# ORTHOGONALITY AND DOMINATION IN UNSTABLE THEORIES

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Abstract. In the first part of the paper we study orthogonality, domination, weight, regular and minimal types in the contexts of rosy and super-rosy theories. Then we try to develop analogous theory for arbitrary dependent theories.

## 1. Introduction and preliminaries

There are several questions that motivated this research. First, it is natural to extend the concepts of domination, regularity and weight to rosy theories (as it has already been done in the simple unstable context). One reason for doing this is "coordinatization" theorems: one would like to analyze an arbitrary type in terms of types that can be studied and classified more easily: regular (admit a pregeometry), minimal, etc. We prove several results of this kind in section 3. These provide a complementary picture to the recent work of Assaf Hasson and the first author [3] where minimal types in super-rosy theories are investigated. For example, the two articles combined throw some light on types in theories interpretable in o-minimal structures.

Another motivation came from our desire to understand and develop the concept of *strong dependence* ([10]). It has recently become clear that this notion is strongly connected to weight. In [11] the second author shows that every strongly dependent type has rudimentary finite generically stable weight. Hence a stable theory is strongly dependent precisely when every type has finite weight. The latter conclusion has also been observed by Adler in [1], as he studied the notion of "burden", which generalizes weight and makes sense in any theory. A related concept (within the context of dependent theories) is investigated by the authors in [8]. A natural question that arises is: given a dependent theory with a good enough independence relation, does strong dependence always imply finite "weight" ? More precisely, we analyze the following two questions in this article. Is thorn-weight finite in a strongly dependent rosy theory? Is there a natural notion of forking weight in an arbitrary dependent theory, and what is the connection to strong dependence? We give a positive answer to the first question in section 2, and address the second one in section 4.

Several directions pursued in this paper require a delicate analysis of existence of mutually indiscernible (sometimes Morley) sequences. Claims of this form are proved in section 2 (in rosy context) and section 4 (for dependent theories). We find these results of interest on their own and believe that they might have further applications. One interesting consequence of our analysis which has several applications beyond the study of weight is that in a dependent theory dividing (say, over an extension base) can be always witnessed by a Morley sequence.

The paper is organized as follows:

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We start by defining notions related to forking and þ-forking, quoting some of the relevant results and proving others that will be needed throughout the paper.

Most of the paper is devoted to understand þ-orthogonality and the role of þ-regular and minimal types in rosy, super-rosy, and finite $U^þ$-rank structures. We show many results analogous to those in stable (and simple) theories, and conclude with a strong decomposition theorem for types of finite rank in rosy theories. As we have already mentioned, this result suggests that analysis of minimal types (as is done e.g. in [3]) leads to understanding of all types in a rosy theory of finite rank (e.g., a theory interpretable in an o-minimal structure).

Section 2 gives proofs of certain basic results on thorn-weight, thorn-domination and regularity. Many of these proofs follow the lines of classical ones, but we still go through them carefully, and where the proofs diverge, we give alternative proofs for the þ-forking context or explain how to bridge the gaps. In this section we also connect thorn-weight to strong dependence and show that every type in a strongly dependent rosy theory has finite thorn-weight.

We have recently learnt that Hans Adler has also written (in an unpublished note) a proof of the fact that in a rosy theory, rudimentarily finite þ-weight implies finite þ-weight (Theorem 2.21). Both his and our proofs of this particular fact are mostly based on Wagner's argument [14] for simple theories, which is itself a generalization of Hyttinen's results [4] in the stable context.

In contrast, the analysis of finite-rank theories in Section 3 is not close to the existing proofs for stable and simple theories. Several useful technical tools applicable in this and related contexts are developed, the main one being Proposition 3.6. We believe that these tools should have many applications.

Finally, in Section 4 we finish the paper by investigating sufficient conditions for existence of mutually indiscernible sequences in dependent theories and draw certain conclusions about the meaning of strong dependence, the behavior of forking and concepts related to weight. In particular it is shown that dividing in a dependent theory can normally be witnessed by a Morley sequence.

1.1. **Notations and Assumptions.** Given a theory $T$, we will work inside its monster model denoted by $\mathfrak{C}$. By "monster" we mean that all cardinals we mention are "small" (i.e. smaller than saturation of $\mathfrak{C}$), all sets are small subsets of $\mathfrak{C}$, all models are small elementary submodels of $\mathfrak{C}$, and truth values of all formulae and all types are calculated in $\mathfrak{C}$. We denote tuples (finite unless said otherwise) by lower case letters $a, b, c$ etc, sets by $A, B, C$ etc, models by $M, N$ etc.

By $a \equiv_A b$ we mean $\text{tp}(a/A) = \text{tp}(b/A)$. Recall that this is equivalent to having $\sigma \in \text{Aut}(\mathfrak{C}/A)$ satisfying $\sigma(a) = b$.

Given an order type $O$, a sequence $I = \langle a_i : i \in O \rangle$ and $j \in O$, we often denote the set $\{a_i : i < j\}$ by $a_{<j}$. Similarly for $a_{\leq j}, a_{>j}$ etc. We also often identify the sequence $I$ with the set $\cup I$; that is, when no confusion should arise we write $\text{tp}(a/I)$ etc.

We will write $a \perp_A B$ for "$\text{tp}(a/AB)$ does not fork over $A$" even if $T$ is not simple. Although non-forking is generally not an independence relation, we still find this notation convenient.

For simplicity we assume $T = T^{eq}$ for all theories $T$ mentioned in this paper.

1.2. **þ-forking.** Since a big part of the paper deals with þ-forking and its properties, we will now define the basic concepts related to this notion. The following definitions and facts can be found in [7].

**Definition 1.1.** *Let $\varphi(x,y)$ be a formula, $b$ be a tuple and $C$ be any set. Then we define the following.*

- $\varphi(x,b)$ strongly divides over $D$ *if $b$ is not algebraic over $D$ and the set*

$$\{\varphi(x,b')\}_{b'\models tp(b/D)}$$

*is $k$-inconsistent for some $k\in\mathbb{N}$.*
- $\varphi(\bar{x},b)$ þ-divides over $C$ *if there is some $D\supset C$ such that $\varphi(x,b)$ strongly divides over $D$.*
- $\varphi(x,b)$ þ-forks over $C$ *if there are finitely many formulas $\psi_1(x,b_1),\dots,\psi_n(x,b_n)$ such that $\varphi(x,b)\vdash\bigvee_i\psi_i(x,b_i)$ and $\psi_i(x,b_i)$ þ-divides over $C$ for $1\le i\le n$.*

We will define a theory to be *rosy* if it does not admit infinite þ-forking chains.

Naturally, we say that a (partial) type þ-divides/forks over a set $A$ if it contains a formula which þ-divides/forks over $A$.

It is convenient to make definition of strong dividing for types slightly more tricky, since by making sure that the strongly dividing type uses “all” the parameters we are able to use algebraic closure much more efficiently. We begin with the following definition in the particular case of a type over a finite set.

**Definition 1.2.** *Let $p(x,b)$ be a (partial) type over a finite tuple $b$. We say that $p(x,b)$ strongly divides over a set $D$ if there is a formula $\varphi(x,b)\in p(x,b)$ which strongly divides over $D$.*

*Remark* 1.1. Notice that the definition of strong dividing (for formulas) implies the following.

(i) If $\varphi(x,a)$ strongly divides over $A$, then for every $b\models\varphi(x,a)$ we have $\mathrm{tp}(a/Ab)$ is algebraic (whereas $\mathrm{tp}(a/A)$ is nonalgebraic).
(ii) $\varphi(x,a)$ strongly divides over $A$ if and only if
  - $a\notin\mathrm{acl}(A)$
  - For every infinite nonconstant indiscernible sequence $\langle a_i : i<\omega\rangle$ in $\mathrm{tp}(a/A)$, we have $\{\varphi(x,a_i)\colon i<\omega\}$ is inconsistent.

(iii) Let $a,b,A$ be such that $a\notin acl(A)$. Then $\mathrm{tp}(b/Aa)$ strongly divides over $A$ if and only if for any $b'\models\mathrm{tp}(b/Aa)$ we have $a\in acl(Ab')$.
(iv) If $\varphi(x,a)$ strongly divides over $A$ and $B\supset A$ is such that $a\notin acl(B)$ then $\varphi(x,a)$ strongly divides over $B$.

*Proof.* (i) Suppose $\varphi(x,a)$ strongly divides over $A$ (so in particular $a\notin\mathrm{acl}(A)$), and let $b\models\varphi(x,a)$. By the definition, there are only finitely many $a_1,\dots,a_{k-1}$ (say, $a_1=a$) in $\mathrm{tp}(a/A)$ such that $\varphi(b,a_i)$. In particular, there are only finitely many realizations of $\mathrm{tp}(a/Ab)$, as required.

(ii) The “only if” direction is clear. For the “if” direction, suppose that $\varphi(x,a)$ does not strongly divide over $A$, but $a\notin\mathrm{acl}(A)$. Then for every $k<\omega$ there is a subset $\{a_1,\dots,a_k\}$ of $\mathrm{tp}(a/A)$ such that $\exists x\bigwedge_{i=1}^k\varphi(x,a_i)$. By compactness, for any cardinal $\mu$ there is a sequence $\langle a_\alpha : \alpha<\mu\rangle$ of realizations of $\mathrm{tp}(a/A)$ such that $\exists x\bigwedge_{i=1}^k\varphi(x,a_{\alpha_i})$ for every $\alpha_1<\ldots<\alpha_k<\mu$. By Fact 1.13 there is such an infinite (nonconstant) indiscernible sequence.

(iii) The "only if" direction follows from (i).

On the other hand, assume that $a \notin \operatorname{acl}(A)$, $\operatorname{tp}(b/Aa)$ does not strongly divide over $A$, but for any $b' \models \operatorname{tp}(b/Aa)$ we have $a \in acl(Ab')$. By (ii), for every formula $\varphi(x,a) \in \operatorname{tp}(b/Aa)$ there is an indiscernible sequence $\langle a_i : i < \omega \rangle$ in $\operatorname{tp}(a/A)$ such that $\{\varphi(x,a_i) \colon i < \omega\}$ is consistent. Let $p(x,a) = \operatorname{tp}(b/Aa)$. By compactness, there is an indiscernible sequence $\langle a_i : i < \omega \rangle$ in $\operatorname{tp}(a/A)$ such that $q(x) = \bigcup_{i<\omega} p(x,a_i)$ is consistent (and, moreover, $a_0 = a$). Let $b' \models q(x)$. Clearly $a = a_0 \notin \operatorname{acl}(Ab')$, since $a_i \equiv_{Ab'} a_0$ for all $i$. This contradicts the assumptions.

(iv) follows easily from (ii).

□

In view of (iii) above, we define in general

**Definition 1.3.** *A type* $\operatorname{tp}(b/B)$ *strongly divides over* $A$ *if* $B$ *is nonalgebraic over* $A$*, but is algebraic over* $Ab'$ *for every* $b' \models \operatorname{tp}(b/A)$.

It may be good to point out that the definition of $p(x,a)$ strong dividing over $A$ is *not* equivalent to having a strong dividing formula. The main point is that strong dividing is quite sensitive to the parameters we name, which is not very common in model theory (it is not closed under elementary equivalence) but which is quite useful when working with þ-forking.

Recall that a formula $\varphi(x,y)$ is called *stable* if it does not have the order property (see [13]).

**Fact 1.2.** *If a stable formula* $\varphi(x,y)$ *witnesses that a type* $p(x,a)$ *forks over* $A$*, then there is a* $\varphi$*-formula witnessing that* $p(x,a)$ *þ-forks over* $A$.

*In particular, in any stable theory the concepts of þ-forking and forking coincide.*

*Proof.* This is Lemma 5.1.1 in [7]. □

As with stable theories, for many of our results we will need the existence of a global rank based on the independence notion, which in this case corresponds to þ-forking.

**Definition 1.4.** *Let* $M$ *be a model. We will define the* $\mathrm{U}^{\text{þ}}$-rank *to be the foundation rank of the order given by the þ-forking relation on types consistent with* $M$*. A theory* $T$ *will be called* super-rosy *whenever the* $U^{þ}$*-rank of any type in any model of* $T$ *is ordinal valued.*

**Fact 1.3.** *Let* $T$ *be a super-rosy theory and let* $a, b, A$ *be subsets of a model* $M$ *of* $T$.

*Then*

$$U^{þ}\left(tp\left(b/aA\right)\right) + U^{þ}\left(tp\left(a/A\right)\right) < U^{þ}\left(tp\left(ab/A\right)\right) < U^{þ}\left(tp\left(b/aA\right)\right) \oplus U^{þ}\left(tp\left(a/A\right)\right).$$

*Proof.* Theorem 4.1.10 in [7]. □

We will need the following easy but important Observation. It will allow us to understand how far we need to extend the types to get þ-dividing from þ-forking and strong dividing from þ-dividing; it will be key for the proof of the decomposition theorem for a type of finite þ-rank in Section 3. The proof is quite close to the proof of Lemmas 3.1, 3.2 and 3.4 in [2]. However, we prove (and need) a slightly different result, so we include a proof.

**Observation 1.4.** *Let* $M$ *be a model of a rosy theory* $T$*, and let* $a, b, A$ *be tuples (and sets) in* $M$*. Then the following hold.*

(i) *Let $p(x,a)$ be a type over $Aa$ which þ-forks over $A$. Then there is a non-þ-forking extension $p(x,a,a')$ of $p(x,a)$ such that $p(x,a')$ þ-divides over $A$.*
(ii) *Let $a,b$ be elements and $A$ be a set such that $tp(b/Aa)$ þ-divides over $A$. Then there is some $e$ such that $b \mathop{\smile\!\!\!\!\!\!|}^{þ}_{Aa} e$ and such that $tp(b/Aea)$ strongly divides over $Ae$.*
*In particular, if $\mathrm{tp}(b/Aa)$ is a type of ordinal valued $U^{þ}$-rank, then $U^{þ}(tp(b/Aae)) < U^{þ}(tp(b/Ae))$.*

*Proof.* (i) Let $p(x,a)$ be as in statement (i) of the Observation and let $b \models p(x,a)$.

By definition, there are finitely many formulas $\varphi_i(x,a_i)$ such that

$$p(x,a) \vdash \bigvee_{i=1}^{n} \varphi_i(x,a_i)$$

and $\varphi(x,a_i)$ þ-divides over $A$. By extension of þ-independence we know that there are $a_1',\dots,a_n' \models \mathrm{tp}(a_1\dots a_n/Aa)$ such that $b \mathop{\smile\!\!\!\!\!\!|}^{þ}_{Aa} a_1'\dots a_n'$.

So $\mathrm{tp}(b/Aaa_1'\dots a_n') \vdash \varphi_m(x,a_m')$ for some $m$; defining $a' := a_m'$ and $p(x,a,a') := tp(b/Aaa')$, we get by construction that $p(x,a,a')$ satisfies the statement of the Observation.

(ii) Let $a,b$ and $A$ be as in statement (ii) of the Observation. By definition of þ-dividing there is some $e'$ and some $\varphi(x,a) \in tp(b/Aa)$ such that $\varphi(x,a)$ strongly divides over $Ae'$. Note that in particular $a \notin \mathrm{acl}(Ae')$.

Let $e \models \mathrm{tp}(e'/Aa)$ be such that $b \mathop{\smile\!\!\!\!\!\!|}^{þ}_{Aa} e$. Since $e \models \mathrm{tp}(e'/Aa)$, strong dividing is preserved. Moreover, $a \notin \mathrm{acl}(Ae)$, hence by the definition (alternatively, by Remark 1.1(v)), $\mathrm{tp}(b/Aae)$ strongly divides over $Ae$.

□

Finally, we will prove the following well known fact.

**Fact 1.5.** *Let $a \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} B$. Then there is a þ-Morley sequence $I$ over $B$ based on $A$ starting with $a$.*

*Proof.* First, construct a non-þ-forking sequence $I' = \langle a_i' : i < \mu\rangle$ in $\mathrm{tp}(a/B)$ based on $A$ starting with $A$ by the standard construction, that is, $a_0' = a$, $a_i' \equiv_B a$, $a_i' \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} Ba_{<i}'$. Moreover, make $\mu$ large enough so that using Erdös-Rado (more precisely, Fact 1.13, see also Remark 1.14) one can find $I$ which is an $\omega$-sequence, $B$-indiscernible and every $n$-type of $I$ over $B$ "appears" in $I'$. Clearly $I$ is a þ-Morley sequence over $B$ based on $A$. Moreover, since every element of $I'$ satisfies $\mathrm{tp}(a/B)$, so does every element of $I$, so by applying an automorphism over $B$ we may assume that $I$ starts with $a$. □

1.3. **Dependent theories and generically stable types.** Recall that a theory $T$ is called *dependent* if there does not exist a formula which exemplifies the independence property. We are mostly going to use the following equivalent definition:

**Fact 1.6.** *$T$ is dependent if and only if there do not exist an indiscernible sequence $I = \langle a_i : i < \lambda\rangle$, a formula $\varphi(x,y)$ and $\bar b$ such that both*

$$\{i \colon \models \varphi(a_i,b)\}$$

*and*

$$\{i\colon \models \neg\varphi(a_i, b)\}$$

*are unbounded in* $\lambda$.

In section 4, we will work with the classical Shelah's notions of dividing, forking and splitting from [13]. Definitions and a quick summary of properties can be found in section 2 of [11]. In particular, we will use the following easy (but important) consequence of dependence (due to Shelah, [9]).

**Fact 1.7.** *(T dependent) Strong splitting implies dividing (and therefore forking).*

We will now define and give the basic properties of generically stable types.

**Definition 1.5.** *We call a type* $p \in \mathrm{S}(A)$ generically stable *if every Morley sequence in it is an indiscernible set.*

The following key properties of generically stable types can be found in [11]:

**Fact 1.8.** *(T dependent)*

(i) $p \in \mathrm{S}(A)$ *is generically stable if and only if some Morley sequence in* $p$ *is an indiscernible set.*

(ii) *Let* $p$ *be a generically stable type. If* $p \in \mathrm{S}(A)$, *then it is definable over* $\mathrm{acl}(A)$. *If* $p$ *is definable over* $A$ *(e.g.* $A = \mathrm{acl}(A)$*), then* $p$ *is stationary (in the sense that it has unique non-forking extensions).*

(iii) *Let* $p$ *be generically stable. Non-forking defines on the set of realizations of* $p$ *a stable independence relation (that is, a relation satisfying all the axioms of a stable independence relation).*

Note that generic stability is not necessarily closed under extensions.

1.4. **Strong dependence and dp-minimality.** The following definitions were motivated by the notions of strong dependence of Shelah (see e.g. [10]) and appear in [11] and [8]. In the definitions below we denote tuples by $\bar{x}, \bar{a}$ (in order to stress the difference between singletons and finite tuples of arbitrary length).

**Definition 1.6.**

(i) *A* randomness pattern *of depth* $\kappa$ *for a (partial) type* $p$ *over a set* $A$ *is an array* $\langle \bar{b}^\alpha_i\colon \alpha < \kappa, i < \omega\rangle$ *and formulae* $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ *for* $\alpha < \kappa$ *such that*

  (a) *The sequences* $I^\alpha = \langle \bar{b}^\alpha_i\colon i < \omega\rangle$ *are mutually indiscernible over* $A$, *that is,* $I^\alpha$ *is indiscernible over* $AI^{\neq\alpha}$.

  (b) $\mathrm{len}(\bar{b}^\alpha_i) = \mathrm{len}(\bar{y}_\alpha)$

  (c) *for every* $\eta \in {}^\kappa\omega$, *the set*

$$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_\eta(\alpha)\colon \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i)\colon \alpha < \kappa, i < \omega, i \neq \eta(\alpha)\}$$

  *is consistent with* $p$.

(ii) *A (partial) type* $p$ *over a set* $A$ *is called* strongly dependent *if there do not exist formulae* $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ *for* $\alpha < \omega$ *and sequences* $\langle \bar{b}^\alpha_i\colon i < \omega\rangle$ *for* $\alpha < \omega$ *mutually indiscernible over* $A$ *such that for every* $\eta \in {}^\omega\omega$, *the set*

$$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_{\eta(\alpha)}\colon \alpha < \omega\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i)\colon \alpha < \omega, i \neq \eta(\alpha)\}$$

*is consistent with p.*

*In other words, p is called strongly dependent if there does not exist a randomness pattern for p of depth $\kappa = \omega$.*

(iii) Dependence rank *(dp-rk) of a (partial) type p over a set A is the supremum of all $\kappa$ such that there exists a randomness pattern for p of depth $\kappa$.*

(iv) *A (partial) type over a set A is called* dp-minimal *if dp-rank of p is 1.*

*In other words, p is dp-minimal if there does not exist a randomness pattern for p of depth 2.*

(v) *A theory is called strongly dependent/dp-minimal if the partial type $x = x$ is (here x is a singleton).*

(vi) *Let T be dependent. A type p is called* strongly stable *if it is strongly dependent and generically stable.*

*Remark* 1.9. Note that Shelah basically shows in [10] Observation 1.7 that if there exists a type $p(\bar{x})$ which is not strongly dependent, then there exists such a type $p'(x)$ with $x$ being a *singleton*. Therefore if there exists an non-strongly dependent type, then $T$ is not strongly dependent and the definitions above make sense.

Note that if in the definition of a randomness pattern all formulae are the same, we get the independence property:

**Observation 1.10.** *A theory T is dependent if and only if it does not admit a randomness pattern of some/any infinite depth with $\varphi_\alpha(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})$ for all $\alpha$ if and only if T does not admit a randomness pattern of depth $|T|^+$.*

*Proof.* By compactness. ☐

A related notion, which will be convenient for us to consider, was investigated by Adler in [1]. We are going to use a slightly different terminology (some of it comes from [8]).

**Definition 1.7.**

(i) *A* dividing pattern *of depth $\kappa$ for a (partial) type p over a set A is an array $\langle \bar{b}^\alpha_i : \alpha < \kappa, i < \omega \rangle$ and formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \kappa$ such that*

(a) *The sequences $I^\alpha = \langle \bar{b}^\alpha_i : i < \omega \rangle$ are mutually indiscernible over A, that is, $I^\alpha$ is indiscernible over $AI^{\neq\alpha}$.*

(b) $\mathrm{len}(\bar{b}^\alpha_i) = \mathrm{len}(\bar{y}_\alpha)$

(c) *for every $\eta \in {}^\kappa\omega$, the set*

$$\{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_\eta(\alpha) : \alpha < \kappa\}$$

*is consistent with p.*

(d) *for every $\alpha < \kappa$ there exists $k_\alpha < \omega$ such that the set*

$$\{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i) : i < \omega\}$$

*is $k_\alpha$-inconsistent with p.*

(ii) *A (partial) type p over a set A is called* strong *if there does not exist a dividing pattern for p of depth $\kappa = \omega$.*

(iii) *A theory is called strong if every finitary type is strong.*

*Remark* 1.11. Note that by mutual indiscernibility in clause (c) of the definition of a dividing pattern it is enough to demand that the set

$$\{\varphi_\alpha(\bar{x}, \bar{b}_0^\alpha) \colon \alpha < \kappa\}$$

is consistent with $p$.

The reader is encouraged to have a look in [1] for the discussion of strong theories. A theory is strong and dependent if and only is it is strongly dependent (as suggested by the name), and this is mostly the case we are interested in in this article; but there are also strong theories which are simple unstable, and even $SOP_2$.

The following easy Lemma was proven by the authors in [8] in order to establish the connection between randomness and dividing patterns. It is also implicit in some proofs in [1]. We include the proof for completeness.

**Lemma 1.12.** (i) *Let $p(x)$ be a type over a set $A$, let $I = \langle b_i \rangle_{i \in O}$ be a sequence indiscernible over $A$, and let $\varphi(x, y)$ be a formula such that $p(x) \cup \varphi(x, b_i)$ is consistent for some (all) $i$ and $\{\varphi(x, b_i)\}_{i \in O}$ is $k$-inconsistent for some $k \in \mathbb{N}$. Then*

$$p(x) \cup \{\varphi(x, b_l)\} \cup \{\neg\varphi(x, b_i)\}_{i \neq l}$$

*is consistent for all $l$.*

(ii) *Let $p(x)$ be a type over a set $A$, $n < \omega$ and let $\langle b_i^\alpha \colon \alpha < n, i < \omega \rangle$, $\{\varphi_\alpha(x, y_\alpha) \colon \alpha < n\}$ be a dividing pattern for $p$ over $A$ of depth $n$. Then there exists a randomness pattern for $p$ over $A$ of depth $n$; in fact, the randomness pattern is given by the same array and collection of formulae.*

(iii) *Clause* (ii) *holds also when the depth $n < \omega$ is replaced with any cardinal $\kappa$.*

*Proof.* (i) Without loss of generality let us assume that $O = \mathbb{Q}$ and $l = 0$. Assume also that $k$ is *minimal* such that the set $\Delta = \{\varphi(x, b_i) \colon i \in \mathbb{Q}\}$ is $k$-inconsistent. By the assumptions $k > 1$.

By indiscernibility it is enough to show that the set

$$\{\varphi(x, b_0)\} \cup \{\neg\varphi(x, b_i) \colon i \in \mathbb{Z}, i \neq 0\}$$

is consistent. Since $\Delta$ is $(k-1)$-consistent, the set

$$\{\varphi(x, b_0)\} \cup \{\varphi(x, b_{\frac{1}{i+1}}) \colon 1 < i < k\}$$

is consistent, realized by some $d$. But $\Delta$ is $k$-inconsistent, so clearly

$$d \models \{\varphi(x, b_0)\} \cup \{\varphi(x, b_{\frac{1}{i+1}}) \colon 1 < i < k\} \cup \{\neg\varphi(x, b_i) \colon i \in \mathbb{Z}, i \neq 0\}$$

and we are done.

(ii) A very similar proof (working with $\bigwedge_\alpha \varphi_\alpha(x, b_0^\alpha)$ instead of $\varphi(x, b_0)$) is left to the reader.

(iii) By clause (ii) and compactness.

□

1.5. **Morley-Erdös-Rado.** We will make use of the following classical result (originally due to Morley, although often is referred to as "Erdös-Rado argument", since it is an easy consequence of Erdös-Rado theorem and compactness):

**Fact 1.13.** *Let $\lambda$ be a cardinal. Then there exists $\mu > \lambda$ such that for every set $A$ of cardinality $\lambda$ and a sequence of tuples $\langle a_i : i < \mu \rangle$ there exists an $\omega$-type $q(x_0, x_1, \cdots)$ of an $A$-indiscernible sequence such that for every $n < \omega$ there exist $i_1 < i_2 < \ldots < i_n < \mu$ such that the restriction of $q$ to the first $n$ variables equals $\mathrm{tp}(a_{i_1} \ldots a_{i_n}/A)$.*

*We will sometimes denote $\mu$ as above by $\mu(\lambda)$.*

*Remark* 1.14. Let $A$ be a set, $A \subseteq B$, $I$ an $A$-indiscernible sequence. Then there exists $I'$, $I' \equiv_A I$, $I'$ indiscernible over $B$.

*Proof.* First extend $I$ to be long enough so that Fact 1.13 can be applied to it with $\lambda = |B| + |T|$. Then there exists $I'$ indiscernible over $B$ such that every $n$-type of $I'$ over $B$ "appears" in $I$. In particular $I'$ has the same type over $A$ as $I$ (since $I$ was $A$-indiscernible and $A \subseteq B$). ☐

## 2. Þ-orthogonality and Þ-regularity in rosy structures.

The first part of this section is devoted to develop the analogue notions of domination, orthogonality, weight and regularity in the þ-forking context and the properties such notions have under different hypothesis. In the mean time we will show the relation with strong dependence.

Throughout the section we will assume that $T$ is rosy.

**Definition 2.1.** *We define the following.*

- *Two types $p(x)$ and $q(x)$ are* weakly þ-orthogonal *if they are defined over a common domain $B$ and for every tuple $a \models p'$ and $b \models q'$ we have $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} b$. This is denoted by $p \perp^{þ}_{w} q$.*
- *Two types $p$ and $q$ are* þ-orthogonal *if every non-þ-forking extensions $p'$ and $q'$ of $p$ and $q$ respectively to a common domain are weakly þ-orthogonal. This is denoted by $p \perp^{þ} q$.*
- *Let $A$ be a set, $a, b$ tuples. We say that $a$* þ-dominates $b$ over $A$ *if for every $c$ we have $b \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} c$ implies $a \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} c$. In this case we write $b \lhd^{þ}_{A} a$.*
- *We say that $a, b$ are* th-domination equivalent over $A$ *if they dominate each other over $A$. Clearly, this is an equivalence relation. In this case we write $a \bowtie^{þ}_{A} b$.*
- *Let $p(x)$ and $q(x)$ be types over $A$ and $B$ respectively. We will say that $p(x)$* þ-dominates *$q(x)$ if there are $a, b$ realizations of $p$ and $q$ respectively such that $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} B$, $b \mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} A$ and $b \lhd^{þ}_{A \cup B} a$. If $A = B$, we say that $p$ þ-dominates $q$* over $A$.
- *We say that types $p$ and $q$ are* þ-equidominant *if there are non-forking extensions $p', q'$ of $p, q$ respectively to a common domain $C$ and realizations $a' \models p', b' \models q'$ which are domination equivalent over $C$. In this case we write $p \bowtie^{þ} q$.*

*Remark* 2.1. Note that equidominance is not (in general) an equivalence relation on types. Note also that if two types dominate each other, they are not necessarily equidominant (even if the domination is over the same set of parameters $A$), not even in stable theories. The problem is that whereas dominance on elements (over a set $A$) is transitive, dominance on types is generally not. See section 5.2 of [14] for further discussion of this matter and examples.

Now we define þ-pre-weight and þ-weight of a type $p$. We will denote them by $\mathrm{pwt}^{þ}(p)$ and $\mathrm{wt}^{þ}(p)$. Note that Fact 1.2 implies that in stable theories þ-weight coincides with the usual notion of weight.

**Definition 2.2.**

- *Let $p(x)$ be any type over some set $A$. We will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses* $\mathrm{pwt}^{þ}(p(x)) \geq n$ *(*þ-pre-weight of $p$ *is at least $n$) if $a \models p(x)$, $\langle b_i \rangle_{i=1}^n$ is $A$-þ-independent and $a \not\perp_A^{þ} b_i$ for all $i, j$. If $n$ is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses* $\mathrm{pwt}^{þ}(p(x)) = n$ *and that $p$* has þ-pre-weight *$n$.*
- *We say that a type $p$ has* finite *þ-pre-weight if* $\mathrm{pwt}^{þ}(p) < \omega$. *We say that a type $p$ has* rudimentary finite *þ-pre-weight if one can not find an infinite witness $\{b_i : i < \omega\}$ as in (i) above.*
- *Let $p(x)$ be any type over some set $A$. We will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses* $\mathrm{wt}^{þ}(p(x)) \geq n$ *(*þ-weight of *$p$ is at least $n$) if $a \models p(x)$, $a \perp_A^{þ} B$, $\langle b_i \rangle_{i=1}^n$ is $B$-þ-independent and $a \not\perp^{þ} b_i$ for all $i, j$. If $n$ is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i=1^n}$ witnesses* $\mathrm{wt}^{þ}(p(x)) = n$ *and that $p$ has* þ-weight *$n$.*
- *We say that a type $p$ has* finite *þ-weight if* $\mathrm{wt}^{þ}(p) < \omega$. *We say that a type $p$ has* rudimentary finite *þ-weight if every non-þ-forking extension of $p$ has rudimentary finite pre-weight.*

It follows from the definition that $\mathrm{wt}^{þ}(p) \geq n$ if and only if there exists a non-þ-forking extension of $p$ with þ-pre-weight at least $n$.

Notice also that one could define infinite þ-pre-weight and weight as usual, but we will be concerned only with finite þ-weights in this paper.

2.1. **Finite þ-weight and strong dependence.** Let us first make the obvious connections between þ-weight and the notion of *cc*-þ-forking studied in [8].

We recall the definitions. Note that the variable $x$ is a *singleton*.

**Definition 2.3.**

(i) *We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set $A$ witness $n$-*crisscrossed strong-dividing *($n$-cc-strong-dividing) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ strong divides over $A$ and $\bar{a}^i \perp_A^{þ} \langle \bar{a}^j \rangle_{j \neq i}$ for all $i$.*

(ii) *We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set $A$ witness $n$-*crisscrossed þ-dividing *($n$-cc-þ-dividing) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ þ-divides over $A$ and $\bar{a}^i \perp_A^{þ} \langle \bar{a}^j \rangle_{j \neq i}$ for all $i$.*

(iii) *We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set $A$ witness $n$-*crisscrossed þ-forking *($n$-cc-þ-forking) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ þ-forks over $A$ and $\bar{a}^i \perp_A^{þ} \langle \bar{a}^j \rangle_{j \neq i}$ for all $i$.*

(iv) *We say that $T$ admits $n$-cc-þ-forking (or þ-dividing or strong dividing) if there exists a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ witnessing $n$-cc-þ-forking (or þ-dividing or strong forking) over $A$.*

(v) *Let $p$ be a 1-type over a set $A$. We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ witnesses $n - cc$* þ-forking *(or þ-dividing or strong dividing)* in *$p$ if $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$, $A$ witness $n$-cc-þ-forking (or þ-dividing or strong dividing) and the formula $\bigwedge_i \varphi_i(x, \bar{a}^i)$ is consistent with $p$.*

(vi) *We say that a type $p \in S_1(A)$ admits $n$-cc-þ-forking (or þ-dividing or strong dividing) if there exists a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ witnessing $n$-cc-þ-forking (or þ-dividing or strong dividing) in $p$.*

*Remark* 2.2. The following follow straight from the definitions.

(i) $T$ admits $n$-*cc*-þ-forking if and only if there exists a set $A$ and a type $p \in S(A)$ which admits $n$-*cc*-þ-forking.
(ii) Let $T$ be rosy Then a type $p \in S_1(A)$ does not admit $n$-*cc*-þ-forking if and only if it has pre-þ-weight less than $n$.
(iii) So a rosy $T$ does not admit $n$-*cc*-þ-forking if and only if every 1-type has pre-þ-weight less than $n$ if and only if every 1-type has þ-weight less than $n$.

The following is shown in [8]. For completeness we include the proof in the appendix, see A.1.

**Fact 2.3.** *The following are equivalent for any $p \in S_1(A)$.*

(i) *$p$ admits $n$-cc-þ-forking.*
(ii) *$p$ admits $n$-cc-þ-dividing.*
(iii) *There is an extension $p(x, B)$ of $p(x)$ such that $p(x, B)$ admits $n$-cc-strong dividing.*

The main goal of this subsection is to characterize, in rosy theories, strong dependence in terms of the þ-pre-weight. In order to do this, we will need to prove existence of mutually þ-Morley sequences. The procedures will also bring some light as to what is needed in order to characterize strong dependence within dependent theories (or Adler's "strongness" within arbitrary theories) in terms of weight with respect to some independence notion.

**Observation 2.4.** *Let $\{I^i : i < n\}$ be sequences such that $I^i$ is a þ-Morley sequence over $AI^{<i}$ based on $A$. Then $I^i$ is a non-þ-forking sequence over $AI^{\neq i}$ based on $A$.*

*Proof.* We need to prove that $\bar{a}^i_j \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{<j} I^{\neq i}$ where we define $I^i = \langle \bar{a}^i_j : j < \mu_i \rangle$.

By the assumptions, $\bar{a}^i_j \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{<j} I^{<i}$ for all $i, j$. Hence by transitivity and finite character of þ-forking, we have $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{þ}_A I^{\leq i}$ for all $i$, in particular $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{\leq j} I^{<i}$ for all $i, j$. By transitivity again, combining $\bar{a}^i_j \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{<j} I^{<i}$ and $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{\leq j} I^{<i}$, we have $\bar{a}^i_{\leq j} \mathop{\smile\!\!\!\!\!\!|}^{þ}_A I^{\neq i}$.

Therefore, since $\bar{a}^i_j \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{<j}$, we get $\bar{a}^i_j \mathop{\smile\!\!\!\!\!\!|}^{þ}_A \bar{a}^i_{<j} I^{\neq i}$, as required. ☐

**Lemma 2.5.** *Let $\{\bar{a}^i : i < n\}$ be a set of tuples and let $\{I^i : i < n\}$ be sequences such that*

- *For each $i < n$ the sequence $I^i$ is $AI^{<i}a^{>i}$-indiscernible.*
- *$I^i$ starts with $\bar{a}^i$.*

*Then there exist sequences $\{J^i : i < n\}$ such that*

- *For each $i < n$ the sequence $J^i$ is $AJ^{\neq i}$-indiscernible.*
- *$I^i \equiv_{Aa^i} J^i$. So in particular, $J^i$ starts with $\bar{a}^i$.*

*Moreover, if $I^i$ are þ-Morley sequences over $AI^{<i}a^{>i}$ based on $A$, then we can make $J^i$ Morley over $AI^{\neq i}$ based on $A$.*

*Proof.* Exactly the same construction is used to prove both parts of the Lemma. To avoid being repetitive, we will prove the "moreover" part. The proof of the first part is the same, except that without the extra assumptions we cannot get the stronger conclusion. So assume the $I^i$'s is a þ-Morley sequence over $AI^{<i}a^{>i}$ based on $A$.

We need to make sure that $I^i$ can be made indiscernible over $AI^{\neq i}$ and not only over $AI^{<i}a^{>i}$. So assume that $\text{len}(I^i) = \mu_i = \mu(\sum \mu_{<i} + |A| + |T|)$ as in Fact 1.13. We will make our way "backwards", that is, by downward induction on $i$, starting with $i = n$.

Assume that for $\ell > i$ we have $I^\ell$ are þ-Morley $\omega$-sequences over $AI^{\neq \ell}$ based on $A$, whereas for $\ell \leq i$ we still have $I^\ell$ of length $\mu_\ell$ which are þ-Morley sequences over $AI^{<\ell}\bar{a}^{>\ell}$ based on $A$, non-þ-forking over $AI^{\neq \ell}$ (we have the last assumption by Observation 2.4).

By Fact 1.13 we can find $J^i$ which is an indiscernible $\omega$-sequence over $AI^{\neq i}$ such that every $n$-type of $J^i$ over $AI^{\neq i}$ "appears" in $I^i$. So in particular $J^i$ has the same type over $AI^{<i}\bar{a}^{>i}$ as $I^i$. Moreover, since þ-forking has finite character, $J^i$ is non-þ-forking over $AI^{\neq i}$.

Notice that given a finite tuple $\bar{b}$ in $J^i$ the question of whether for some $\bar{\alpha} = \alpha_1 < \ldots < \alpha_k < \omega$ and $\bar{\beta} = \beta_1 < \ldots < \beta_k < \omega$ we have $\bar{a}^\ell_{\bar{\alpha}} \equiv_{\bar{b}I^{\neq \ell, i}} \bar{a}^\ell_{\bar{\beta}}$ amounts to the same question over *some* $\bar{b}'$ in $I^i$. Since these were indiscernible, we get that for any $\ell > i$ the sequence $I^\ell$ is still indiscernible over $AI^{\neq \ell, i}J^i$. Using a similar argument one can also make sure that for $\ell \neq i$ we still have that $I^\ell$ is a non-þ-forking sequence over $AI^{\neq \ell, i}J^i$.

So the $J^i$ satisfies all the requirements, except for the fact that we need the first element of it to be $\bar{a}^i$. Note, though, that the first element of $J^i$ has the same type over $I^{<i}\bar{a}^{>i}$ as $\bar{a}^i$. So applying an automorphism over $I^{<i}\bar{a}^{>i}$, we obtain a new $J^i$ that starts with $\bar{a}^i$ and new $I^\ell$ for $\ell > i$ which satisfy all the required properties, completing the proof of the inductive step.

□

**Lemma 2.6.** *Let $\{\bar{a}^i : i < n\}$ be a set of tuples which is þ-independent over a set $A$. Then there exist sequences $\{I^i : i < n\}$ such that*

- *For each $i < n$ the sequence $I^i$ is a þ-Morley sequence over $AI^{\neq i}$ based on $A$. So $I^i$ is $AI^{\neq i}$-indiscernible and $I^i \mathop{\smile\!\!\!\!\!\mid}^{þ}_A I^{\neq i}$.*
- *$I^i$ starts with $\bar{a}^i$.*

*Proof.* We construct sequences $I^i$ such that $I^i$ is a þ-Morley sequence over $AI^{<i}\bar{a}^{>i}$ based on $A$. By Lemma 2.5, this is enough in order to obtain the desired conclusion. The construction is by induction on $i < n$.

The case $i = 0$ follows from Fact 1.5.

So let $i > 0$, and assume that $I^{<i}$ already exist. Note that $I^0 \mathop{\smile\!\!\!\!\!\mid}^{þ}_A \bar{a}^{>0}$, $I^1 \mathop{\smile\!\!\!\!\!\mid}^{þ}_A I^0\bar{a}^{>1}$ hence $I^0 I^1 \mathop{\smile\!\!\!\!\!\mid}^{þ}_A a^{>1}$. Continuing like this we see that $I^{<i} \mathop{\smile\!\!\!\!\!\mid}^{þ}_A \bar{a}^{\geq i}$. By symmetry and transitivity $\bar{a}^i \mathop{\smile\!\!\!\!\!\mid}^{þ}_A I^{<i}\bar{a}^{>i}$, and we can apply Fact 1.5 again.

□

We are now able to prove that strong dependence implies boundedness (by $\omega$) of *cc*-strong dividing patterns and of þ-weight.

**Proposition 2.7.** *If a type $p$ admits a $n$-cc-strong-dividing witness then $dp - rk(p) \geq n$.*

*Proof.* Let $\langle \psi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set $A$ witness $n$-*cc*-strong dividing, that is, $\bigwedge_i \psi_i(x, \bar{a}^i)$ is consistent with $p$, $\psi_i(x, \bar{a}^i)$ strongly divides over $A$ and $\bar{a}^i \mathop{\smile\!\!\!\!\!\mid}^{þ}_A \langle \bar{a}^j \rangle_{j \neq i}$ for all $i$.

By the definition of strong dividing $\bar{a}^i \notin \operatorname{acl}(A)$. Since $\{\bar{a}^i : i < n\}$ is þ-independent, we can build as in Lemma 2.6 sequences $I^i = \langle \bar{a}^i_j : j < \omega \rangle$ such that:

- $I^i$ is a þ-Morley sequence over $AI^{\neq i}$ based on $A$.
- $\bar{a}^i_0 = \bar{a}^i$.

For each $i < n$ and $k < \omega$ denote $\psi^k_i(x) = \psi^k_i(x, \bar{a}^i_{<k}) = \bigwedge_{j<k} \psi_i(x, \bar{a}^i_j)$. Note that since $\psi_i(x, \bar{a}^i_0)$ strongly divides over $A$, for some $k < \omega$ the formula $\psi^k_i(x)$ is inconsistent.

So we clearly have a þ-dividing pattern (see Definition 1.7) for $p$ of depth $n$; applying Lemma 1.12(ii), we are done.

□

**Theorem 2.8.** *If $T$ is strongly dependent (and rosy) then every (finitary) type has rudimentarily finite þ-weight. If $T$ is dp-minimal then every 1-type has þ-weight 1.*

*Moreover, the conclusion is true if we just assume that $T$ is strong and rosy.*

*Proof.* This now follows easily from Remark 2.2, Fact 2.3 and Proposition 2.7.

For the "moreover" part note that Lemma 2.6 only assumes rosiness, and in Proposition 2.7 we show, in fact, existence of a dividing pattern. □

The rest of the section will be devoted to show the equivalence between "rudimentarily finite" þ-weight and "finite þ-weight".

2.2. **Basic properties.** Here we list the basic properties of þ-weight. Some of the results and the proofs in this subsection are very similar, and sometimes completely analogous to the results in simple theories (see section 5.2 of [14]).

**Lemma 2.9.** *The following hold.*

(i) *If $a \mathop{\smile\!\!\!\!\!|}^{þ}_A b$ then $\operatorname{wt}^{þ}(a/A) = \operatorname{wt}^{þ}(a/Ab)$.*

(ii) $\operatorname{wt}^{þ}(ab/A) \leq \operatorname{wt}^{þ}(a/A) + \operatorname{wt}^{þ}(b/A)$. *Equality holds whenever $a \mathop{\smile\!\!\!\!\!|}^{þ}_A b$.*

*Proof.* The proofs are the same as proofs of Lemmas 5.2.3 and 5.2.4 in [14], replacing instances of forking for þ-forking. □

The following is very easy:

**Observation 2.10.** *Suppose that $a$ is þ-dominated by $b$ over a set $A$, and $A' \supseteq A$ is such that $a \mathop{\smile\!\!\!\!\!|}^{þ}_A A', b \mathop{\smile\!\!\!\!\!|}^{þ}_A A'$. Then $a$ is þ-dominated by $b$ over $A'$.*

**Observation 2.11.** *Suppose $a \lhd^{þ}_A b$. Then* $\operatorname{wt}^{þ}(a/A) \leq \operatorname{wt}^{þ}(b/A)$.

*Proof.* Assume that $\operatorname{wt}^{þ}(a/A) \geq n$. Then there are $A', \{c_i : i < n\}$ witnessing this; that is, $a \mathop{\smile\!\!\!\!\!|}^{þ}_A A'$, $\{c_i : i < n\}$ is an $A'$-þ-independent set, and $a \not\mathop{\smile\!\!\!\!\!|}^{þ}_A c_i$ for all $i$. Let $b' \equiv_{Aa} b$ be such that $b' \mathop{\smile\!\!\!\!\!|}^{þ}_{Aa} A'$. So $ab' \mathop{\smile\!\!\!\!\!|}^{þ}_A A'$, hence by the previous Observation $a$ is dominated by $b'$ over $A'$. So $b' \mathop{\smile\!\!\!\!\!|}^{þ}_{A'} c_i$ for all $i$. In particular, $\operatorname{wt}^{þ}(b/A) \geq \operatorname{wt}^{þ}(b'/A') \geq n$, as required. □

**Observation 2.12.**

- *If $p, q \in S(A)$ are not þ-weakly orthogonal and $\operatorname{pwt}^{þ}(q) = 1$, then $p$ dominates $q$ over $A$.*
- *The relation $p \not\perp_A q$ is an equivalence relation on types over $A$ of þ-pre-weight 1.*

*Proof.* Easy (see 5.2.11 and 5.2.12 in [14]). □

The following two lemmas are easy but very useful.

**Lemma 2.13.** *Assume $b \triangleleft^þ_A a$. Then there exists $B$ containing $A$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ (hence $b \mathop{\smile\!\!\!\!\!\!\mid}_A B$) such that $ab \triangleleft^þ_B a$.*

*Proof.* We try to choose by induction on $\alpha < |T|^+$ an increasing and continuous sequence of sets $A_\alpha$ such that $A_0 = A$ and for all $\alpha$ we have:

- $ab \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_{A_\alpha} A_{\alpha+1}$
- $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A_\alpha} A_{\alpha+1}$ (hence $b \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A_\alpha} A_{\alpha+1}$)

By local character of þ-independence, there is $\alpha < |T|^+$ such that it is impossible to choose $A_{\alpha+1}$. Denote $B = A_\alpha$. It is easy to see that the all the requirements are satisfied. □

**Lemma 2.14.** *Assume that $ab \triangleleft^þ_A a$, $a \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_A c$, $b \mathop{\smile\!\!\!\!\!\!\mid}^þ_A c$ and $\mathrm{wt}^þ(\mathrm{tp}(c/A)) = 1$. Then $bc \triangleleft^þ_A a$.*

*Proof.* Assume $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} d$. Since $ab \triangleleft^þ_A a$, we have $ab \mathop{\smile\!\!\!\!\!\!\mid}^þ_A d$, hence $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{Ab} d$. Let $A' = Ab$. Then $c \mathop{\smile\!\!\!\!\!\!\mid}^þ_A A'$ and $c \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} a$ (otherwise, by transitivity $c \mathop{\smile\!\!\!\!\!\!\mid}^þ_A ab$). Since $\mathrm{wt}^þ(c/A) = 1$, clearly $c \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} d$ (otherwise, remembering that $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} d$, we would get that $a, d$ witness $\mathrm{pwt}^þ(c/A') \geq 2$). Hence $bc \mathop{\smile\!\!\!\!\!\!\mid}^þ_A d$, as required. □

**Observation 2.15.** *Let $p \in \mathrm{S}(B)$ and $a, B, b_1, \ldots, b_n$ witness $\mathrm{pwt}^þ(p) = n$ Then $a \triangleleft^þ_B b_1 \ldots b_n$.*

*Proof.* Assume $c \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_B a$ and $c \mathop{\smile\!\!\!\!\!\!\mid}^þ_B b_1 \ldots b_n$. Then the set $\{c, b_1, \ldots, b_n\}$ is $B$-þ-independent, and it witnesses $\mathrm{pwt}^þ(a/B) \geq n+1$, a contradiction. □

2.3. **From rudimentarily finite to finite.** We will now prove that if a type has rudimentarily finite þ-weight, it has finite þ-weight. As with stable theories, in order to show this we found it necessary to prove the very interesting fact that a type of (rudimentarily) finite þ-weight is þ-equidominant with a finite free product of þ-weight 1 types.

A good start would be showing that every type of rudimentarily finite weight is "related" (in terms of non-þ-orthogonality) to þ-weight-1 types. The following two lemmas generalize Hyttinen's results from [4] on types in a stable theory, and we adapt his technique to the rosy context.

**Lemma 2.16.** *Let $p \in \mathrm{S}(A)$, and assume that*

(i) *$a, A', \{b_1, \ldots, b_n\}$ witness $\mathrm{wt}^þ(p) \geq n$. That is, $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_A A'$, $\{b_1, \ldots, b_n\}$ are þ-independent over $A'$ and $a \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} b_i$ for all $i$.*

(ii) *There is no $C$ extending $A'$ such that the following three conditions hold:*
   (a) *$a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} C$*
   (b) *$b_1 b_2 \ldots b_{n-1} \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} C b_n$*
   (c) *$b_n \not\mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} C$.*

*Then*

(1) *Whenever $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} c$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A' b_n} c$, we have $b_n \mathop{\smile\!\!\!\!\!\!\mid}^þ_{A'} c$.*

(2) *If, furthermore,* $\mathrm{wt}^{\text{þ}}(\mathrm{tp}(b_n/A')) > 1$*, then there are* $B$ *and* $b'_n, b'_{n+1}$ *such that* $a, B, \{b_1, \dots, b_{n-1}, b'_n, b'_{n+1}\}$ *witness* $\mathrm{wt}^{\text{þ}}(p) \geq n+1$.

*Proof.*

(1) Assume $b_n \not\perp^{\text{þ}}_{A'} c$ but $a \perp^{\text{þ}}_{A'} c$ and $a \perp^{\text{þ}}_{A'b_n} c$. Without loss of generality $c \perp^{\text{þ}}_{A'b_n a} b_1 \dots b_{n-1}$, hence $c \perp^{\text{þ}}_{A'b_n} b_1 \dots b_{n-1}$. Let $C = A'c$. It is easy to see that (a),(b),(c) above hold for $C$ (e.g. (b) holds by symmetry and transitivity), contradicting assumption (ii) of the Lemma.

(2) Assume $\mathrm{wt}^{\text{þ}}(\mathrm{tp}(b_n/A')) > 1$. This means that there are $B \supseteq A'$ and $c, d$ such that
   - $b_n \perp^{\text{þ}}_{A'} B$
   - $c \perp^{\text{þ}}_{B} d$
   - $b_n \not\perp^{\text{þ}}_{B} c$ and $b_n \not\perp^{\text{þ}}_{B} d$

   Without loss of generality $ab_1 \dots b_{n-1} \perp^{\text{þ}}_{A'b_n} Bcd$. It is easy to see that the assumptions of the Lemma still hold after replacing $A'$ with $B$. So part (1) holds as well. In particular, since $b_n \not\perp^{\text{þ}}_{B} c$ and $b_n \not\perp^{\text{þ}}_{B} d$, whereas $a \perp^{\text{þ}}_{Bb_n} c$ and $a \perp^{\text{þ}}_{Bb_n} d$, so $a \not\perp^{\text{þ}}_{B} c$ and $a \not\perp^{\text{þ}}_{B} d$. Choosing $b'_n = c, b'_{n+1} = d$, we are done.

□

**Lemma 2.17.** *Let* $p \in S(A)$ *be a type of rudimentarily finite þ-weight. Then* $p$ *is non-þ-orthogonal to a type of þ-weight 1.*

*Moreover, suppose that* $a \models p, B = \{b_i : i < m\}, d$ *are such that* $a, A, \{b_i : i < m\} \cup \{d\}$ *witness* $\mathrm{wt}^{\text{þ}}(p) \geq m+1$. *Then there exist* $D \supseteq A$ *and* $d'$ *such that*

- $\mathrm{wt}^{\text{þ}}(d'/A') = 1$
- $a, D, \{b_i : i < m\} \cup \{d'\}$ *witness* $\mathrm{wt}^{\text{þ}}(p) \geq m+1$.

*Proof.* By considering a non-þ-forking extension it is clear that the Lemma follows from the "moreover" part.

We will prove that if the conclusion fails we can witness that $p$ has rudimentary infinite þ-weight, thus contradicting the hypothesis of the Lemma.

Assume towards a contradiction that the conclusion fails and construct by induction on $n \geq m$ sets $A_n$ , $B_n$ and tuples $d_n$ such that

- $B_n = \{b_i : i < n\}$, so $|B_n| = n$
- $A_m = A$, $B_m = B$, $d_m = d$
- The sequences $\langle A_n : n < \omega\rangle$ and $\langle B_n : n < \omega\rangle$ are increasing
- $a, A_n, B_n = \{b_i : i < n\} \cup \{d_n\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$.

The case $n = m$ is given, so suppose we have $A_n$, $B_n$ and $d_n$ as above.

By local character of þ-independence, we can replace $A$ by $A'$ satisfying the assumptions of Lemma 2.16 with $b_n$ there replaced by our $d$: if given some $A'$ there exists a $C$ as in (ii) of Lemma 2.16 above, it satisfies all the requirements of $A'$ in (i), so we can replace $A'$ with $C$ and continue; local character of þ-forking and the fact that $d \not\perp^{\text{þ}}_{A'} C$ guaranties that the process will eventually stop. So by Lemma 2.16 (and the assumption towards contradiction), we can "split" $d$ into two elements $b_n$ and $d_{n+1}$, that is, find $A_{n+1}, b_n, d_{n+1}$ such that $a, A_{n+1}, \{b_i : i < n\} \cup \{d_{n+1}\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$, as required.

Let $A_\omega = \bigcup_{n<\omega} A_n$, $B_\omega = \bigcup_{n<\omega} B_n$. Clearly, $B_\omega$ is an *infinite witness* for $\mathrm{wt}^{\text{þ}}(p) \geq \aleph_0$, contradicting $p$ having rudimentarily finite weight.

Since this construction contradicts our hypothesis, we know that for some $n$ we have $\mathrm{wt}^{\text{þ}}(d_n/A_n) = 1$. But then $D = A_n, d' = d_n$ satisfy the conditions required in the conclusion of the Lemma. □

We are finally ready to prove that a type of rudimentarily finite þ-weight has finite þ-weight. The proof will be based on Observation 2.18, but first we make the following (temporary) definition.

**Definition 2.4.** *Let $p = \mathrm{tp}(a/A)$ be any type.*

*We will say that a witness $a, A, \{b_i : i < m\}$ is a* nice witness *of $\mathrm{wt}^{\text{þ}}(p) \geq m$ if $ab_0 \dots b_{m-1} \lhd_A a$ and $\mathrm{wt}^{þ}(b_i/A) = 1$ for all $i$.*

*We will say that a witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{þ}(p) \geq m$ to $\mathrm{wt}^{þ}(p) > m$* is contained *in a witness $a, A', \{b_i : i < n\}$ of $\mathrm{wt}^{þ}(p) \geq n$ if $A \subset A'$ and $m \leq n$. We say that the first witness if* properly contained *in the second one if $m < n$.*

*We will say that a (nice) witness is* maximal *if it is not properly contained in any other (nice) witness.*

**Observation 2.18.** *Let $p = \mathrm{tp}(a/A)$ be a type of rudimentarily finite weight.*

*Then every witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{þ}(p) \geq m$ is contained in a* maximal *witness $a, A', \{b_i : i < n\}$.*

*Even more, every nice witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{þ}(p) \geq m$ is contained in a witness $a, A', \{b_i : i < n\}$ to $\mathrm{wt}^{þ}(p) \geq n$ which is maximal among all nice witnesses.*

*Proof.* The proof is precisely the same as the proof of Lemma 2.17 above:

If there is no maximal witness, then we can construct by induction on $n < \omega$ increasing witnesses $A_n, B_n = \{B_i : i < n\}$; taking the unions of these sets, get a contradiction. □

Notice that, a priori, this does not mean, that every such maximal witness has the same size, or that there are no different such witnesses of finite unbounded cardinalities so that the þ-weight of $p$ could still be infinite.

The proof of the following lemma shows that the size of any nice maximal witness (in particular with $\mathrm{wt}^{\text{þ}}(b_i) = 1$) is the same finite number $n$, which must *a posteriori* be equal to $\mathrm{wt}^{\text{þ}}(p)$; that every type of rudimentary finite weight has finite weight follows as an easy corollary.

**Lemma 2.19.** *Let $p$ be a type of rudimentarily finite þ-weight. Then any maximal nice witness $a, A', \{b_i : i < m\}$ of $\mathrm{wt}^{þ}(p) \geq m$ satisfies $a \bowtie^{þ}_{A'} b_0 \dots b_{n-1}$.*

*Proof.* Let $a, A'$ and $b_0 \dots b_{n-1}$ be as in the statement of the lemma. It is clearly enough to make sure that $a \lhd^{\text{þ}}_{A'} b_0 \dots b_{n-1}$.

So suppose $a \not\perp^{\text{þ}}_{A'} c$ but $b_0 \dots b_{n-1} \perp^{\text{þ}}_{A'} c$. Then by definition $a, A', B \cup \{c\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$. By Lemma 2.17 there are $D, c'$ such that $a, D, B \cup \{c'\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$ and $\mathrm{wt}^{\text{þ}}(c'/D) = 1$. By Lemma 2.14, $b_0 \dots b_{n-1} c' \lhd^{\text{þ}}_{A'} a$. By Lemma 2.13 we may assume $ab_0 \dots b_{n-1} c' \lhd^{\text{þ}}_{A'} a$. So $a, A', b_0 \dots b_{n-1} c'$ is a nice witness of $\mathrm{wt}^{\text{þ}}(p) \geq n+1$, contradicting the maximality of $a, A', b_0 \dots b_{n-1}$. □

The following easy observation shows that nice witnesses exist.

**Observation 2.20.** *Let $p = \mathrm{tp}(a/A)$ a nonalgebraic type of rudimentarily finite weight. Then there exists a nice witness of* $\mathrm{wt}^{þ}(p) \geq 1$.

*Proof.* By Lemma 2.17 we can find $b$, $\mathrm{wt}^{þ}(b/A') = 1$ where $A'$ is the domain over which $a \not\downarrow^{þ}_{A'} b$. Since $\mathrm{wt}^{þ}(b/A') = 1$ and $a \not\downarrow^{þ}_{A'} b$, Observation 2.12 implies that $b \lhd^{þ}_{A} a$. Finally, we can assume $ab \lhd^{þ}_{A'} a$ by Lemma 2.13, which finishes the proof. □

We have finally reached our goal.

**Theorem 2.21.** *Let $p \in S(A)$ be a nonalgebraic type of rudimentarily finite þ-weight. Then* $\mathrm{wt}^{þ}(p) < \aleph_0$ *and $p$ is þ-equidominant with a finite free product of þ-weight-1 types.*

*More precisely, there exist $a, A', \{b_i : i < n\}$ such that*

- $a, A', \{b_i : i < n\}$ *witness that* $\mathrm{wt}^{þ}(p) \geq n$
- $\mathrm{wt}^{þ}(b_i/A') = 1$ *for all $i$*
- $a \bowtie^{þ}_{A'} b_0 \ldots b_{n-1}$.

*Proof.* Let $a, A'$, $B = \{b_i : i < n\}$ be such that

(i) $a, A', \{b_i : i < n\}$ witness that $\mathrm{wt}^{þ}(p) \geq n$
(ii) $\mathrm{wt}^{þ}(b_i/A') = 1$ for all $i$
(iii) $aB \lhd^{þ}_{A'} a$
(iv) $\{b_i : i < n\}$ is maximal satisfying (i),(ii) and (iii). In other words, if there are $A'' \supseteq A'$, $B'' \supseteq B$ satisfying (i), (ii) and (iii), then $B'' = B$.

In other words, $a, A', B$ is a maximal nice witness for $\mathrm{wt}^{þ}(p) \geq n$. It is easy to see that such $A', B$ exist: Observation 2.20 gives us a nonempty $B_0$ satisfying (i),(ii) and (iii). Since $p$ has rudimentary finite weight, by Observation 2.18 we know that $B_0$ is contained in a maximal $B$, as required in (i) – (iv) above.

By Lemma 2.19, $a \bowtie^{þ}_{A'} b_0 \ldots b_{n-1}$. By Lemma 2.9 and Observation 2.11 it follows that $p$ has finite weight $n$. □

Reading carefully the proof of the Theorem, we obtain the following more precise statement.

**Corollary 2.22.** *Let $p$ be a type of rudimentarily finite þ-weight. Then* $\mathrm{wt}^{þ}(p) = n$ *for some $n < \omega$, and any maximal nice witness $a, A', \{b_i : i < m\}$ of* $\mathrm{wt}^{þ}(p) \geq m$ *satisfies $m = n$ and* $a \bowtie^{þ}_{A'} b_0 \ldots b_{n-1}$.

**Corollary 2.23.** *In a strongly dependent (and even strong) rosy theory, every type has finite þ-weight.*

*Proof.* By Theorem 2.8 and Theorem 2.21. □

2.4. **þ-regular types.** We will finish this section by understanding some implications of the above results to þ-regular types. The definition is the analogue of the definition of regular types in the stable and simple context.

**Definition 2.5.** *A type $r(x)$ over $A$ is* þ-regular *if for any $B \supset A$ then given any þ-forking extension $q(x)$ of $r(x)$ and a non-þ-forking extension $p(x)$ of $r(x)$ if $r, q$ are over $B$ then $q(x)$ is weakly þ-orthogonal to $r(x)$.*

The following desired property of þ-regular types follows as an easy corollary of the definition of þ-regularity and the results we have so far in this section.

**Corollary 2.24.** *A þ-regular type of finite þ-weight has þ-weight 1.*

*Proof.* Suppose not, and let $p \in S(A)$ be a þ-regular type of þ-weight at least 2. Without loss of generality (since a non-þ-forking extension of a þ-regular type is þ-regular), there exists a witness $a, \{b_1, b_2\}$ for $\mathrm{pwt}^{þ}(p) \geq 2$. Moreover, by Lemma 2.17 we may assume that $\mathrm{wt}^{þ}(\mathrm{tp}(b_1/A)) = \mathrm{wt}^{þ}(\mathrm{tp}(b_2/A)) = 1$.

Let $a'$ be such that $\mathrm{tp}(a'/Ab_1) = \mathrm{tp}(a/Ab_1)$, $a' \mathop{\smile\!\!\!\!\!\!|}^{þ}_{Ab_1} b_2$. Then clearly $a' \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} b_2$ (as $b_1, b_2$ are independent over $A$).

Now notice:

- $a \mathop{\smile\!\!\!\!\!\!|}^{þ}_{Ab_2} a'$: The type $p$ is þ-regular, so $\mathrm{tp}(a/Ab_2)$ and $\mathrm{tp}(a'/Ab_2)$ are weakly þ-orthogonal.
- $b_1 \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{Ab_2} a$: We know $b_1 \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} b_2$ and $b_1 \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} a$.
- $b_1 \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{Ab_2} a'$: This follows from $a' \equiv_{Ab_1} a$ (so $a' \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} b_1$), and $b_1 \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} b_2$.

So we have a witness for $\mathrm{wt}^{þ}(\mathrm{tp}(b_1/Ab_2) \geq 2$, but this type is a non-þ-forking extension of $\mathrm{tp}(b_1/A)$, a contradiction. ☐

We will conclude by pointing out the following unsurprising but important property of a regular type:

**Observation 2.25.** *Let $p \in S(A)$ be a þ-regular type. Define (as usual) for a tuple $\bar{c}$ of realizations of $p$*

$$\mathrm{cl}_p(\bar{c}) = \{a \models p \colon a \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} \bar{c}\}$$

*Then $(p^{\mathfrak{C}}, \mathrm{cl}_p)$ is a pregeometry.*

*Proof.* The proof is quite easy and it is the same as the standard proof of the analogue result for (forking) regular types. ☐

*Remark* 2.26. We should mention that the converse of Observation 2.25 is true assuming stability of $p(x)$ (see [6] for a definition). In the general –rosy– context, however, we have been unable to either prove it or show a counterexample.

## 3. Super-rosy theories and types of finite $\mathrm{U}^{þ}$-rank

3.1. **Exchange and decomposition in types of finite weight.** The goal of this section is proving that under reasonable assumptions, any type can be "decomposed" into a finite product of "geometric" types. Recall that in Theorem 2.21 we in particular proved the following.

**Theorem 3.1.** *Let $p \in S(A)$ be such that $\mathrm{wt}^{þ}(p) = n$. Then there exists a set $B$, $A \subseteq B$, and $b_1, \ldots, b_n$ þ-independent over $B$ such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $\mathrm{wt}^{þ}(b_i/B) = 1$.*

We will improve this statement by replacing þ-weight-1 types in the conclusion by regular (in the super-rosy context) and þ-minimal (in the finite rank context) types.

**Lemma 3.2** (Exchange Lemma). *Let $a, b_1, \dots, b_n$ be a þ-weight 1 witness of* $\mathrm{wt}^þ(\mathrm{tp}(a/A)) = n$. *Let $q$ be a type with* $\mathrm{dom}(q) \supset A$ *such that $q$ is not þ-orthogonal to $tp(b_n/A)$ and* $\mathrm{wt}^þ(q) = 1$. *Then there is some $b \models q$ and some $B$ such that $a, B, \langle b_1, \dots, b_{n-1}, b\rangle$ witness $tp(a/A)$ has þ-weight $n$.*

*Moreover, if $a \bowtie^þ_A b_1 \dots b_n$, then we can find $B$ such that both $a \bowtie^þ_B b_1 \dots b_{n-1} b_n$ and $a \bowtie^þ_B b_1 \dots b_{n-1} b$.*

*Proof.* Let $b' \models q$, $B'$ be such that $b_n \mathop{\smile\!\!\!\!\!|}^þ_A B'$, $b' \mathop{\smile\!\!\!\!\!|}^þ_A B'$ and $b' \not\mathop{\smile\!\!\!\!\!|}^þ_{B'} b_n$ (such $b'$ and $B'$ exist as $tp(b_n/A)$ and $q$ are not þ-orthogonal).

Without loss of generality $Bb \mathop{\smile\!\!\!\!\!|}^þ_{Ab_n} ab_1 \dots b_{n-1}$. In particular, $ab_1 \dots b_n \mathop{\smile\!\!\!\!\!|}^þ_A B$ and $b_1 \dots b_{n-1} \mathop{\smile\!\!\!\!\!|}^þ_B b_n b$, and so the set $\{b, b_1, \dots, b_{n-1}\}$ is independent over $B$.

Now if $a \mathop{\smile\!\!\!\!\!|}^þ_B b$, then $a, b$ witness $\mathrm{wt}^þ(tp(b_n/B)) \geq 2$ which contradicts our assumptions (via Lemma 2.9). So $a \not\mathop{\smile\!\!\!\!\!|}^þ_B b$ and by the definition $a, B, \langle b_1, \dots, b_{n-1}, b\rangle$ witnesses $tp(a/A)$ has þ-weight $n$.

For the "moreover" part, assume that $a \bowtie^þ_A b_1 \dots b_n$. Recall that by Observations 2.13 and 2.10 we may assume $ab_1 \dots b_n \lhd^þ_A a$ (that is, first replace $A$ with some $A'$ such that $a \mathop{\smile\!\!\!\!\!|}^þ_A A'$, $b_1 \dots b_n \mathop{\smile\!\!\!\!\!|}^þ_A A'$, and $ab_1 \dots b_n \lhd^þ_{A'} a$, and then find $B$), hence (by 2.10 again) $ab_1 \dots b_n \lhd^þ_B a$. By Lemma 2.14, $b_1 \dots b_{n-1} b \lhd^þ_B a$. Finally by Observation 2.15 we have $b_1 \dots b_{n-1} b_n \bowtie^þ_B a$ and $b_1 \dots b_{n-1} b \bowtie^þ_B a$, as required. □

3.2. **þ-regularity and decomposition in the super-rosy case.** As in the super-stable case, we first prove the existence of "many" þ-regular types in a super-rosy theory, which makes the theory of þ-regular types relevant. We will also point out that all super-rosy types in a rosy theory have finite þ-weight (hence the results of the previous section apply in the super-rosy context).

**Proposition 3.3.** *Let $T$ be super-rosy. Then every type $p$ with domain $A$ is non-þ-orthogonal to a þ-regular type $q$ with domain $B \supset A$.*

*Proof.* The proof is a variation of the proof of Proposition 5.1.11 in [14].

Let $\mathcal{P}$ be the set of types $r$ such that $dom(r) = B \supset A$ and $r$ is not weakly þ-orthogonal to $p$ and let $q$ be a type in $\mathcal{P}$ of minimal $\mathrm{U}^þ$-rank. Let $a', b$ be realizations of $p, q$ respectively such that $a' \mathop{\smile\!\!\!\!\!|}^þ_A B$ and $a' \not\mathop{\smile\!\!\!\!\!|}^þ_B b$.

Suppose $q$ is not þ-regular so there is some $c', b', C'$ such that $c', b' \models q$, $b' \mathop{\smile\!\!\!\!\!|}^þ_B C'$, $c' \not\mathop{\smile\!\!\!\!\!|}^þ_B C'$ and $b' \not\mathop{\smile\!\!\!\!\!|}^þ_{C'} c'$.

Since $tp(b'/B) = tp(b/B) = q$ there is an automorphism fixing $B$ and sending $c', b', C'$ to elements $c, b, C$ and let $a \models p$ realize a non-þ-forking extension of $tp(a'/bB)$ to $bBC$. So $a \mathop{\smile\!\!\!\!\!|}^þ_{bB} C$ and, since $b \mathop{\smile\!\!\!\!\!|}^þ_B C$, we have by transitivity that $ab \mathop{\smile\!\!\!\!\!|}^þ_B C$ which implies that $a \mathop{\smile\!\!\!\!\!|}^þ_B C$; it follows that $a \mathop{\smile\!\!\!\!\!|}^þ_A C$ (recall that $a' \mathop{\smile\!\!\!\!\!|}^þ_A B$ and $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$).

Notice also that $a \not\mathop{\smile\!\!\!\!\!|}^þ_C b$ (as $a \not\mathop{\smile\!\!\!\!\!|}^þ_B b$ and $a \mathop{\smile\!\!\!\!\!|}^þ_B C$).

So we have $a \mathop{\smile\!\!\!\!\!|}^þ_A C$, $a \not\mathop{\smile\!\!\!\!\!|}^þ_C b$, $b \mathop{\smile\!\!\!\!\!|}^þ_B C$, $c \not\mathop{\smile\!\!\!\!\!|}^þ_B C$ and $b \not\mathop{\smile\!\!\!\!\!|}^þ_C c$. In particular

$$\mathrm{U}^þ(tp(c/C)) < \mathrm{U}^þ(tp(c/B)) = \mathrm{U}^þ(tp(b/B))$$

and

$$\mathrm{U}^{\text{þ}}(tp(b/Cc)) < \mathrm{U}^{\text{þ}}(tp(b/C)) = \mathrm{U}^{\text{þ}}(tp(b/B));$$

by minimality of $\mathrm{U}^{\text{þ}}(tp(b/B))$ (among all types in $\mathcal{P}$) we have that $tp(c/C)$ and $tp(b/Cc)$ are not in $\mathcal{P}$; so in particular $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{C} c$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Cc} b$. By transitivity $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{C} bc$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{C} b$ a contradiction. ☐

**Proposition 3.4.** *Let $p(x)$ be a type such that*

$$U^{þ}(p) = \sum_{i=1}^{k} \omega^{\alpha_i} n_i.$$

*Then $p$ has þ-weight at most $\sum_{i=1}^{k} n_i$.*

*Proof.* This is word by word the same proof as Theorem 5.2.5 in [14] using the þ-forking version of Lascar's inequalities (Fact 1.3). ☐

As an easy corollary we obtain the following theorem which strengthens Theorem 3.1 in the super-rosy context.

**Theorem 3.5.** *The following hold.*

- *Any super-rosy type has finite þ-weight.*
- *Let $T$ be super-rosy, $p \in \mathrm{S}(A)$. Then $\mathrm{wt}^{þ}(p) = n$ for some $n < \omega$ and there exists a set $B$, $A \subseteq B$, and $b_1, \ldots, b_n$ þ-independent over $B$ such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $\mathrm{tp}(b_i/B)$ are þ-regular.*

*Proof.* The first item follows immediately from Proposition 3.4.

To prove the second item, notice first that $\mathrm{wt}^{\text{þ}}(p)$ is finite by Proposition 3.4. Now apply Theorem 3.1 combined with existence of þ-regular types (Proposition 3.3) and the Exchange Lemma (Lemma 3.2), recalling that by Corollary 2.24 þ-regular types have þ-weight 1. ☐

## 3.3. Types of finite $\mathrm{U}^{\text{þ}}$-rank.

The following proposition is a remarkably interesting result with many consequences in theories of finite $\mathrm{U}^{\text{þ}}$-rank.

**Proposition 3.6.** *The following hold.*

- *Let $p(x) = tp(b/A)$ be any type such that $U^{þ}(p) = \alpha + 1$. Then there are tuples $a, e$ such that $U^{þ}(tp(b/Aa)) = \alpha$, $b \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} e$, $b \mathop{\smile\!\!\!\!\!\mid}^{þ}_{Aa} e$, $tp(b/Aa)$ strongly divides over $Ae$ and $U^{þ}(tp(a/Ae)) = 1$.*
- *If $p(x) = tp(b/A)$ is any type of þ-rank $\alpha + 1$ then there is a non-þ-forking extension $tp(b/Ae)$ of $p$ and a tuple $a \in \mathrm{acl}(Abe)$ such that $tp(a/Ae)$ is minimal.*

*Proof.* The second item follows immediately from the first one. To prove the first item, notice that we can choose $a$ so that $p(x, a)$ is a þ-dividing extension of $p(x)$ and $\mathrm{U}^{\text{þ}}(tp(b/Aa)) = \alpha$. By Observation 1.4(ii) there is some $e$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Aa} e$ and $tp(b/Aae)$ strongly divides over $Ae$ so $a \in acl(Abe)$. Note that

$$\alpha = \mathrm{U}^{\text{þ}}(tp(b/Aa)) = \mathrm{U}^{\text{þ}}(tp(b/Aae)) < \mathrm{U}^{\text{þ}}(tp(b/Ae)) \le \mathrm{U}^{\text{þ}}(tp(b/A)) = \alpha + 1$$

hence $\mathrm{U}^{\text{þ}}(tp(b/Ae)) = \mathrm{U}^{\text{þ}}(tp(b/A)) = \alpha + 1$; in particular, $b \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A} e$. By Lascar's inequalities we know that

$$\mathrm{U}^{\text{þ}}(tp(ba/Ae)) = \mathrm{U}^{\text{þ}}(tp(b/Ae)) + \mathrm{U}^{\text{þ}}(tp(a/Abe)) = \alpha + 1 + 0 = \alpha + 1$$

and

$$\mathrm{U}^{\text{þ}}(b/Aae) + \mathrm{U}^{\text{þ}}(tp(a/Ae)) \leq \mathrm{U}^{\text{þ}}(tp(ba/Ae)) \leq \mathrm{U}^{\text{þ}}(b/Aae) \oplus \mathrm{U}^{\text{þ}}(tp(a/Ae)).$$

So

$$\alpha + \mathrm{U}^{\text{þ}}(tp(a/Ae)) \leq \alpha + 1 \leq \alpha \oplus \mathrm{U}^{\text{þ}}(tp(a/Ae)).$$

and the result follows. □

Notice that Proposition 3.6 provides the inductive step, in theories of finite $\mathrm{U}^{\text{þ}}$-rank, for any property which is closed under non-þ-forking restrictions and coordinatized types (in the sense that if a type $p$ is coordinatized by types having the property, then $p$ must have the property). This has nice consequences (it was strongly used, for example, in [3]). Some of the more direct consequences include the following.

**Corollary 3.7.** *Let $p(x)$ be any type of finite $U^{\text{þ}}$-rank. Then $p(x)$ is non-þ-orthogonal to a þ-minimal type.*

*Proof.* By Proposition 3.6 given $p(x) = tp(b/A)$ of finite $\mathrm{U}^{\text{þ}}$-rank, there is a non-þ-forking extension $tp(b/Ae)$ and an element $a \in \mathrm{acl}(Abe)$ such that $tp(a/Ae)$ is þ-minimal. Clearly $tp(b/Ae)$ and $tp(a/Ae)$ are non-þ-weakly orthogonal. □

**Corollary 3.8.** *Let $p \in \mathrm{S}(A)$ be a type of finite $U^{\text{þ}}$ rank. Then $\mathrm{wt}^{\text{þ}}(p) = n$ for some finite $n$ and there is a set $B$, $A \subseteq B$, and $b_1, \ldots, b_n$ independent over $B$ such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $U^{\text{þ}}(tp(b_i/B)) = 1$.*

*Proof.* The fact that a type of finite $\mathrm{U}^{\text{þ}}$-rank has finite þ-weight follows from Proposition 3.4. The rest of the assertions follow from Theorem 3.1 using the Exchange Lemma and Corollary 3.7. □

We will conclude this section by making some remarks about Proposition 3.6.

At first glance, it would appear that one could coordinatize a non-þ-forking extension of any type of finite $\mathrm{U}^{\text{þ}}$-rank by repeatedly applying the Proposition. However, this would prove a coordinatization theorem in the stable case, which is known not to be true, as the following example shows.

**Example 3.9.** *Let $\mathcal{L} := \{L, E\}$ be such that $L$ is a ternary relation and $E$ a binary relation and let $T$ be the theory that states that $E$ is an equivalence relation with infinitely many infinite classes and such that $L$ defines an affine space on each $E$-class (so $L(x,y,z) \Rightarrow E(x,y) \wedge E(x,z) \wedge E(y,z)$). A natural model $M$ of this theory is a sheaf of affine planes indexed by a line, where $E(x,y)$ if and only if $x$ and $y$ are in the same plane and $L(x,y,z)$ happens whenever $x, y, z$ are collinear points in the same $E$-class.*

*Let $g$ be $\varnothing$-generic $E$-class in $M$ and $a$ and $g$-generic point in $g$. The conclusion of Proposition 3.6 applied to the type $tp(a/g)$ can be seen in the following way: Let $b$ be any point in $g$ such that $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{g} b$ and let $l$ be the line through $a$ and $b$ (so that $l := \{x \mid L(x,a,b)\}$). Then $tp(a/gb)$ is a non-forking extension of $tp(a/g)$, $L \in \mathrm{acl}(ab)$ and $tp(l/ge)$ is a þ-minimal type.*

*Going back to coordinatization, if we try to coordinatize $tp(a/\varnothing)$ the first step is $tp(a/g)$ and $tp(g/\varnothing)$. The next step, however, would be to coordinatize the non-þ-forking extension $tp(a/gb)$ of $tp(a/g)$. But $b \not\downarrow^{þ}_{\varnothing} a$ (and the reader can check that this is true for every possible $b$ we can choose) so this does not help at all in trying to coordinatize $tp(a/\varnothing)$, nor any non-þ-forking extension of it. In fact, it is not hard to check that $tp(a/\varnothing)$ cannot be coordinatized in terms of þ-minimal types.*

The example above shows a stable (even super-stable) example where no coordinatization is possible, and it shows the limitations of Proposition 3.6 to get a full coordinatization for super-rosy theories. The main issue there is that we have no control over the parameter $e$ we need to get from þ-dividing to strong dividing. In the affine space, for example, this $e$ cannot be overlooked nor can we have any control as to where it comes from.

This has two main consequences. On the one hand, once we try to use Proposition 3.6 inductively and coordinatize $tp(b/Aae)$ then the $f$ we need can be taken to be such that $b \downarrow^{þ}_{Aa} f$ but there is no hope that we can find it such that $b \downarrow^{þ}_{A} f$. The second consequence is that we can only coordinatize a non-þ-forking extension of types of rank $\alpha + i$ in types of rank $\alpha$ and rank $i$ when $i = 1$, but we cannot do the same for $i > 1$ without further assumptions.

It seems that this lack of control over the choice of $e$ could be somewhat solved if we had extra assumptions (definable choice seems to be the right notion), but even this assumption seems to not be enough to get any coordinatization-like result beyond possibly the finite $\mathrm{U}^{þ}$-rank case. However, coordinatization is such a useful tool, and the connections with definable choice are so unclear, that even results assuming finite $\mathrm{U}^{þ}$-rank would be quite interesting.

## 4. Indiscernibles in dependent theories, strong dependence and weight.

Theorem 2.8 states that if a rosy theory is strongly dependent then every type has rudimentary finite (and hence finite) þ-weight. It is natural to ask whether an analogous notion of weight exists in a general setting (for example, an arbitrary dependent theory). It has been established that non-forking plays an important role in the study of dependent theories. One might wonder, therefore, whether there a notion of weight based on non-forking which behaves well in dependent theories. One desired property of such a notion would be: $T$ is strongly dependent if and only if every type has rudimentarily finite (and possibly finite) weight.

A possible notion of weight satisfying the property mentioned above was studied by the authors in [8]. One drawback of that notion is that it "measures" weight of a type with respect to Morley sequences (and not elements). Although by [12] we know that a Morley sequence is precisely what is needed in a dependent theory in order to determine a global invariant type (so the definitions in [8] are quite natural), we were (and still are) curious whether the definition of weight using Morley sequences is equivalent to the classical notion.

The answers to these questions are still unclear and they have motivated further research, such as [12], [5]. We have discovered that in order to make sense of a notion of weight based on non-forking, one needs to understand under which conditions there exist mutually indiscernible sequences starting with given elements (and to what extent one can "determine" the types of those sequences). Let us explain more precisely what we mean.

Suppose one defined "weight" as usual (like in stable theories; that is, take the definitions in section 2 and replace þ-forking by forking). Recall that one ingredient of the proof of Theorem

2.8 was showing that given a þ-independent set of elements (tuples), there exist mutually indiscernible sequences starting with those elements. A natural question whether an analogous result holds for non-forking is answered positively by Theorem 4.5 below. This is, unfortunately, not enough in order to prove a result similar to Theorem 2.8: since we do not have any control over the types of those Morley sequences, it is not clear why they should exemplify dividing (recall that in the case of þ-forking life was easier, as we could work with strong dividing, which is exemplified by any infinite indiscernibly sequence in the type). Of course, if $T$ were stable (or even simple), there would be no problem, since any Morley sequence would exemplify dividing.

The discussion above leads to the following two questions:

*Question* 4.1. To which extent can we "control the types" of the mutually indiscernible sequences constructed in Theorem 4.5? More precisely: what must we assume about the set $\{b_i : i < \alpha\}$ such that for every indiscernible sequences $I_i$ starting with $b_i$ respectively, there are indiscernible sequences $I'_i \equiv_{b_i} I_i$ such that $I'_i$ is indiscernible over $I'_{\neq i}$?

*Question* 4.2. To which extent do Morley sequences exemplify dividing in a dependent theory?

It was shown in [11] that if the types realized by the $b_i$'s in Question 4.1 are generically stable, then it is enough to assume non-forking independence. This was the main ingredient in the proof of the main theorem of Section 8 there: in a strongly dependent theory, every type has a rudimentarily finite generically stable weight (below we give a much easier proof of this result based on Theorem 4.8 - see Theorem 4.12). We could not establish (and in fact, it is still open) whether assuming non-forking independence is enough given arbitrary $b_i$ with or without assuming the theory is dependent, but some progress in this direction has been made, and the results appear in the second half of this section. It has become clear in subsequent works ([12], [5]) that Question 4.1 is related to so-called "strict non-forking" defined by Shelah in [9].

Concerning Question 4.2, we prove that, although it is not the case that *every* Morley sequence exemplifies dividing, there normally *are* such sequences. This fact has several consequences, some of which we investigate.

In this section we are going to assume that $T$ *is dependent* unless said otherwise.

We will work with classical notions of dividing, forking and splitting. We assume that the reader is familiar with all of these. Recall that $a \perp_A B$ stands for "$\text{tp}(a/AB)$ does not fork over $A$".

.

## 4.1. **Existence of Morley and mutually indiscernible sequences.** Let us start with the following easy lemma.

**Lemma 4.1.** *(No need of dependence).*

(i) *Assume* $A \subseteq C$ *and* $a \perp_A C$ *(that is,* $\text{tp}(a/C)$ *does not fork over* $A$*). Then there exists an* $C$*-indiscernible sequence* $I = \langle a_i : i < \omega \rangle$ *with* $a_0 = a$*. Such* $I$ *can be chosen to be a Morley sequence in* $\text{tp}(a/C)$ *based on* $A$*.*

(ii) *If in addition* $C \subseteq D$*, then there is* $D' \equiv_C D$ *such that* $I$ *is a Morley sequence in* $\text{tp}(a/D')$ *based on* $A$*.*

*Proof.* (i) Let $\mu$ be "big enough" (that is, so that Fact 1.13 can be applied for $\lambda = |C|+|T|$). Using existence of non-forking extensions, we can construct a sequence $I' = \langle a'_i : i < \mu \rangle$ in $\mathrm{tp}(a/AC)$ based on $A$ such that

- $a'_i \equiv_{Ca_{<i}} a_j$ for every $j > i$
- $a'_i \downfree_A Ca_{<i}$;

(note that if e.g. $T$ is dependent and $A = \mathrm{bdd}(A)$, we are done, since this is also a non-splitting sequence, hence indiscernible.)

By Fact 1.13 there is an $\omega$-sequence $I = \langle a_i : i < \omega \rangle$ indiscernible over $C$ such that every $n$-type of $I$ over $C$ "appears" in $I'$. In particular, this sequence is still based on $A$ because forking $a_i \not\downfree_A Ca_{<i}$ is a property of the type $\mathrm{tp}(a_{\leq i}/C)$. Since $a_0 \equiv_C a'_i$ for some $i < \mu$ and $a'_i \equiv_C a'_0 = a$, there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/C)$ taking $a_0$ to $a$; by replacing $I$ with the image of $I$ under $\sigma$, (which is still a Morley sequence over $C$ based on $A$) we may assume $a_0 = a$.

(ii) Since $a \downfree_A C$, by existence of non-forking extensions there exists $a' \equiv_C a$ such that $a' \downfree_A D$. So there is an automorphism over $C$ taking $a'$ to $a$ and $D$ to $D'$; now apply clause (i).

□

Although most properties of non-forking identifying it as an independence relation is stable or simple theories are generally false in our contexts, some things can still be said. We will refer to the fact below as "transitivity on the left".

**Fact 4.2.** *Let $A, B$ be sets and assume that $I = \langle a_i : i < \lambda \rangle$ is a non-forking sequence based on $A$ (that is, $a_i \downfree_A Ba_{<i}$ for all $i < \lambda$). Then $I \downfree_A B$, that is, $\mathrm{tp}(I/AB)$ does not fork over $A$.*

*Proof.* This is Claim 5.16 in [9]. □

**Corollary 4.3.** *Let $\{A_i : i < \lambda\}$ be a non-forking (independent) set over $A$, that is, $A_i \downfree_A A_{\neq i}$ for all $i$. Then for every $W, U \subseteq \lambda$ disjoint we have $A_{\in W} \downfree_A A_{\in U}$.*

*Proof.* Monotonicity and transitivity on the left. □

**Observation 4.4.** *Suppose $I$ is an indiscernible sequence over $A$ and $B \downfree_A I$. Then $I$ is indiscernible over $AB$.*

*Proof.* By Fact 1.7 $\mathrm{tp}(B/AI)$ does not split strongly over $A$. Recall that this implies that for every $\bar{a}_1, \bar{a}_2 \in I$ which are on the same $A$-indiscernible sequence we have $B\bar{a}_1 \equiv_A B\bar{a}_2$, which is precisely what we want. □

We proceed to the main results of this section. The first theorem allows us to construct mutually indiscernible (Morley) sequences when started with a non-forking sequence.

**Theorem 4.5.** *Let $T$ be a dependent theory, $A$ a set, and let $\{a_i : i < \kappa\}$ be a set of tuples satisfying $a_i \downfree_A a_{<i}$. Then there are mutually $A$-indiscernible infinite sequences $I_i$ (that is, $I_i$ is indiscernible over $A \cup \bigcup\{I_j : j \neq i\}$), each $I_i$ starts with $a_i^0 = a_i$. Moreover, if $\kappa = k$ is finite, then $I_i \downfree_A I_{<i}$ for all $i$ and for $i > 0$ we have that $I_i$ is a Morley sequence in $\mathrm{tp}(a_i/Aa_{<i})$ based on $A$, and if $\mathrm{tp}(a_0/A)$ does not fork over $A$, then we can get $I_0$ to be a Morley sequence in $\mathrm{tp}(a_0/A)$ over $A$.*

*Proof.* Note that by compactness it is enough to prove the theorem when $\kappa = k < \omega$; we will prove this by induction on $k$. Clearly, there is nothing to prove for $k = 1$.

So assume $\langle a_i : i < k+1 \rangle$ are given, $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_{<i}$. By the induction hypothesis there are $\langle I'_i : i < k \rangle$ mutually indiscernible, $I'_i$ starts with $a_i$, $I'_i$ is a Morley sequence over $I'_{<i}$ based on $A$.

By Lemma 4.1 (ii) with $D = A \cup \bigcup \langle I'_i : i < k \rangle$ and $C = Aa_{<k}$, there are $\langle I_i : i < k+1 \rangle$ satisfying

- $\langle I_i : i < k \rangle \equiv_{Aa_{<k}} \langle I'_i : i < k \rangle$. So in particular these are mutually $A$-indiscernible sequences starting with $a_i$; all the non-forking requirements are preserved too.
- $I_k$ is a Morley sequence in $\mathrm{tp}(a_k/AI_{<k})$ based on $A$ starting with $a_k$. So in particular it is indiscernible over $AI_{<k}$.

Now we are going to use that $T$ is dependent. By Fact 4.2 we know that $I_k \mathop{\smile\!\!\!\!\!\!|}_A I_{<k}$, so in particular $I_k \mathop{\smile\!\!\!\!\!\!|}_{AI_{<i}I_{i\in(i,k)}} I_i$ for all $i < k$. By the induction hypothesis $I_i$ is indiscernible over the base $AI_{<i}I_{\in(i,k)}$. By Observation 4.4 this implies that $I_i$ is indiscernible over $AI_{<i}I_{>i}$, as required. □

Although we find the theorem above interesting on its own, it will normally not be enough for our applications, since we will often be interested in starting with given indiscernible sequences (e.g. exemplifying dividing) and "make" them mutually indiscernible, that is, find mutually indiscernible sequences of the same type keeping a part of the original configuration (e.g. the first elements). We make several steps in that direction.

*Remark* 4.6. The reader should be aware that related results can be found in Shelah [9] (e.g. Claim 5.13). However, we believe that Claim 5.13(1) is wrong as stated there and the assumptions of Claim 5.13(2) are too strong for what we are hoping for, so we prefer not to rely on Shelah's work here.

**Lemma 4.7.** *Let $I = \langle a_i : i < \omega \rangle$ be an indiscernible sequence over $Ab$ such that $\mathrm{tp}(I/Ab)$ does not fork over $A$. Then there exists a Morley sequence $\langle I_\alpha : \alpha < \omega \rangle$ over $Ab$ based on $A$ with $I_0 = I$ such that for every $\alpha$ we have $I_\alpha$ is indiscernible over $AbI_{\neq\alpha}$ (here $b$ can be an empty, finite or infinite tuple).*

*Proof.* Let $\mu$ be a cardinal. We construct by induction on $\alpha < \mu$ a sequence $I_\alpha$ such that

- $I_0 = I$
- $I_\alpha \equiv_{Ab} I$
- $I_\alpha \mathop{\smile\!\!\!\!\!\!|}_A bI_{<\alpha}$
- $I_i$ is indiscernible over $AbI_{\neq i}$ for all $i \leq \alpha$.

For $\alpha = 0$ there is nothing to do (note that we are using $I \mathop{\smile\!\!\!\!\!\!|}_A Ab$). Assume that we have $I_{<\alpha}$ as above. Let $I'_\alpha$ be such that

- $I'_\alpha \equiv_{Ab} I$
- $I'_\alpha \mathop{\smile\!\!\!\!\!\!|}_A I_{<\alpha}b$

Clearly, we may assume that $I'_\alpha$ is as long as we wish, hence by Fact 1.13 there exists an $\omega$-sequence $I_\alpha$ which is indiscernible over $AI_{<\alpha}b$ and every $n$-type of $I_\alpha$ over $AI_{<\alpha}b$ "appears" in $I'_\alpha$. Clearly $I_\alpha \equiv_{Ab} I$. By finite character of forking $I_\alpha \mathop{\smile\!\!\!\!\!\!|}_A bI_{<\alpha}$. By monotonicity, for every

$\beta < \alpha$ we have $I_\alpha \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}_{AbI_{<\alpha,\neq\beta}} I_\beta$. Since $I_\beta$ is indiscernible over $AbI_{<\alpha,\neq\beta}$, by Observation 4.4, we have $I_\beta$ is indiscernible over $AbI_{\leq\alpha,\neq\beta}$, as required.

So we obtain a sequence $\langle I_\alpha : \alpha < \mu\rangle$ which is a non-forking sequence over $Ab$ (based on $A$) of mutually indiscernible sequences over $A$, starting with $I_0$. Choosing $\mu$ big enough, using Fact 1.13 as in the proof of Lemma 4.1, we may assume in addition that $\langle I_\alpha : \alpha < \omega\rangle$ is also $Ab$-indiscernible, i.e. a Morley sequence over $Ab$ based on $A$. □

**Theorem 4.8.** *Any instance of dividing over $A$ which is witnessed by a sequence $I$ such that $tp(I/A)$ does not fork over $A$, can always be witnessed by a Morley sequence.*

*Moreover, if $I := \langle a_i\rangle_{i\in\omega}$ is indiscernible over $Ab$ and such that $\{\varphi(x, a_i)\}_{i\in\omega}$ is $k$-inconsistent and $tp(I/Ab)$ does not fork over $A$, then there is a Morley sequence $\langle a^i\rangle_{i\in\omega}$ over $Ab$ based on $A$ such that $a^0 = a_0$ and $\{\varphi(x, a^i)\}_{i\in\omega}$ is inconsistent.*

*Proof.* We prove the "moreover" part.

Assume that $I = \langle a_i : i < \omega\rangle$ is an $Ab$-indiscernible sequence such that $\{\varphi(x, a_i) : i < \omega\}$ is $k$-inconsistent for some $k \in \mathbb{N}$. Denote $a = a_0$.

It is clearly enough to find a Morley sequence as in the statement of the theorem such that $tp(a/Ab) = tp(a^0/Ab)$. So assume towards a contradiction that given any Morley sequence $\langle a^i : i < \omega\rangle$ over $Ab$ based on $A$ with $tp(a/A) = tp(a^0/A)$ we have that

$$\bigwedge_{i<\omega} \varphi(x, a^i)$$

is consistent.

Let $a^0_i := a_i$ and let $I^0 := \langle a_i : i < \omega\rangle$. By Lemma 4.7 there is a Morley sequence of sequences $\langle I^j : j < \omega\rangle$ over $Ab$ based on $A$ such that $I^j$ is indiscernible over $AbI^{\neq j}$; let $I^j := \langle a^j_i : i < \omega\rangle$.

*Claim* 4.9. Let $\eta : \omega \to \omega$ be a function. Then the sequence $\langle a^i_{\eta(i)} : i < \omega\rangle$ is a Morley sequence over $Ab$ based on $A$.

*Proof.* It is clear from the construction that $\langle a^i_0\rangle$ is a Morley sequence over $Ab$ based on $A$. But it follows easily by induction over $n$ (using mutual indiscernibility) that

$$\mathrm{tp}\left(a^0_{\eta(0)}, a^1_{\eta(1)}, \ldots, a^n_{\eta(n)}/Ab\right) = \mathrm{tp}\left(a^0_0, a^1_0, \ldots, a^n_0/Ab\right)$$

□

We will now prove that given any function $\eta : \omega \to \omega$ the type

$$\left\{\left(\varphi\left(x, a^i_j\right)\right)^{j=\eta(i)} : i, j < \omega\right\}$$

is consistent, thus contradicting Observation 1.10.

The proof will again be by induction. Let

$$p^n(x) := \bigwedge_{i=0}^{n} \bigwedge_{j<\omega} \left(\varphi\left(x, a^i_j\right)\right)^{j=\eta(i)} \wedge \bigwedge_{i=n+1}^{\omega} \varphi\left(x, a^i_{\eta(i)}\right);$$

Since $\langle a^i_{\eta(i)} : i < \omega\rangle$ is a Morley sequence for all $\eta$, our hypothesis implies that $p^{-1}(x) := \bigwedge_{i=n+1}^{\omega} \varphi\left(x, a^i_{\eta(i)}\right)$ is consistent.

Assume that $p^n$ is consistent. Since $I^{n+1}$ is indiscernible over $A^{n+1} = AI^{\neq n+1}$, and since $I = I^0$ witnesses that $\varphi(x, a_0)$ divides over $A$, it follows that $\varphi(x, a_0^{n+1})$ divides over $A^{n+1}$ witnessed by $I^{n+1}$. By Lemma 1.12(i) with $p^n(x)$ here standing for $p(x)$ there (and the induction hypothesis), we have

$$p^{n+1}(x) := \bigwedge_{i=0}^{n+1} \bigwedge_{j<\omega} \left(\varphi\left(x, a_j^i\right)\right)^{j=\eta(i)} \wedge \bigwedge_{i \geq n+2} \varphi\left(x, a_{\eta(i)}^i\right)$$

is consistent, which completes the induction.

By compactness

$$\left\{\left(\varphi\left(x, a_j^i\right)\right)^{j=\eta(i)} : i, j < \omega\right\}$$

is consistent which contradicts dependence of $T$, see Observation 1.10. ☐

*Remark* 4.10. Even though one can easily construct examples where forking does not satisfy "existence", these are almost always quite artificial. In most of the theories one works with it is always the case that $tp(a/A)$ does not fork over $A$. In such cases Theorem 4.8 just states that any instance of dividing can be witnessed with a Morley sequence.

As a consequence, we obtain the following (quite desirable) property of generically stable types:

**Corollary 4.11.** *Suppose that $\varphi(x, b)$ divides over a set $A$ as exemplified by an $A$-indiscernible sequence $I$ with $\text{tp}(I/A)$ does not fork over $A$. Assume furthermore that $\text{tp}(b/A)$ is generically stable. Then any Morley sequence in $\text{tp}(b/A)$ exemplifies that $\varphi(x, b)$ divides over $A$.*

*Proof.* By Theorem 4.8 we know that there exists such a Morley sequence; but by stationarity of the generically stable type $\text{tp}(b/A)$ over $\text{acl}(A)$, clearly any Morley sequence will work. ☐

4.2. **Strong dependence and finite weight.** We will finish this paper by pointing out some results that follow from strong dependence and the results we have proved so far.

Let us first recall the classical concept of weight (we will give the definition without assuming anything on the theory; of course, it does not always give rise to a well-behaved notion).

As we mentioned before, some of the partial results we get arise when we restrict the definition of weight to certain kind of types (for example, generically stable ones). All of the definitions can be given and studied with either forking or þ-forking. However, forking is clearly the right notion for generically stable types (see [11]) so from now on we will just work with the standard classification theory notions (forking, splitting and dividing).

**Definition 4.1.**

- *Let $p(x)$ be any type over some set $A$. We will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses* $\text{pwt}(p(x)) \geq n$ *(*forking pre-weight of *$p$ is at least $n$) if $a \models p(x)$, $\langle b_i \rangle_{i=1}^n$ is $A$-independent and $a \not\perp_A b_i$ for all $i, j$. If $n$ is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses* $\text{pwt}(p(x)) = n$ *and that $p$* has forking pre-weight *$n$.*
- *Let $p(x)$ be any type over some set $A$. We will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses* $\text{wt}(p(x)) \geq$ *$n$ (*forking weight of *$p$ is at least $n$) if $a \models p(x)$, $a \perp_A B$, $\langle b_i \rangle_{i=1}^n$ is $B$-independent sequence and $a \not\perp b_i$ for all $i, j$. If $n$ is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses* $\text{wt}(p(x)) = n$ *and that $p$ has* forking weight *$n$.*

Similar definitions can be given requiring that the types $\mathrm{tp}(b_i/A)$ above are generically stable (see Fact 1.8), obtaining *generically stable pre-weight and weight of* $p$ denoted $\mathrm{gstpw}(p)$ and $\mathrm{gstw}(p)$ respectively. See section 8 of [11] for precise definitions.

The following follows easily from the results we have so far.

**Theorem 4.12.** *Assume $T$ is strongly dependent. Then every finitary type over a model (or just over an extension base) has rudimentary finite generically stable pre-weight.*

*Proof.* This was original proved in section 8 of [11]; however, having established Corollary 4.11, the proof of Theorem 4.12 becomes much easier than the original one given in [11]. Indeed, just like in a stable theory, given an instance of pre-weight $k$, as exemplified by $\{b_i\colon i<k\}$ realizing generically stable types, we can simply construct "mutually" Morley sequences $I_i$ starting with $b_i$, which by stationarity will be mutually indiscernible. By Corollary 4.11 they exemplify dividing, thus form a dividing pattern. □

We would like to generalize Theorem 4.12 to forking weight. For notational simplicity, let us concentrate on randomness patterns of depth 2 (analogous statements for larger depth will follow by a simple induction).

The following theorem weakens the assumptions of Theorem 4.12 somewhat, requiring only one of the types to be generically stable.

**Theorem 4.13.** *Let $a, b, c$ be elements and $A$ be a subset of a model $M$ of a dependent theory $T$. If $\mathrm{tp}(b/A)$ is generically stable, $b \mathop{\smile\!\!\!\!\!\mid}_A c$, $\mathrm{tp}(a/Ab)$ divides over $Ac$, and $\mathrm{tp}(a/Ac)$ divides over $Ab$, then there is a randomness pattern of depth 2 for $\mathrm{tp}(a/A)$. In particular, $T$ is not dp-minimal.*

*Proof.* Let $a, b, c, A$ be as in the statement of the theorem. By definition of dividing and Theorem 4.8, for every cardinal $\mu$ there is a sequence $I := \langle b_i : i<\mu\rangle$ which is Morley over $Ac$ and a formula $\varphi(x, y)$ such that $\models \varphi(a, b_0)$ and $\{\varphi(x, b_i)\}_{i\in\omega}$ is $k$-inconsistent for some $k\in\mathbb{N}$.

Since $b_0 \mathop{\smile\!\!\!\!\!\mid}_A c$ and $\mathrm{tp}(b_0/A) = \mathrm{tp}(b_i/A)$ is generically stable, it follows that $\mathrm{tp}(b_0/Ac) = \mathrm{tp}(b_i/Ac)$ is generically stable for all $i$. Using transitivity (Corollary 4.13 or Theorem 7.6 in [11]), it is easy to prove that $\langle b_i\rangle_{i\in\omega}$ is also a Morley sequence over $A$. So in particular, by right transitivity

$$\langle b_i\rangle_{i\in\omega} \mathop{\smile\!\!\!\!\!\mid}_A c$$

By definition of dividing, there is an $Ab_0$-indiscernible sequence $J := \langle c_j : j<\omega\rangle$ and a formula $\psi(x, y)$ such that $c = c_0$, $\models \psi(a, c_0)$ and $\{\psi(x, s_i)\}_{i\in\omega}$ is $k'$-inconsistent for some $k'\in\mathbb{N}$; by taking the maximum, we may assume that $k = k'$.

By extension, there is some $J' \equiv_{Ac_0} J$ such that $I \mathop{\smile\!\!\!\!\!\mid}_A J'$. We may assume that $I$ is as long as we want, so by Fact 1.13 there is an $\omega$-sequence $I' \equiv_{Ac_0} I$ which is indiscernible over $J'$ such that every $n$-type of $I'$ over $AJ'$ "appears" in $I$. In particular, it is still the case that $I' \mathop{\smile\!\!\!\!\!\mid}_A J'$. Moreover, since $I$ is indiscernible over $Ac_0$, we have (denoting $I' = \langle b'_i : i<\omega\rangle$) $c_0b'_0 \equiv_A c_0b_0 = cb$, in particular, $\exists x \varphi(x, b'_0) \wedge \psi(x, c_0)$ is consistent with $\mathrm{tp}(a/A)$, whereas the formulas $\varphi(x, b'_0)$ and $\psi(x, c_0)$ divide over $Ac_0$ and $Ab'_0$ respectively, exemplified by the sequences $I'$, $J$.

Note that $I'$ is indiscernible over $AJ'$ and by Observation 4.4 (since $I' \downarrow_A J'$), $J'$ is. Lemma 1.12 implies that we obtain a randomness pattern of depth 2 for $\mathrm{tp}(a/A)$ as required. ☐

We will finish this section by partial results which do not assume anything on the types. The following result addresses the question about the possible assumptions on indiscernible sequences (e.g., exemplifying dividing and pre-weight $k$; we concentrate on $k = 2$) are sufficient for achieving results such as constructing a randomness pattern. They are not strict generalizations of Theorems 4.12 or 4.13 because we need to include requirements on the sequences, and not just their first elements. However, it has the advantage of removing the generic stability assumptions completely.

**Proposition 4.14.** *Let $I = \langle a_i : i < \omega \rangle$, $J = \langle b_j : j < \omega \rangle$ be $A$-indiscernible sequences such that $I \downarrow_A b_0$ and $b_0 \downarrow_A a_0$. Then there exist $I', J'$ mutually $A$-indiscernible with $I' \equiv_{Aa_0} I$, $J' \equiv_{Aa_0} J$.*

*Proof.* Since $b_0 \downarrow_A a_0$, there is $b_0'' \downarrow_A I$ satisfying $b_0'' \equiv_{Aa_0} b_0$. Applying an automorphism over $Aa_0$ taking $b_0''$ to $b_0$, we obtain a new sequence $I'$ with the same type over $Aa_0$ as $I$; so without loss of generality $I = I'$, and in addition to our assumptions we have $b_0 \downarrow_A I$, hence $I$ is $Ab_0$-indiscernible.

Since $I \downarrow_A b_0$, there is $I'' \equiv_{Ab_0} I$ satisfying $I'' \downarrow_A J$. Applying an automorphism taking the first element of $I''$ to $a_0$ (fixing $Ab_0$), we obtain new sequences $I', J'$ starting with $a_0, b_0$ respectively, satisfying the same type over $A$ as $I, J$ respectively. So without loss of generality $I = I', J = J'$.

Prolonging $I$ and applying Fact 1.13, we get $I''$ indiscernible over $AJ$, "similar" to $I$ over $AJ$; in particular, $I''$ is $Ab_0$ - indiscernible (and has the same type over $Ab_0$ as $I$) and $I'' \downarrow_A J$. Applying an automorphism $\sigma$ over $Ab_0$ taking $I''$ onto (an initial segment of) $I$, we get, denoting $J' = \sigma(J)$:

- $J' \equiv_{Ab_0} J$
- $I$ is indiscernible over $AJ'$
- $I \downarrow_A J'$, hence $J'$ is indiscernible over $AI$

This finishes the proof. ☐

The following is easy now:

**Corollary 4.15.** *Let $p \in \mathrm{S}(A)$, $I = \langle a_i : i < \omega \rangle$, $J = \langle b_j : j < \omega \rangle$ be $A$-indiscernible sequences such that $I \downarrow_A b_0$ (e.g. $I$ is a Morley sequence over $Ab_0$ based on $A$) and $J \downarrow_A a_0$ (or just $b_0 \downarrow_A a_0$) such that*

- *$I, J$ exemplify that $\varphi(x, a_0)$, $\psi(x, b_0)$ divide over $A$*
- *$\varphi(x, a_0) \wedge \psi(x, b_0)$ is consistent with $p(x)$*

*Then there exists a dividing pattern and a randomness pattern in $p$. In particular, $p$ is not dp-minimal.*

The results above seem to suggest that for discussion of weight in dependent theories it is not enough to look just at the first elements of the sequences of a dividing pattern (the usual notion of forking weight). So let us conclude with the following notion of *splitting weight* which behaves quite nicely.

**Definition 4.2.**

- *Let $p(x)$ be any type over some set $A$. We will say that $a, \langle b_i^0 b_i^1 \rangle_{i=1}^n$ witnesses split $-$ $\mathrm{pwt}(p(x)) \geq n$ (*splitting pre-weight of *$p$ is at least $n$) if $a \models p(x)$, $\langle b_i^0 b_i^1 \rangle_{i=1}^n$ is $A$-independent and $a \not\perp_A b_i^0 b_i^1$ for all $i$ in a very strong way, namely:*
  ◇ *There exists a formula $\varphi_i(x,y)$ such that*
  $$\models \varphi_i(a, b_i^0) \wedge \neg\varphi_i(a, b_i^1)$$
  .
  *If $n$ is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses split $-$ $\mathrm{pwt}(p(x)) = n$ and that $p$* has splitting pre-weight $n$.
- *Let $p(x)$ be any type over some set $A$. We will say that $a, B, \langle b_i^0 b_i^1 \rangle_{i=1}^n$ witnesses split $-$ $\mathrm{wt}(p(x)) \geq n$ (*splitting weight of *$p$ is at least $n$) if $a \models p(x)$, $\langle b_i^0 b_i^1 \rangle_{i=1}^n$ is $A$-independent and*
  ◇ *There exists a formula $\varphi_i(x,y)$ such that*
  $$\models \varphi_i(a, b_i^0) \wedge \neg\varphi_i(a, b_i^1)$$
  .
  *If $n$ is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses split $-$ $\mathrm{wt}(p(x)) = n$ and that $p$ has* splitting weight $n$.

**Observation 4.16.** *Let $p \in \mathrm{S}(A)$. Then $p$ is strongly dependent if and only if its splitting pre-weight is rudimentary finite.*

*Proof.* The "if" direction is clear. For the "only if" direction, using Theorem 4.5, one can construct an array of mutually indiscernible (Morley) sequences starting with $b_i^0 b_i^1$. The rest is easy. □

The next observation (whose proof is easy and similar to the previous one) connects dependence in general to weight:

**Observation 4.17.** *A theory $T$ is dependent if and only if every type has a bounded splitting weight.*

We see that it is unnecessary to assume (as we did in Proposition 4.14) that $I \perp_A b_0$, it is enough to look at the first two elements of the sequence $I$; but this seems to be important (in case $b_0$ is not generically stable). In a sense, what we do is replacing in the dividing/randomness pattern the formulas $\varphi_i(x, y_i)$ with $\varphi'_i(x, y_i^0 y_i^1) = [\varphi_i(x, y_i^0) \equiv \varphi_i(x, y_i^1)]$ and considering a dividing pattern (= witness for high pre-weight) with respect to these new formulas.

We would like to finish by remarking that results in this section pretend to be a first approach to characterize strong dependence by a notion of finite weight. A complete result of this type would be quite interesting and, we believe, very useful. However, it is not clear that plain forking is the right notion for this. As we mentioned before, it seems that strict non-forking is the right way to go, and we refer the reader to [12] and [5] for more details.

## Appendix A. cc-forking and cc-dividing

The following theorem is proven in [8]. We include the proof here for the sake of completeness.

**Theorem A.1.** *The following are equivalent for any $p \in S_1(A)$.*

(i) *$p$ admits $n$-cc-þ-forking.*
(ii) *$p$ admits $n$-cc-þ-dividing.*
(iii) *There is an extension $p(x,B)$ of $p(x)$ such that $p(x,B)$ admits $n$-cc-strong dividing.*

*Proof.* Any witness for $n$-*cc*-strong dividing is a witness of $n$-*cc*-þ-dividing and any witness for $n$-*cc*-þ-dividing is a witness of $n$-*cc*-þ-forking. We will prove the other implications for $n = 2$. The general case will follow by a straightforward induction on $n$ using the properties of þ-forking in rosy theories.

(i) $\Rightarrow$(ii). Let

$$\{\varphi_i(x,\bar a), \psi(x,\bar b)\}, A$$

be a $2-cc$-þ-forking witness for $p$. By definition there are finitely many formulas $\varphi_i(x,\bar a'_i)$, $\psi_j(x,\bar b'_j)$ and tuples $\bar c', \bar d'$ such that $\varphi(x,\bar a) \vdash \bigvee_{i=1}^{k_a} \varphi_i(x,\bar a'_i)$, $\psi(x,\bar b) \vdash \bigvee_{j=1}^{k_b} \psi_j(x,\bar b'_j)$ and $\varphi_i(x,\bar a'_i)$ strongly divides over $A\bar c'$ and $\psi_j(x,\bar b'_j)$ strongly divides over $A\bar d'$.

By hypothesis $\bar a \mathop{\smile\!\!\!\!\!\mid}^{þ}_A \bar b$ so by extension of þ-independence we can find $\bar a'' \models tp(\bar a/A\bar b)$ such that $\bar a'' \mathop{\smile\!\!\!\!\!\mid}_A \langle \bar b'_j\rangle \bar d' \bar b$. Let $\langle \bar a''_i\rangle, \bar c''$ be images of $\langle \bar a'_i\rangle, \bar c'$ under an automorphism that fixes $A, \bar b$ and sends $\bar a$ to $\bar a''$.

Using extension on the other side there are $\bar b''\langle \bar b''_j\rangle \bar d'' \models tp(\bar b\langle \bar b'_j\rangle \bar d'/A\bar a'')$ such that

$$\langle \bar a''_i\rangle \bar c'' \bar a'' \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} \bar b''\langle \bar b''_j\rangle \bar d''.$$

But $tp(\bar a''\bar b''/A) = tp(\bar a''\bar b/A) = tp(\bar a\bar b/A)$ so by applying an automorphism over $A$, we can find $\langle \bar a_i\rangle$, $\bar c$, $\langle \bar b_j\rangle$, $\bar d$ such that $\langle \bar a_i\rangle \bar c\bar a \mathop{\smile\!\!\!\!\!\mid}_A \langle \bar b_j\rangle \bar d\bar b$.

So in particular we have $\mathrm{tp}(\langle \bar a_i\rangle \bar c/A\bar a) = \mathrm{tp}(\langle \bar a'_i\rangle \bar c'/A\bar a)$ and $\mathrm{tp}(\langle \bar b_i\rangle \bar d/A\bar b) = \mathrm{tp}(\langle \bar b'_i\rangle \bar d'/A\bar b)$.

Therefore

$$\varphi(x,\bar a) \models \bigvee_{i=1}^{k_a} \varphi_i(x,\bar a_i), \quad \psi(x,\bar b) \models \bigvee_{i=1}^{k_a} \psi_i(x,\bar b_i), \tag{1}$$

and

$$\begin{array}{l} \varphi(x,\bar a_i) \text{ strongly divides over } A\bar c \text{ for all } i, \\ \psi(x,\bar b_j) \text{ strongly divides over } A\bar d \text{ for all } j \end{array} \tag{2}$$

Since $\varphi(x,\bar a) \wedge \psi(x,\bar b)$ is consistent with $p$, it is clear from (1) that the conjunction $\varphi_i(x,\bar a_i) \wedge \psi_j(x,\bar b_j)$ is consistent with $p$ for some $i,j$. By monotonicity of þ-forking independence we know that $\bar a_i \mathop{\smile\!\!\!\!\!\mid}_A \bar b_j$ so (2) implies that $(\varphi(x,\bar a_i), \psi(x,\bar b_j)), A$ is a witness for *cc*-þ-dividing.

(ii)$\Rightarrow$ (iii). Once again we will prove the case $n = 2$.

Let

$$\{\varphi_i(x,\bar a), \psi(x,\bar b)\}, A$$

be a $2-cc$-þ-dividing witness for $p$. Let $D'$ and $E'$ be supersets of $A$ such that $\varphi(x,\bar a)$ strong divides over $D'$ and $\psi(x,\bar b)$ strong divides over $E'$. Since $\bar a \mathop{\smile\!\!\!\!\!\mid}^{þ}_A \bar b$ we can, by extension (as in the proof of (i) $\Rightarrow$(ii)), find $D, E$ satisfy types $tp(D'/A\bar{a})$ and $tp(E'/A\bar{b})$ respectively and such that $\bar{a}D \downfree^{þ}_{A} \bar{b}E$; so in particular $\varphi(x,\bar{a})$ strong divides over $D$ and $\psi(x,\bar{b})$ strong divides over $E$, $\bar{a} \downfree^{þ}_{D} E$, $\bar{b} \downfree^{þ}_{E} D$, and $\bar{a} \downfree^{þ}_{D\cup E} \bar{b}$.

Since by definition $\bar{a} \notin acl(D)$ and $\bar{b} \notin \mathrm{acl}(E)$ we get that $\bar{a}, \bar{b} \notin \mathrm{acl}(ED)$: e.g., $\bar{a} \notin \mathrm{acl}(D)$, but $\bar{a} \downfree^{þ}_{E} D$, so $\bar{a} \notin \mathrm{acl}(ED)$.

So

$$\begin{array}{l} \varphi(x,\bar{a}) \text{ strongly divides over } E \cup D, \\ \psi(x,\bar{b}) \text{ strongly divides over } E \cup D, \\ \bar{a} \downfree^{þ}_{D\cup E} \bar{b}. \end{array} \tag{3}$$

Let $B := D \cup E$ and let $p(x, B, \bar{a}, \bar{b})$ be a non-þ-forking extension of $p(x) \cup \{\varphi(x,\bar{a}) \cup \psi(x,\bar{b})\}$ and let $p(x, B)$ be the restriction of $p(x, B, \bar{a}, \bar{b})$ to $B$. All the conditions in the definition of $2 - cc$-strong dividing are satisfied which completes the proof of the theorem. $\square$

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