \overline{\text{EM}(J, \Phi)}$ .

*Proof.* By instability of the pair  $(\varphi, \psi)$  and the standard Erdős-Rado argument

⊗ 2.5.1. There exists an EM-blueprint  $\Phi$  proper for  $\mathfrak{C}$  such that if  $I$  is a linear order and  $M = \text{EM}(I, \Phi)$  with skeleton  $\langle \bar{a}_i \bar{b}_i : i \in I \rangle$ , then in  $M$  we have

$$\bullet i < j \in I \Rightarrow \varphi(\bar{a}_i, \bar{b}_j) \wedge \psi(\bar{a}_j, \bar{b}_i)$$

Denote  $M = \overline{\text{EM}(I, \Phi)}, N = \overline{\text{EM}(J, \Phi)}$  and let  $f: M \rightarrow N$ . Assume by contradiction that  $f$  is a  $(\{\varphi, \psi\}, \frac{c}{3})$ -embedding.

Let  $\langle \bar{a}_i \bar{b}_i : i \in I \rangle, \langle \bar{a}'_j \bar{b}'_j : j \in J \rangle$  be the skeletons of  $\text{EM}(I, \Phi), \text{EM}(J, \Phi)$  respectively. Since  $I$  is  $\theta$ -unembeddable into  $J$  (and we can identify  $I$  with  $\langle \bar{a}_i \bar{b}_i : i \in I \rangle$ ), there exist  $i_0, i_1, l_0, l_1 \in I$  with  $i_0 < i_1, l_1 < l_0$  and  $f(\bar{a}_{i_0} \bar{b}_{i_0}) f(\bar{a}_{i_1} \bar{b}_{i_1}) \sim f(\bar{a}_{l_0} \bar{b}_{l_0}) f(\bar{a}_{l_1} \bar{b}_{l_1})$ , in particular

$$f(\bar{a}_{i_0}) f(\bar{b}_{i_0}) f(\bar{a}_{i_1}) f(\bar{b}_{i_1}) \equiv f(\bar{a}_{l_0}) f(\bar{b}_{l_0}) f(\bar{a}_{l_1}) f(\bar{b}_{l_1})$$

By ⊗ 2.5.1 we have  $M \models \varphi(\bar{a}_{i_0}, \bar{b}_{i_1})$  and  $M \models \psi(\bar{a}_{l_0}, \bar{b}_{l_1})$ . By the assumption towards contradiction,

$$N \models \varphi^{[\frac{c}{3}]}(f(\bar{a}_{i_0}), f(\bar{b}_{i_1}))$$

and

$$N \models \psi^{[\frac{c}{3}]}(f(\bar{a}_{l_0}), f(\bar{b}_{l_1}))$$

But since

$$f(\bar{a}_{i_0})f(\bar{b}_{i_0})f(\bar{a}_{i_1})f(\bar{b}_{i_1}) \equiv f(\bar{a}_{l_0})f(\bar{b}_{l_0})f(\bar{a}_{l_1})f(\bar{b}_{l_1})$$

we also get e.g.

$$N \models \varphi^{\lfloor \frac{\varepsilon}{3} \rfloor}(f(\bar{a}_{l_0}), f(\bar{b}_{l_1}))$$

which is clearly a contradiction to the pair  $(\varphi, \psi)$  being  $c$ -contradictory. QED<sub>2.5</sub>

We conclude that relatively weak notions of categoricity imply stability or just  $c$ -additive stability.

- Corollary 2.6.** (i) *Assume that there exists  $\lambda > |L|$  such that for every  $M, N \in K^c$ , every  $\Delta \subseteq L$  finite and every  $\varepsilon > 0$  there exists  $f: M \rightarrow N$  which is a  $(\Delta, \varepsilon)$ -additive isomorphism. Then  $\mathfrak{C}$  is stable.*
- (ii) *In (i) it is enough to assume that for every  $\Delta \subseteq L$  finite there exists  $\varepsilon < \frac{\text{diam}(\Delta)}{3}$  and  $f: M \rightarrow N$  which is a  $(\Delta, \varepsilon)$ -additive isomorphism.*
- (iii) *In (i) it is enough to assume that for every  $\Delta$  and  $\varepsilon > 0$  there exists  $\lambda > |L|$  such that for every  $M, N$  of density  $\lambda$  there exists a  $(\Delta, \varepsilon)$ -additive isomorphism from  $M$  to  $N$ .*
- (iv) *Similarly, it is enough to assume that for every  $\Delta$  there exist  $\varepsilon < \frac{\text{diam}(\Delta)}{3}$  and  $\lambda > |L|$  such that for every  $M, N$  of density  $\lambda$  there exists a  $(\Delta, \varepsilon)$ -additive isomorphism from  $M$  to  $N$ .*
- (v) *Assume that for every  $\Delta \subseteq L$  finite of diameter  $\geq c$  there exists  $e < \frac{c}{3}$  and  $f: M \rightarrow N$  which is a  $(\Delta, e)$ -add-isomorphism. Then  $\mathfrak{C}$  is  $c$ -additively stable.*
- (vi) *Again, in (v) it is enough to assume the weaker version: for every  $\Delta$  of diameter  $\geq c$  there exists  $e$  and  $\lambda > |L|$ .*

*Proof.* Straightforward by Proposition 2.5 and Fact 2.2. QED<sub>2.6</sub>

### 3. THE CONTINUOUS CASE AND BANACH SPACES

The following definition of “continuous truth” is strongly related to continuous model theory studied in [BU] (and was, in fact, a motivation for the second author’s interest in the subject). But since we are working here in a much more general setting than continuous first order logic, we will not require any background or use any results from [BU] (except some notations, which we introduce explicitly).

**Definition 3.1.** (i) Given a formula  $\varphi(\bar{x})$  and  $\bar{a} \in \mathfrak{C}$ , we define the *continuous truth value* of  $\varphi(\bar{a})$  by

$$\varphi(\bar{a}) = \inf\{\varepsilon: \mathfrak{C} \models \varphi^{\lfloor \varepsilon \rfloor}(\bar{a})\}$$

Statements of the form  $[\varphi(\bar{a}) = \varepsilon]$ ,  $[\varphi(\bar{a}) \leq \varepsilon]$ ,  $[\varphi(\bar{a}) < \varepsilon]$  have the obvious meaning, and we will refer to them as *conditions*. Conditions of the form  $[\varphi(\bar{x}) = \varepsilon]$  or  $[\varphi(\bar{x}) \leq \varepsilon]$  are called *closed* while conditions of the form  $[\varphi(\bar{x}) < \varepsilon]$  are *open*. We will say that a tuple  $\bar{a}$  *satisfies* the condition  $[\varphi(\bar{x}) \leq \varepsilon]$  (or  $[\varphi(\bar{x}) < \varepsilon]$ , etc) if  $[\varphi(\bar{a}) \leq \varepsilon]$ ; sometimes we write  $\bar{a} \models [\varphi(\bar{x}) \leq \varepsilon]$ .

- (ii) Let  $\mathfrak{C}$  be a momspace. We say that a formula  $\varphi(\bar{x})$  *has weak negations* if for every  $c > 0$ , the closed condition  $[\varphi(\bar{x}) \geq c]$  is type-definable.

*Remark 3.2.* (i) Note that  $\varphi(\bar{a})$  is just the distance between  $\bar{a}$  and the closed set  $\varphi^c$ .  
(ii) Let  $M \in K$ ,  $\varphi(\bar{x})$  a formula. Then for every  $\bar{a} \in M$  we have  $\varphi^M(\bar{a}) = \varphi^c(\bar{a})$ , that is, the continuous truth value does not depend on where it is being computed. In particular, if  $N = \text{mcl}(M) \in K^c$ , then  $\varphi^M(\bar{a}) = \varphi^N(\bar{a})$  for all  $\bar{a} \in M$ .

**Definition 3.3.** (i) Let  $\varphi(\bar{x})$  be a formula,  $c \leq 0$ . We say that  $\varphi$  does not have the *c-additive order property* (or is *c-additively stable*, *c-add-stable*) if for every indiscernible sequence  $\langle \bar{a}_i \bar{b}_i : i < \lambda \rangle$  long enough we have

$$i < j \Rightarrow |\varphi(\bar{a}_i, \bar{b}_j) - \varphi(\bar{a}_j, \bar{b}_i)| \leq c$$

- (ii) Let  $\varphi(\bar{x})$  be a formula,  $c \geq 1$ . We say that  $\varphi$  does not have the  *$\varepsilon$ -multiplicative order property* (or is  *$\varepsilon$ -multiplicatively stable*,  *$\varepsilon$ -mult-stable*) if for every indiscernible sequence  $\langle \bar{a}_i \bar{b}_i : i < \lambda \rangle$  long enough we have

$$i < j \Rightarrow \frac{1}{c} \leq \frac{\varphi(\bar{a}_i, \bar{b}_j)}{\varphi(\bar{a}_j, \bar{b}_i)} \leq c$$

where we stipulate  $\frac{0}{0} = 1$ ,  $\frac{c}{0} = \infty$  for  $c > 0$ .

**Observation 3.4.** (i) If a formula  $\varphi(\bar{x}, \bar{y})$  is *c-add-unstable*, then  $\mathfrak{C}$  is *c-add-unstable* exemplified by a pair  $(\varphi^{[r]}(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$  for some  $r \geq 0$  and  $\psi$  with  $\mathbf{d}(\varphi, \psi) > r + c$ .

- (ii)  $\mathfrak{C}$  is *c-add-stable* if and only if every formula is *c-add-stable*.

*Proof.* Note that  $\varphi(\bar{a}, \bar{b}) > e$  means that the formula  $\varphi^{[e]}(\bar{a}, \bar{b})$  does not hold, and therefore by the definition of a monster metric space there exists  $\psi(\bar{x}, \bar{y})$  such that  $\psi(\bar{a}, \bar{b})$  holds and the pair  $(\varphi^{[e]}, \psi)$  is contradictory. So if e.g.  $\varphi(\bar{a}_i, \bar{b}_j) \leq r$  and  $\varphi(\bar{a}_j, \bar{b}_i) > r + c$ , then there exists  $\psi(\bar{x}, \bar{y})$  such that  $(\varphi^{[r+c]}, \psi)$  is a contradictory pair and  $\psi(\bar{a}_j, \bar{b}_i)$  holds. The rest should be clear. QED<sub>3.4</sub>

**Definition 3.5.** (i) Let  $\varphi(\bar{x})$  be a formula,  $M, N \in K$ ,  $f: M \rightarrow N$ ,  $c \geq 0$ . We say that  $f$  is a *( $\varphi, c$ )-additive isomorphism* (*( $\varphi, c$ )-add-isomorphism*) if for every  $\bar{a} \in M$  we have  $|\varphi(\bar{a}) - \varphi(f(\bar{a}))| \leq c$ .

- (ii) Let  $\varphi(\bar{x})$  be a formula,  $M, N \in K$ ,  $f: M \rightarrow N$ ,  $c \geq 1$ . We say that  $f$  is a *( $\varphi, c$ )-multiplicative isomorphism* (*( $\varphi, c$ )-mult-isomorphism*) if for every  $\bar{a} \in M$  we have

$$\frac{1}{c} \leq \frac{\varphi(\bar{a})}{\varphi(f(\bar{a}))} \leq c$$

where we stipulate  $\frac{0}{0} = 1$ ,  $\frac{c}{0} = \infty$  for  $c > 0$ .

- (iii) Let  $\varphi(\bar{x})$  be a formula,  $M, N \in K$ ,  $f: M \rightarrow N$ . We define the  *$\varphi$ -additive norm* of  $f$  by

$$\|f\|_{\varphi, \text{add}} = \inf\{\bar{c}: f \text{ is a } (\varphi, \bar{c})\text{-add-isomorphism}\}$$

(could be  $\infty$ ).

- (iv) Let  $\varphi(\bar{x})$  be a formula,  $M, N \in K$ ,  $f: M \rightarrow N$ . We define the  $\varphi$ -multiplicative norm of  $f$  by

$$\|f\|_{\varphi, \text{mult}} = \inf\{\bar{c}: f \text{ is a } (\varphi, c)\text{-mult-isomorphism}\}$$

(could be  $\infty$ ).

- (v) Let  $\varphi(\bar{x})$  be a formula,  $c \geq 0$ . We say that  $\mathfrak{C}$  is  $(\varphi, c)$ -additively categorical in a cardinality  $\lambda$  if for every  $M, N \in K^c$  of density  $\lambda$  there is  $f: M \rightarrow N$ ,  $\|f\|_{\varphi, \text{add}} \leq c$ .
- (vi) Let  $\varphi(\bar{x})$  be a formula,  $c \geq 1$ . We say that  $\mathfrak{C}$  is  $(\varphi, c)$ -multiplicatively categorical in a cardinality  $\lambda$  if for every  $M, N \in K^c$  of density  $\lambda$  there is  $f: M \rightarrow N$ ,  $\|f\|_{\varphi, \text{mult}} \leq c$ .

**Corollary 3.6.** *Let  $c \geq 0$ .*

- (i) *If a formula  $\varphi(\bar{x}, \bar{y})$  is  $c$ -add-unstable then for every  $\lambda > |L|$  there exists a sequence of models  $\langle M_i : i < 2^\lambda \rangle$  of density character  $\lambda$  such that for  $i \neq j$  and for every  $f: M_i \rightarrow M_j$  we have  $\|f\|_{\varphi, \text{add}} > \frac{c}{3}$ .*
- (ii) *Assume that  $\mathfrak{C}$  is  $(\varphi, c)$ -additively categorical in some  $\lambda > |L|$ . Then  $\varphi$  is  $3 \cdot c$ -additively stable.*
- (iii) *Assume that for every  $c > 0$  there is  $\lambda > |L|$  such that  $\mathfrak{C}$  is  $(\varphi, c)$ -additively categorical in  $\lambda$ . Then  $\varphi$  is stable.*

*Proof.* This is basically a restatement of Proposition 2.5 and Corollary 2.6. Note that one has to use Observation 3.4 in order to obtain EM-blueprint  $\Phi$  as in 2.5.1. QED<sub>3.6</sub>

We would like to formulate the multiplicative analogue of Corollary 3.6. It will be more convenient for us to deal with this in the ‘‘Hausdorff’’ case. Since our main aim is connecting weak multiplicative categoricity for classes of normed spaces with quantifier-free stability (so  $\varphi(x, y) = \|x + y\|$ , that is,  $\|x + y\| = 0$ ), existence of weak negations is a reasonable assumption.

**Corollary 3.7.** *Let  $c \geq 1$ ,  $\varphi(\bar{x}, \bar{y})$  a formula with weak negations.*

- (i) *If  $\varphi(\bar{x}, \bar{y})$  is  $c$ -mult-unstable then for every  $\lambda > |L|$  there exists a sequence of models  $\langle M_i : i < 2^\lambda \rangle$  of density character  $\lambda$  such that for  $i \neq j$  and for every  $f: M_i \rightarrow M_j$  we have  $\|f\|_{\varphi, \text{mult}} > \sqrt{c}$ .*
- (ii) *Assume that  $\mathfrak{C}$  is  $(\varphi, c)$ -multiplicatively categorical in some  $\lambda > |L|$ . Then  $\varphi$  is  $c^2$ -mult-stable.*
- (iii) *Assume that for every  $c > 1$  there is  $\lambda > |L|$  such that  $\mathfrak{C}$  is  $(\varphi, c)$ -mult-categorical in  $\lambda$ . Then  $\varphi$  is stable.*

*Proof.* (ii) and (iii) clearly follow from (i).

(i) We just have to make sure that the proof of Proposition 2.5 can be adjusted. So suppose  $\varphi$  has the  $c$ -multiplicative order property, that is, there are arbitrary long

indiscernible sequences  $\langle \bar{a}_i : i < \theta \rangle$  satisfying e.g. for some  $r$  and  $\varepsilon > 0$

$$i < j \Rightarrow \varphi(\bar{a}_i, \bar{a}_j) \leq r, \varphi(\bar{a}_j, \bar{a}_i) \geq r \cdot c + \varepsilon$$

(for simplicity of notation, we add dummy variables in order to combine two sequences into one; of course, this can be avoided, as in the proof of Proposition 2.5).

By the assumption on  $\varphi$ , we can find an EM-blueprint such that the inequalities above are satisfied by the elements of the skeleton of  $\text{EM}(I, \Phi)$  for every linear order  $I$  (so  $\psi$  in 2.5.1 is replaced with the “weak negation”  $[\varphi(\bar{x}, \bar{y}) \geq r \cdot c + \varepsilon]$ ).

Now repeating the proof of Proposition 2.5, given  $\lambda > |L|$  we get a sequence  $\langle M_\alpha : \alpha < 2^\lambda \rangle$  of models in  $K^c$  such that

- $M_\alpha = \overline{\text{EM}(I_\alpha, \Phi)}$
- $\alpha \neq \beta \Rightarrow I_\alpha$  is not  $\theta$ -embeddable into  $I_\beta$  ( $\theta$  as in 2.5).

Let  $f: M_\alpha \rightarrow M_\beta$ . Assuming  $\|f\|_{\varphi, \text{mult}} \leq \sqrt{c}$ , we will get  $i_0 < i_1, l_1 < l_0$  such that

- $M_\alpha \models [\varphi(\bar{a}_{i_0}, \bar{a}_{i_1}) \leq r]$  and therefore  $M_\beta \models [\varphi(f(\bar{a}_{i_0}), f(\bar{a}_{i_1})) \leq r\sqrt{c}]$
- $M_\alpha \models [\varphi(\bar{a}_{l_0}, \bar{a}_{l_1}) \geq rc + \varepsilon]$  and therefore  $M_\beta \models [\varphi(f(\bar{a}_{l_0}), f(\bar{a}_{l_1})) \geq r\frac{c}{\sqrt{c}} + \frac{\varepsilon}{\sqrt{c}}]$
- $f(\bar{a}_{i_0})f(\bar{a}_{i_1}) \equiv f(\bar{a}_{l_0})f(\bar{a}_{l_1})$

which when put together is clearly a contradiction. QED<sub>3.7</sub>

The following particular case is of especial interest to us in this context.

**Definition 3.8.** Assume that  $\mathfrak{C}$  is a *normed* structure (over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , maybe with extra-structure). We say that  $\mathfrak{C}$  is *c-categorical* in a cardinal  $\lambda$  for  $c \geq 1$  if for every two models  $M, N$  of density  $\lambda$  there exists a linear isomorphism  $f: M \rightarrow N$  such that for every  $a \in M$  we have  $\frac{1}{c} \leq \frac{\|a\|_M}{\|f(a)\|_N} \leq c$  (where  $\frac{0}{0} = 1, \frac{r}{0} = \infty$  for  $r \neq 0$ ).

*Remark 3.9.* Note that if  $f$  is as in the definition above, it is a  $(\varphi(\bar{x}), c)$ -multiplicative isomorphism for any  $\varphi(\bar{x}) = \varphi(x_{<k})$  of the form  $\varphi(\bar{x}) = \|\sum_{i < k} r_i x_i\|$  (where  $r_i \in \mathbb{F}$ ).

So by Corollary 3.7 and the previous remark we obtain the following:

**Corollary 3.10.** *Let  $\mathfrak{C}$  be a normed structure over  $\mathbb{F}$  (possibly with extra-structure), c-categorical in  $\lambda > |L|$ . Then every formula  $\varphi(\bar{x}, \bar{y}) = \varphi(x_{<k}, y_{<\ell})$  of the form*

$$\varphi(\bar{x}, \bar{y}) = \left\| \sum_{i < k} r_i x_i + \sum_{j < \ell} s_j y_j \right\|$$

is  $c^2$ -mult-stable.

#### 4. JUNGE-ROSENTHAL STABILITY

After hearing the statement of Corollary 4.5, Ward Henson asked whether the proof can be modified in order to obtain a stronger version of stability. In this section we give a positive answer to Henson’s question.

The following definition in the context of Banach spaces is due to Junge and Rosenthal (although to the best of our knowledge the original paper was never finished). One reference is [JKO]. We will not make use of the name ‘‘asymptotically symmetric’’ suggested in [JKO] (since the definition below is simply a very natural generalization of stability).

For the sake of simplicity of presentation, we only deal with the multiplicative case (and assume existence of weak negations). The additive analogue can be developed similarly.

**Definition 4.1.** Let  $\varphi(\bar{x}_0, \dots, \bar{x}_{n-1})$  be a formula with weak negations,  $c \geq 1$ . We say that  $\varphi(\bar{x}_0, \dots, \bar{x}_{n-1})$  is  $(c, n)$ -multiplicatively stable ( $(c, n)$ -mult-stable or just  $c$ -mult-stable, since  $n$  is clear from  $\varphi$ ) if for every indiscernible sequences  $\langle \bar{a}_{i,0} \dots \bar{a}_{i,n-1} : i < \omega \rangle$  and permutations  $\sigma, \pi$  of  $n$  we have

$$\frac{1}{c} \leq \frac{\varphi(\bar{a}_{\sigma(0),0} \dots \bar{a}_{\sigma(n-1),n-1})}{\varphi(\bar{a}_{\pi(0),0} \dots \bar{a}_{\pi(n-1),n-1})} \leq c$$

The following is a modification of Fact 2.2.

**Fact 4.2.** Let  $\sigma$  be a permutation of  $n$  and let  $\theta_\sigma(x_0 \dots x_{n-1}, y_0 \dots y_{n-1})$  be the following formula in the language of order:

$$\theta_\sigma = (x_0 < x_1 < \dots < x_{n-1}) \wedge (y_{\sigma(0)} < y_{\sigma(1)} < \dots < y_{\sigma(n-1)})$$

Let  $\lambda > |L|$  regular,  $\mu \geq \lambda$ . Then there exists a family  $\langle I_\alpha : \alpha < 2^\lambda \rangle$  of linear orders of cardinality  $\mu$  which are pairwise  $\theta_\sigma(\bar{x}, \bar{y})$ -unembeddable into each other.

*Proof.* By Claim 2.25 in [Sh:e], III.

QED<sub>4.2</sub>

**Theorem 4.3.** Let  $c \geq 1$ ,  $\varphi(\vec{x}) = \varphi(\bar{x}_0, \dots, \bar{x}_{n-1})$  a formula with weak negations.

- (i) If  $\varphi(\vec{x})$  is  $c$ -mult-unstable then for every  $\lambda > |L|$  regular and every  $\mu \geq \lambda$  there exists a sequence of models  $\langle M_i : i < 2^\lambda \rangle$  of density character  $\mu$  such that for  $i \neq j$  and for every  $f: M_i \rightarrow M_j$  we have  $\|f\|_{\varphi, \text{mult}} > \sqrt{c}$ .
- (ii) Assume that  $\mathfrak{C}$  is  $(\varphi, c)$ -multiplicatively categorical in some  $\lambda > |L|$ . Then  $\varphi$  is  $c^2$ -mult-stable.
- (iii) Assume that for every  $c > 1$  there is  $\lambda > |L|$  such that  $\mathfrak{C}$  is  $(\varphi(\vec{x}), c)$ -mult-categorical in  $\lambda$ . Then  $\varphi(\vec{x})$  is stable (that is,  $(n, 1)$ -stable).

*Proof.* As in the proof of Corollary 3.7, (ii) and (iii) follow from (i), whereas for (i) we should modify the proof of Corollary 3.7 appropriately. So suppose  $\varphi(\vec{x})$  is  $c$ -mult-unstable, that is, there are arbitrary long indiscernible sequences  $\langle \bar{a}_{i,0} \dots \bar{a}_{i,n-1} : i < \theta \rangle$  satisfying e.g. for some  $r, \varepsilon > 0$  and a permutation  $\sigma$  of  $n$

$$i_0 < i_2 < \dots < i_{n-1} \Rightarrow \varphi(\bar{a}_{i_0,0}, \dots, \bar{a}_{i_{n-1},n-1}) \leq r, \quad \varphi(\bar{a}_{i_{\sigma(0)},0}, \dots, \bar{a}_{i_{\sigma(n-1)},n-1}) \geq r \cdot c + \varepsilon$$

We can find an EM-blueprint such that the inequalities above are satisfied by the elements of the skeleton of  $\text{EM}(I, \Phi)$  for every linear order  $I$ .

Now applying Fact 4.2, given  $\lambda > |L|$  we get a sequence  $\langle M_\alpha : \alpha < 2^\lambda \rangle$  of models in  $K^c$  such that

- $M_\alpha = \overline{\text{EM}(I_\alpha, \Phi)}$
- $\alpha \neq \beta \Rightarrow I_\alpha$  is not  $\theta_\sigma$ -embeddable into  $I_\beta$

The rest is exactly as in the proof of Corollary 3.7.

QED<sub>4.3</sub>

*Remark 4.4.* Note that Theorem 4.3 is not quite a generalization of Corollary 3.7: here we only get  $2^\lambda$  models of density  $\lambda$  if  $\lambda$  is *regular* (otherwise, we get  $2^\mu$  for every regular  $\mu < \lambda$ ), whereas Corollary 3.7 gives the maximal number in every  $\lambda > |L|$ . The reason is that Theorem 4.3 relies on an easier straightforward argument given in [Sh:e], chapter III, whereas in Corollary 3.7 we could apply a more sophisticated analysis of [Sh:e], chapter VI.

Let us conclude with a precise statement of Theorem 1.1, which we think of as one of the main results of the paper. It follows immediately from Theorem 4.3.

**Corollary 4.5.** *Let  $\mathfrak{C}$  be a normed structure over  $\mathbb{F}$  (possibly with extra-structure),  $c$ -categorical in  $\lambda > |L|$ . Then  $\mathfrak{C}$  is  $c^2$ -asymptotically symmetric, that is, for every  $n$ , the formula*

$$\varphi(x_1, \dots, x_n) = \|x_1 + \dots + x_n\|$$

*is  $c^2$ -mult-stable.*

## REFERENCES

- [Ben05] I. Ben-Yaacov, *Uncountable dense categoricity in cats*, Journal of Symbolic Logic, volume 70 (2005), no. 3, 829-860.
- [BU] I. Ben-Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, submitted.
- [Io] J. Iovino, *Stable Banach spaces and Banach space structures I. Fundamentals*, pp. 77–95 in Models, Algebras, and Proofs (Bogotá, 1995), edited by C. Montenegro and X. Caicedo, vol. 203 of Lecture Notes in Pure and Applied Mathematics, Dekker, New York, 1999.
- [JKO] M. Junge, D. Kutzarova, E. Odell, *On Asymptotically Symmetric Banach Spaces*, Studia Math. 173 (2006), 203-231.
- [KM] Krivine, J.-L., B. Maurey, *Espaces de Banach stables*, Israel Journal of Mathematics, vol. 39 (1981), 273–95
- [Sh:e] S. Shelah, *Nonstructure theory* (a book), preprint.
- [ShUs837] S. Shelah and A. Usvyatsov, *Model theoretical stability and categoricity in complete metric spaces*, submitted

SAHARON SHELAH, MATHEMATICS DEPARTMENT, HEBREW UNIVERSITY OF JERUSALEM, 91904 GIVAT RAM, ISRAEL,

SAHARON SHELAH, DEPARTMENT OF MATHEMATICS, HILL CENTER-BUSCH CAMPUS RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY 110 FRELINGHUYSEN RD PISCATAWAY, NJ 08854-8019, USA

ALEXANDER USVYATSOV, UNIVERSIDADE DE LISBOA, CENTRO DE MATEMÁTICA E APLICAÇÕES FUNDAMENTAIS, AV. PROF. GAMA PINTO, 2, 1649-003 LISBOA, PORTUGAL

URL: <http://www.math.ucla.edu/~alexus>