

STABILITY AND UNIDIMENSIONALITY IN MONSTER METRIC SPACES

ALEXANDER USVYATSOV

ABSTRACT. This article investigates stability for monster metric spaces introduced in [ShUs837] without the additional assumption of the language L being countable. We prove that any monster metric space which is categorical in any cardinality bigger than $|L|$ is 0^+ -stable in $|L|$ and unidimensional. In particular, the result holds for uncountable continuous theories.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. In [ShUs837] Shelah and the author prove the analogue of Morley’s categoricity theorem for the context of compact monster metric spaces. A natural question is: can the proof be generalized to uncountable languages? It seems that the answer is no: the last section of [ShUs837] relies heavily on countability of L , and a new, more general approach is needed if this assumption is dropped. But most results can still be extended using similar techniques, by applying a more delicate analysis. This is the purpose of this article.

Although all the main theorems here do not require compactness assumptions and hold in any homogenous monster equipped with a metric (“monster metric space” in the terminology of [ShUs837]), we work (for the sake of simplicity and clarity) in the context of metric cats introduced by Itai Ben-Yaacov in [Ben05], equivalently, compact monster metric spaces. This context is still very distinct from continuous first order model theory developed in [BU] and [BBHU]. Continuous theories are closely connected to metric cats whose space of types is a Hausdorff topological space (as shown in [BU]); but without this assumption many of classical model theoretic tools fail. This is why techniques we use are mostly borrowed from homogenous model theory.

In subsection 1.2 we recall basic definitions and facts on metric cats and monster metric spaces. The reader is referred to [ShUs837] for the proofs.

In section 2 we discuss “local” stability in cats (i.e. stability for a given formula or a given contradictory pair of formulae). As the language is not necessarily countable, working “locally” will sometimes make our life simpler. We show several equivalent versions of local stability and extend results from [Ben03a]. Then we use local stability in order to connect topological stability (0^+ -stability) to nonsplitting and classical stability in homogenous model theory.

Date: February 1, 2007.

In section 3 we introduce average types of indiscernible sequences and show that they are well-behaved.

Section 4 is devoted to generalizing some results from [ShUs837] to uncountable languages. We show that if the class of almost elementary complete submodels of a metric cat is categorical in some power bigger than $|L|$, or even weakly categorical (see Definition 4.1), then it is 0^+ -stable in $|L|$, and therefore classically stable. Then we prove that a weakly categorical metric cat is unidimensional. In particular, we obtain the following weak form of a categoricity transfer theorem: if \mathfrak{C} is a metric cat which is categorical in some power bigger than $|L|$ (or even weakly categorical), then it has exactly one $|L|^+$ -saturated model of each density bigger than $|L|$.

In order to extend the categoricity theorem to the class of *all* almost elementary submodels, one needs to show that categoricity implies non-existence of non- $|L|^+$ -saturated models in high cardinalities. This is done in [Us0x] for elementary continuous classes, but the proof uses classical model theoretic tools and would not go through in the more general context of arbitrary metric cats.

1.2. Preliminaries. We begin with the definition of a metric cat, see [Ben05] for more details.

Fix a “big enough” cardinal $\lambda^* = (\lambda^*)^{\aleph_0}$.

Definition 1.1. A *metric cat* is a metric structure of cardinality λ^* which is compact for positive existential formulae.

Remark 1.2. (i) See e.g. [ShUs837], Definitions 2.4(1), 2.13, 2.15(2) for notions mentioned in the definition above.

(ii) If \mathfrak{C} is a metric cat then it is a monster metric space as defined in [ShUs837], Definition 2.17; moreover, it is compact (Definition 2.21 there) for $\Delta(\mathfrak{C}) =$ positive existential formulae. See also [ShUs837], Example 2.24(5). So all results proven in [ShUs837] apply to metric cats.

(iii) Note that by Observation 2.23 in [ShUs837], “positive existential” in Definition 1.1(2) above means precisely what one would expect: the closure of the atomic formulae under conjunction, disjunction and existential quantification.

From now on, \mathfrak{C} will denote a metric cat (unless stated otherwise). All elements, tuples, sets, will be taken inside \mathfrak{C} . The finite diagram of \mathfrak{C} will be denoted by D . By “types” we will always mean D -types, which for metric cats is just the collection of all positive existential types. By “formula”, we always mean a positive existential formula.

Let K be the class of all almost elementary submodels of the monster \mathfrak{C} , K^c the class of complete such (as defined in [ShUs837], Definition 2.19). We will call elements of K *premodels* and elements of K^c *models*.

See e.g. section 1 of [ShUs837] on definitions of basic concepts of finite diagrams, (D, λ) -homogeneity of models, etc. Note that for metric cats (D, λ) -homogeneity is equivalent to λ -saturation for positive existential types (by the choice of D and Remark

1.2 in [ShUs837]), and that (D, \aleph_1) homogeneity implies completeness (by Observation 3.10 in [ShUs837]), so any (D, λ) -homogeneous premodel (for $\lambda > \aleph_0$) is a model.

We recall the definition of approximations of formulae and types (see more in [ShUs837], section 3):

Definition 1.3. (i) For a formula $\varphi(\bar{x})$ possibly with parameters, we define $\varphi^{[\varepsilon]}(\bar{x}) = \exists \bar{x}'(\varphi(\bar{x}') \wedge \mathbf{d}(\bar{x}, \bar{x}') \leq \varepsilon)$. So $\varphi^{[\varepsilon]}(\bar{x}, \bar{a}) = (\exists \bar{x}')[\mathbf{d}(\bar{x}, \bar{x}') \leq \varepsilon \wedge \varphi(\bar{x}', \bar{a})]$.
 (ii) For a type p , define $p^{[\varepsilon]} = \{(\bigwedge_{\ell < n} \varphi_\ell)^{[\varepsilon]} : n < \omega \text{ and } \varphi_0, \dots, \varphi_{n-1} \in p\}$.

Recall that a pair of formulae (maybe with parameters) $(\varphi(\bar{x}), \psi(\bar{x}))$ is called *contradictory* if $\varphi^{\mathfrak{e}} \cap \psi^{\mathfrak{e}} = \emptyset$. A pair $(\varphi(\bar{x}), \psi(\bar{x}))$ is called ε -*contradictory* (where $\varepsilon > 0$ as usual) if $\mathbf{d}(\varphi^{\mathfrak{e}}, \psi^{\mathfrak{e}}) > \varepsilon$.

Fact 1.4. *A pair $(\varphi(\bar{x}), \psi(\bar{x}))$ is contradictory if and only if it is ε -contradictory for some $\varepsilon > 0$.*

Proof. See [ShUs837], Claim 3.7.

QED_{1.4}

Fact 1.5. *Let p be a type over a set A , $\varepsilon > 0$. Then*

$$p^{[\varepsilon]} = \{\bar{a} : \exists \bar{a}'(\mathbf{d}(\bar{a}, \bar{a}') \leq \varepsilon \text{ and } \bar{a}' \models p)\}$$

Proof. See [ShUs837], Claim 3.8.

QED_{1.5}

Fact 1.6. *Let M be a model (i.e. $M \in K^c$, so it is complete) which is not (D, λ) -homogeneous. Then for some $A \subseteq M$ of cardinality less than λ , $p \in S_1(A)$ and $\varepsilon > 0$, M omits $p^{[\varepsilon]}$.*

Proof. See [ShUs837], Corollary 3.14.

QED_{1.6}

2. STABILITY

We recall the definitions from [ShUs837], [Ben05]. We use the notations from [ShUs837] in order not to get the reader confused between the two notions of λ -stability (the topological one and the classical one used in homogeneous model theory).

Definition 2.1. (i) A homogeneous metric monster $(\mathfrak{C}, \mathbf{d})$ is called 0^+ - λ -stable if for all $A \subseteq \mathfrak{C}$ of cardinality λ there exists $B \subseteq \mathfrak{C}$ of cardinality λ such that each $p \in S_D(A)$ is realized in $\text{mcl}(B) = \bar{B}$ (i.e., the topological closure of B).
 (ii) We call a homogeneous metric monster *classically* λ -stable if it is λ -stable in the sense of [Sh3], i.e. for every $A \subseteq \mathfrak{C}$ of cardinality λ , the cardinality of $S_D(A)$ is at most λ .

Fact 2.2. *For a metric cat \mathfrak{C} the following are equivalent.*

- (i) \mathfrak{C} is 0^+ - λ -stable
- (ii) For each $A \subseteq \mathfrak{C}, |A| \leq \lambda$, there exists $B, |B| \leq \lambda$ such that for each $\varepsilon > 0$ and $p \in S_D(A), p^{[\varepsilon]}$ is realized in B (B almost realizes all types over A).

- (iii) *There are no $A, |A| \leq \lambda, \langle p_i : i < \lambda^+ \rangle$ a sequence of members of $\mathbf{S}_D(A)$, i.e., complete D -types over A and $\varepsilon > 0$ such that $p_i^{[\varepsilon]} \cap p_j^{[\varepsilon]} = \emptyset$ for all $i, j < \lambda^+$.*

Proof. See Claim 4.3 in [ShUs837].

QED_{2.2}

Let us recall several basic definitions which are relevant for local stability:

- Definition 2.3.**
- (i) Let $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ be a contradictory pair. We say that it has the *order property* if there exist indiscernible sequences $\langle \bar{a}_i : i < \omega \rangle$ and $\langle \bar{b}_i : i < \omega \rangle$ such that $\varphi(\bar{a}_i, \bar{b}_j)$ for $i < j$ and $\psi(\bar{a}_i, \bar{b}_j)$ for $i > j$.
 - (ii) We say that a formula $\varphi(\bar{x}, \bar{y})$ has the order property if for some $\psi(\bar{x}, \bar{y})$ contradictory to φ , the pair (φ, ψ) has it.
 - (iii) For a contradictory pair $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$, a complete (φ, ψ) -type over a set A is a restriction of a complete type over A to formulae of the form $\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{b})$ for $\bar{a}, \bar{b} \in A$.
 - (iv) For a formula $\varphi(\bar{x}, \bar{y})$, a complete φ -type over a set A is a restriction of a complete type over A to formulae of the form $\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{b})$ for $\bar{a}, \bar{b} \in A$ and ψ contradictory to φ .
 - (v) We say that a contradictory pair $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ is *classically stable* in a power λ if for every set A of cardinality λ there are only λ -many $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ -types. Similarly for a formula φ being classically stable in λ .
 - (vi) We say that a type $p \in S(B)$ *splits* over $A \subseteq B$ if there is a contradictory pair $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$, $\bar{a}, \bar{b} \in B$ which have the same type over A such that $\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{b}) \in p$. In this case we also say that p $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ -splits over A .

Proposition 2.4. *Assume φ does not have the order property. Then every φ -type over a $(D, 2^{|L|^+})$ -homogeneous model does not split over a set of cardinality $2^{|L|}$.*

Proof. Assume a certain φ -type p over M splits over every subset of M of cardinality $\leq 2^{|L|}$. Let $\bar{c} \in \mathfrak{C}$ realize p .

We define sequences $\langle \bar{a}_i, \bar{b}_i, \bar{c}_i : i < 2^{|L|^+} \rangle$ in M by induction on i as follows:

By the assumption, p splits over $\{\bar{a}_j \bar{b}_j \bar{c}_j : j < i\}$ witnessed by, say, the pair $(\varphi(\bar{x}, \bar{a}_i), \psi_i(\bar{x}, \bar{b}_i))$.

So the pair (φ, ψ_i) is contradictory, and we have $\varphi(\bar{c}, \bar{a}_i)$ and $\psi_i(\bar{c}, \bar{b}_i)$, while for $j < i$, either $\varphi(\bar{c}_j, \bar{a}_i) \wedge \varphi(\bar{c}_j, \bar{b}_i)$ or $\psi_{i,j}(\bar{c}_j, \bar{a}_i) \wedge \psi_{i,j}(\bar{c}_j, \bar{b}_i)$ for some $\psi_{i,j}$ which is contradictory to φ .

Now choose $\bar{c}_i \models p \upharpoonright \{\bar{a}_j \bar{b}_j \bar{c}_j : j < i\} \cup \{\bar{a}_i \bar{b}_i\}$. So $\varphi(\bar{c}_i, \bar{a}_j)$ and $\psi_j(\bar{c}_i, \bar{b}_j)$ for all $j \leq i$.

By Erdős-Rado Theorem, without loss of generality we have $\psi_i = \psi_{i,j} = \psi$ for all $i < |L|^+, j < i$ (for those j for which $\psi_{i,j}$ is defined).

Now either $\langle \bar{c}_i \bar{a}_i : i < |L|^+ \rangle$ or $\langle \bar{c}_i \bar{b}_i : i < |L|^+ \rangle$ exemplifies φ -order property in M .

QED_{2.4}

Observation 2.5. If φ does not have the order property, then there exists a cardinal λ such that φ is λ -classically stable, i.e. over every set A of cardinality λ , there are only λ -many D -types over A .

Proof. Choose e.g. $\lambda = \beth_2(|L|^+)$, so every set A of cardinality λ is contained in a $(D, 2^{|L|^+})$ -homogeneous model M of cardinality λ . Now every type over M does not split over a subset of M of cardinality $2^{|L|}$. For a given such subset B there are at most $\beth_2(|L|)$ -many types over B , and therefore $\beth_2(|L|)$ -many φ -types over M which do not split over B ; there are $\lambda^{2^{|L|}} = \lambda$ such B 's, so λ -many types over M altogether. QED_{2.5}

So we've shown that a formula which does not have the order property is classically stable. The other direction is pretty clear; we will still present it for completeness.

Observation 2.6. Suppose the pair (φ, ψ) exemplifies the order property. Then for every infinite λ there is a set of cardinality λ and λ^+ types over A which are pairwise (φ, ψ) -contradictory.

Proof. By compactness, we can find sequences witnessing the (φ, ψ) -order property of any order type. In particular, we can find such a witness of order type I where I is a linear order of cardinality λ with $\geq \lambda^+$ Dedekind cuts (e.g. choose μ to be the minimal cardinal such that $2^\mu > \lambda$ and consider $I = {}^{\mu > 2}$ with the natural order).

The rest should be clear.

QED_{2.6}

Now by [Ben03a], Proposition 2.2, we can conclude:

Corollary 2.7. *The following are equivalent for a formula φ :*

- (i) φ does not have the order property
- (ii) For some λ , φ is classically λ -stable
- (iii) φ is classically λ -stable for all $\lambda = \lambda^{|L|}$

As in [Ben03a], we call such a formula *stable*. We call a contradictory pair (φ, ψ) *stable* if it is classically λ -stable for some infinite λ , i.e. there are no λ^+ -many pairwise (φ, ψ) -contradictory types over a set of cardinality λ . Clearly, this definition is equivalent to the one in [Ben03a], and a formula φ is stable if for every ψ the pair (φ, ψ) is.

The following is clear now:

Corollary 2.8. *The following are equivalent:*

- (i) \mathfrak{C} is classically stable (classically λ -stable for some λ).
- (ii) Every formula φ is stable.
- (iii) Every contradictory pair (φ, ψ) is stable.
- (iv) \mathfrak{C} is classically λ -stable for all $\lambda = \lambda^{|L|}$.
- (v) \mathfrak{C} is $0^+ - \lambda$ -stable for some λ .

Definition 2.9. We call \mathfrak{C} *stable* if one (all) of the above are satisfied for \mathfrak{C} .

In order to deal with uncountable languages, we will need a better “local” understanding, i.e. a better understanding of a single stable pair.

Remark 2.10. A pair (φ, ψ) is stable if and only if it is classically λ -stable for *all* infinite λ .

Proof. Standard, e.g. by compactness and the fact that for every infinite λ , denoting $\mu = \min\{\mu: 2^\mu > \lambda\}$, we get $2^{<\mu} \leq \lambda$, so there is a binary tree of cardinality $\leq \lambda$ with more than λ branches. QED_{2.10}

Proposition 2.11. *Let (φ, ψ) be a stable pair. Then for any $B \subseteq \mathfrak{C}, p \in \mathbf{S}_D(B)$, p does not (φ, ψ) -split over a finite subset of B .*

Proof. Suppose p (φ, ψ) -splits over every finite subset of its domain. We construct finite sets A_n for $n < \omega$ and elementary maps F_η for $\eta \in {}^\omega 2$ as follows:

- (i) $A_0 = \emptyset$
- (ii) $A_{n+1} = A_n \cup \{\bar{a}_n, \bar{b}_n\}$, where $\varphi(\bar{x}, \bar{a}_n), \psi(\bar{x}, \bar{b}_n)$ exemplify p (φ, ψ) -splits over A_n , i.e. $\bar{a}_n \equiv_{A_n} \bar{b}_n, \varphi_n(\bar{x}, \bar{a}_n), \psi_n(\bar{x}, \bar{b}_n) \in p$ and $\varphi(\mathfrak{C}, \bar{a}_n) \cap \psi(\mathfrak{C}, \bar{b}_n) = \emptyset$
- (iii) for $\eta \in {}^n 2, F_\eta: A_n \rightarrow \mathfrak{C}$ is an elementary mapping
- (iv) for $\eta \in {}^n 2, F_{\eta \frown \langle 0 \rangle}(\bar{a}_n) = F_{\eta \frown \langle 1 \rangle}(\bar{b}_n)$ and $F_{\eta \frown \langle 0 \rangle}, F_{\eta \frown \langle 1 \rangle}$ extend F_η .

The construction is straightforward. Now denote for $\eta \in {}^\omega 2, F_\eta = \bigcup_{n < \omega} F_{\eta \upharpoonright n}, p_\eta^* = F_\eta(p), A = \bigcup \{\text{Rang}(F_\eta) : \eta \in {}^\omega 2\}$; so A is countable. Choose $p_\eta, p_\eta^* \subseteq p_\eta \in \mathbf{S}_D(A)$. Obviously, for $\eta \neq \nu \in {}^\omega 2, p_\eta$ and p_ν are (φ, ψ) -disjoint types, contradicting stability of the pair (φ, ψ) (see Remark 2.10). QED_{2.11}

Corollary 2.12. *Let \mathfrak{C} be stable. Then every type $p \in \mathbf{S}_D(A)$ does not split over a subset of A of cardinality $|L|$. So (in terms of [Sh54]) $\kappa(D) \leq |L|$.*

Proof. The type p does not split over a union of $|L|$ -many (one for each contradictory pair) finite sets. QED_{2.12}

3. AVERAGES OF INDISCERNIBLE SETS

Definition 3.1. (i) Let \mathbf{I} be an indiscernible set in \mathfrak{C} of length $|L|^+$, A a set and Δ a set of formulas. We define the Δ -average type of \mathbf{I} over A , $\text{Av}_\Delta(\mathbf{I}, A)$ as follows: for a Δ -formula $\varphi(\bar{x}, \bar{a})$ over A , $\varphi(\bar{x}, \bar{a}) \in \text{Av}_\Delta(\mathbf{I}, A)$ iff for all but $|L|$ -many $\bar{c} \in \mathbf{I}$, $\varphi(\bar{c}, \bar{a})$ holds.
(ii) If $\Delta = L$, we omit it. If $A = \bigcup \mathbf{I}$, we omit it.

Observation 3.2. Assume \mathfrak{C} is stable and $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ is a contradictory pair of formulas. Then there is no infinite indiscernible sequence \mathbf{I} in \mathfrak{C} and $\bar{b} \in \mathfrak{C}$ such that both $\{\bar{c} \in \mathbf{I}: \varphi(\bar{c}, \bar{b})\}$ and $\{\bar{c} \in \mathbf{I}: \psi(\bar{c}, \bar{b})\}$ are infinite.

Proof. Standard. QED_{3.2}

Lemma 3.3. (i) $\text{Av}(\mathbf{I}, A)$ defined as above is a complete D -type over A .
(ii) For a formula $\varphi(\bar{x}, \bar{a})$ over A , $\varphi(\bar{x}, \bar{a}) \in \text{Av}(\mathbf{I}, A)$ iff for infinitely many $\bar{c} \in \mathbf{I}$, $\varphi(\bar{c}, \bar{a})$ holds.

- Proof.* (i) Clearly, $\text{Av}(\mathbf{I}, A)$ is consistent. Given a formula $\varphi(\bar{x}, \bar{a})$, for every $\psi(\bar{x}, \bar{y})$ such that $(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$ is a contradictory pair, there are only finitely “exceptions”, that is, either $\varphi(\bar{c}, \bar{a})$ or $\psi(\bar{c}, \bar{a})$ holds for cofinitely many $\bar{c} \in \mathbf{I}$ (by Observation 3.2 above). So if for no such $\psi(\bar{x}, \bar{y})$ does $\varphi(\bar{x}, \bar{a})$ belong to $\text{Av}(\mathbf{I}, A)$, then necessarily $\varphi(\bar{x}, \bar{a})$ does (as there are $|L|$ -many exceptions), which implies completeness of $\text{Av}(\mathbf{I}, A)$.
- (ii) If $\varphi(\bar{x}, \bar{a})$ does not belong to $\text{Av}(\mathbf{I}, A)$, then by completeness necessarily $\psi(\bar{x}, \bar{a})$ does for some $\psi(\bar{x}, \bar{y})$ such that (φ, ψ) is a contradictory pair. But then by Observation 3.2 there are only finitely many $\bar{c} \in \mathbf{I}$ satisfying $\varphi(\bar{c}, \bar{a})$, as required.

QED_{3,3}

Proposition 3.4. *Assume $(\mathfrak{C}, \mathbf{d})$ is stable. If M is $(D, |L|^+)$ -homogeneous, $p \in \mathbf{S}_m(M)$, then for some indiscernible $\mathbf{I} \subseteq {}^m M$ of length $|L|^+$, we have $p = \text{Av}(\mathbf{I}, M)$.*

Proof. Without loss of generality $m = 1$.

Let $B \subseteq M$ be of cardinality $|L|$ over which p does not split (see Corollary 2.12). Choose $a_\alpha \in M$ realizing $p \upharpoonright B \cup \{a_\beta : \beta < \alpha\}$ by induction on $\alpha < |L|^+$. Now $\mathbf{I} = \langle a_\alpha : \alpha < \omega_1 \rangle$ is indiscernible over B by [Sh:c] Lemma 2.5.

If $q = \text{Av}(\mathbf{I}, M) \neq p$ still $q \in \mathbf{S}_D(M)$ and we can find $\varphi(x, \bar{b}) \in q, \psi(x, \bar{b}) \in p$ and they are contradictory. So $u = \{\alpha < \omega_1 : \mathfrak{C} \models \varphi[a_\alpha, \bar{b}]\}$ is infinite, and let $v \subseteq u$ be of cardinality \aleph_0 . Let $v \subseteq \alpha(*) < |L|^+$ and choose $a'_\beta \in M(\beta \in [\alpha(*), |L|^+)$ realizing $p \upharpoonright (B \cup \{a_i : i < \alpha(*)\}) \cup \bar{b} \cup \{a'_\gamma : \gamma \in (\alpha(*), \beta)\}$.

Easy contradiction: for a given contradictory pair (φ, ψ) all but finitely many elements of \mathbf{I} have to “make a choice”, see Observation 3.2.

QED_{3,4}

Remark 3.5. (i) Let $\mathbf{I} = \langle \bar{a}_i : i < \alpha \rangle$, where $\langle \bar{a}_i : i \leq \alpha \rangle$ is indiscernible. Then $\bar{a}_\alpha \models \text{Av}(\mathbf{I}, \cup \mathbf{I})$.

(ii) Let \mathbf{I} be indiscernible, $\mathbf{I} = \langle \bar{a}_i : i < \alpha \rangle$ and let $\bar{a}_\alpha \models \text{Av}(\mathbf{I}, \cup \mathbf{I})$. Then $\langle \bar{a}_i : i \leq \alpha \rangle$ is indiscernible.

(iii) It follows from (i) + (ii) that $\mathbf{I} = \langle \bar{a}_i : i < \alpha \rangle \subseteq M$ is a maximal indiscernible sequence (set) in M iff $\text{Av}(\mathbf{I}, \cup \mathbf{I})$ is omitted in M .

(iv) $\varphi(x, \bar{a}_{i_1}, \dots, \bar{a}_{i_n}) \in \text{Av}(\mathbf{I}, \cup \mathbf{I})$ for $\mathbf{I} = \langle \bar{a}_i : i < \delta \rangle$ (δ -limit ordinal) iff $\bar{a}_j \models \varphi(x, \bar{a}_{i_1}, \dots, \bar{a}_{i_n})$ for some/every $j \notin \{i_1, \dots, i_n\}$.

4. $|L|$ -STABILITY AND UNIDEMENSIONALITY

This section is devoted to metric cats which are categorical in a power bigger than the cardinality of the language. We show that possible categoricity (and even weak categoricity) implies 0^+ - $|L|$ -stability and unidimensionality.

The results in this section do not require compactness, and, although we will state them for metric cats, hold for monster metric spaces (see [ShUs837]) with no additional assumptions, with exactly the same proofs. Although in the proof of Theorem 4.6 below we rely on the results and techniques proven and developed in the previous sections, where we worked with metric cats for the sake of simplifying the picture (e.g. 0^+ -stability is

equivalent to classical stability), the same arguments, applied more carefully to monster metric spaces will lead to the desired conclusions without use of compactness (as is done for countable language in e.g. [ShUs837], Claim 4.10).

First we recall the different definitions of categoricity, although the reader easily can proceed reading this section with having only the classical notions in mind. We refer to section 6 of [ShUs837] for the definition of ε -embedding, ε^+ -isomorphism, etc. Note that categoricity is defined only for *complete* models in K and with respect to density character (and not cardinality) of the model.

- Definition 4.1.**
- (i) \mathfrak{C} is called *categorical* in a power λ if every two models in K^c of density character λ are isomorphic.
 - (ii) \mathfrak{C} is called 0^+ -categorical in a power λ if every two models in K^c of density character λ are 0^+ -isomorphic.
 - (iii) \mathfrak{C} is called *possibly categorical* if it is categorical in some $\lambda > |L|$.
 - (iv) \mathfrak{C} is called *possibly 0^+ -categorical* if it is 0^+ -categorical in some $\lambda > |L|$.
 - (v) \mathfrak{C} is called *weakly categorical* if for every $\varepsilon > 0$ there exists $\lambda > |L|$ such that every two models of density character λ are ε^+ -isomorphic.

The proofs in this section use the technique of Ehrenfeucht-Mostowski models generalized to our context in section 5 of [ShUs837]. The proof of the following theorem is pretty standard:

Theorem 4.2. *Let \mathfrak{C} be a metric cat (or even a monster metric space) categorical in some $\lambda > |L|$, or even weakly categorical. Then \mathfrak{C} is stable in $|L|$.*

Proof. First assume that \mathfrak{C} is categorical in $\lambda > |L|$.

If the conclusion fails, let A be a set of cardinality $|L|$, $\varepsilon > 0$ such that there are $|L|^+$ ε -disjoint types $\langle p_i : i < |L|^+ \rangle$ over A . Let M_1 be a model of density λ containing A and $\langle a_i : i < |L|^+ \rangle$ realizations of $\langle p_i : i < |L|^+ \rangle$. Let $M_0 = \overline{\text{EM}(\lambda, \Phi)}$ where Φ is any blueprint of an indiscernible sequence in \mathfrak{C} (see e.g. section 5 of [ShUs837] for an explanation of how to construct $\text{EM}(\lambda, \Phi)$). So by the categoricity assumption, there exists a subset B of M_0 and $\langle b_i : i < |L|^+ \rangle$ in M_0 realizing ε -disjoint types over B .

Without loss of generality, B is a subset of the skeleton \mathbf{I} of M_0 . The type of an element of $\text{EM}(\lambda, \Phi)$ over B is determined by a term $\sigma(x_1, \dots, x_m)$ and an m -type in the language of order over B , so there are $|L|$ -many different types over B realized in $\text{EM}(\lambda, \Phi)$. Therefore for no $\varepsilon > 0$ can there be more than $|L|$ -many ε -disjoint types realized in M_0 , a contradiction.

The proof for the weakly categorical case is similar (given $|L|^+$ -many ε -disjoint types, we choose λ appropriately using the definition of weak categoricity, and proceed as before). QED_{4.2}

The following notion was explored in [Sh3] but not defined there:

Definition 4.3. A (good) finite diagram D is *unidimensional* if for some regular λ there is no (D, λ) -homogeneous model in K which is not λ^+ -homogeneous of cardinality $\geq \lambda^+$.

In [Sh3] it is essentially proven (seed [Sh3], section 6) that:

Theorem 4.4. *Assume D is stable. Then the following are equivalent:*

- (i) D is not unidimensional
- (ii) There is some regular λ such that there are maximally (D, λ) -homogeneous models of arbitrary large cardinalities
- (iii) For all large enough regular $\lambda < \mu$, there is a (D, λ) -homogeneous model M and $\langle a_i : i < \mu \rangle, \langle b_i : i < \lambda \rangle$ mutually indiscernible sequences in M such that $\langle b_i : i < \lambda \rangle$ is a maximal indiscernible sequence in M , i.e., can not be extended in M .

Remark 4.5. We will call \mathfrak{C} *unidimensional* if its finite diagram D is.

Theorem 4.6. (i) *Let $(\mathfrak{C}, \mathbf{d})$ be a metric cat (or, more generally, a monster metric space) with vocabulary $\tau(\mathfrak{C})$, \mathfrak{C} categorical in λ , $\lambda > \aleph_0 + |\tau_{\mathfrak{C}}|$. Then \mathfrak{C} is unidimensional.*

- (ii) *The same is true if \mathfrak{C} is 0^+ -categorical in some λ as above and even weakly categorical.*

Proof. (i) Note that D is stable by Theorem 4.2 and Corollary 2.8, so we can use Theorem 4.4.

Assume the conclusion fails. Choose $0 < \theta_1 \ll \theta_2$ and let M be a model, $M \in K$, $|M| = \theta_2$, M is (D, θ_1) -homogeneous, $\langle a_i : i < \theta_2 \rangle, \langle b_i : i < \theta_1 \rangle$ mutually indiscernible, $\langle a_i : i < \theta_1 \rangle$ cannot be extended in M , so (denoting $\mathbf{I} = \langle a_i : i < \theta_2 \rangle, \mathbf{J} = \langle b_i : i < \theta_1 \rangle$) $\text{Av}(\mathbf{J}, \cup \mathbf{J})$ is omitted in M (see Remark 3.5).

We now expand the language by a predicate P for \mathbf{J} and skolem functions, call the new vocabulary τ' . Let $T' = \text{Th}_{\tau'}(M')$ (where M' is M in the expanded language).

Note that

$\otimes_0 T' \models$ “ P is a τ -indiscernible set”, i.e., for every τ -formula, T' implies that any two tuples from P behave the same.

The type $p = \text{Av}_{\tau}(\mathbf{J}, \cup \mathbf{J})$ is omitted in M , therefore, for some ε , $p^{[\varepsilon]}$ is omitted. If we choose θ_1, θ_2 carefully enough ($\theta_2 \gg \theta_1$) in the beginning, then by the Erdős-Redo theorem (as in the proof of [Sh:c][VIII,5.3], or using [Ben03a] Lemma 1.2) we can choose a diagram of indiscernibles (EM-blueprint) Φ in vocabulary τ' such that for any μ , denoting the skeleton of $\text{EM}(\mu, \Phi)$ by $\mathbf{I}' = \langle a'_i : i < \mu \rangle$, we have

$\otimes_1 \mathbf{I}'$ is a τ' -indiscernible sequence (set), moreover, it is τ' -indiscernible over $P^{M'_0}$

For each $n < \omega$, for some $i_1, \dots, i_n < \theta_2$, a'_0, \dots, a'_{n-1} has the same τ' -type as a_{i_1}, \dots, a_{i_n} .

Denote $M'_0 = \text{EM}_{\tau}(\mu, \Phi)$. Now:

⊗₂ $M'_0 \prec \mathfrak{C}$ (as τ' has skolem functions), so $\text{mcl}(M'_0) \in K$.

⊗₃ $M'_0 \models T'$ (skolem functions, so $M'_0 \equiv M'$)

⊗₄ $P^{M'_0}$ is a τ -indiscernible set (by ⊗₃ and ⊗₀ above)

Note: $|P^{M'_0}| = |L|$. Indeed, each $b \in P^{M'_0}$ is of the form $\sigma(a'_{i_1}, \dots, a'_{i_n})$ for some τ' -term σ . But as \mathbf{I}' is τ' -indiscernible over $P^{M'_0}$, each such b depends only on σ , and there are $|L|$ -many τ' -terms.

Denote $\mathbf{J}' = P^{M'_0}$, a τ -indiscernible set. Denote $p' = \text{Av}_{\tau}(\mathbf{J}', \cup \mathbf{J}')$. Then

⊗₅ $[p']^{[\varepsilon]}$ is omitted in M_0 :

Pick $\sigma(a'_{i_1}, \dots, a'_{i_n}) \in M'_0$. Let j_1, \dots, j_n be such that $a'_{i_1}, \dots, a'_{i_n} \equiv a_{j_1}, \dots, a_{j_n}$. $\sigma(a_{j_1}, \dots, a_{j_n}) \in M'$ does not realize $p'^{[\varepsilon]}$, so for some $\varphi(x) \in p, M' \models \mathbf{d}(\sigma(\bar{a}_j), \varphi) \geq \varepsilon$.

Note: $\varphi(x) = \varphi(x, \bar{c})$, $\bar{c} \in P^{M'}$. Recall that $\varphi(x, \bar{c}) \in p \Leftrightarrow M' \models \varphi(d, \bar{c})$ for some/all $d \in P^{M'}$, $d \cap \bar{c} = \emptyset$ (as $P^{M'} = \mathbf{J}$ is an indiscernible set, and see Remark 3.5). So $M' \models \text{“}\exists \bar{c} \in P \text{ such that”}$

$$[\forall d \in P \setminus \bar{c}, \varphi(d, \bar{c})] \text{ and } [\mathbf{d}(\sigma(\bar{a}_j), \varphi(x, \bar{c})) \geq \varepsilon]$$

Therefore, M'_0 satisfies the same formula with $\sigma(\bar{a}_i)$, which obviously means that $\sigma(\bar{a}'_i)$ does not satisfy $[p']^{[\varepsilon]}$, as required.

We have finished now: let $\mu = \lambda$, so in $\overline{M_0} = \overline{\text{EM}_{\tau}(\lambda, \Phi)}$ we have an indiscernible set of cardinality $|L|$ whose average is omitted (as $(p')^{[\varepsilon]}$ is omitted in M_0), so $\overline{M_0}$ is a non- $(D, |L|^+)$ -homogeneous model in K^c of density λ . But as there is $(D, |L|^+)$ -homogenous model in density λ , categoricity in λ fails, moreover, 0^+ -categoricity fails, as $\overline{M_0}$ is at least $\frac{\varepsilon}{2}$ -distant from any $(D, |L|^+)$ -homogeneous model.

- (ii) Repeat the proof of (i), and at the end choose λ in which, say, $(\frac{\varepsilon}{2})$ -categoricity holds, and get the same contradiction.

QED_{4.6}

REFERENCES

- [Ben03a] I. Ben-Yaacov, *Simplicity in compact abstract theories*, Journal of Mathematical Logic, volume 3 (2003), no.2, 163-191.
- [Ben05] I. Ben-Yaacov, *Uncountable dense categoricity in cats*, Journal of Symbolic Logic, volume 70 (2005), no. 3, 829-860.
- [BBHU] I. Ben-Yaacov, A. J. Berenstein, C. W. Henson, A. Usvyatsov, *Model Theory for Metric Structures*, submitted.
- [BU] I. Ben-Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, submitted.
- [Sh3] S. Shelah, *Finite diagrams stable in power*, Annals Math Logic, volume 2 (1970), 69-118.

- [Sh54] S. Shelah, *The lazy model-theoretician's guide to stability*, *Logique et Analyse*, volume 18 (1975), 241-30.8
- [Sh:c] S. Shelah, *Classification theory and the number of nonisomorphic models*, 2nd ed., North-Holland Publishing Co., Amsterdam, 1990.
- [ShUs837] S. Shelah and A. Usvyatsov, *Model theoretical stability and categoricity in complete metric spaces*, submitted
- [Us0x] A. Usvyatsov, *Categoricity of uncountable continuous theories*, preprint

ALEXANDER USVYATSOV, UNIVERSITY OF CALIFORNIA – LOS ANGELES, MATHEMATICS DEPARTMENT, BOX 951555, LOS ANGELES, CA 90095-1555, USA

URL: <http://www.math.ucla.edu/~alexus>