A NOTE ON UDI'S PAPER "STABLE GROUP THEORY AND APPROXIMATE SUBGROUPS"

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The goal of this note is to exemplify the basic model theoretic technique which is used by Hrushovski in order to obtain Corollary 1.2 in his preprint "On stable group theory and approximate subgroups". The note is written for a talk at the CMAF Logic Seminar. Corrections and comments are welcome.

Here is the statement of the Corollary:

Theorem 1. *("Corollary 1.2") For any $k, \ell, m \in \mathbb{N}$, there are $p < 1$ and $K \in \mathbb{N}$ such that the following holds:*

Let G be a group, $X_0 \subseteq G$ finite, $X = X_0^{-1}X_0$ such that $|X_0X| \leq k|X_0|$. Assume also that at least $100 \cdot p$ percent of l-tuples $\bar{a} = (a_1, \ldots, a_\ell) \in X^\ell$ satisfy $|\bar{a}^X|(= |a_1^X \ldots a_\ell^X|) \geq |X|/m$.

Then there is a subgroup S of G, $S \subseteq X^2$, such as X is covered by at most K cosets of S.

In the talk I will explain some background and motivation for this result.

The main model theoretic tool in the proof is the "Stabilizer Theorem" , Theorem 3.4 in Udi's paper. We state a (quite a bit) simplified version of it (which is precisely what is needed for Corollary 1.2).

Theorem 2. *Let G be a definable group, μ be a definable G-invariant Keisler measure, $X_0 \subseteq G$ a definable set of positive measure, $X = X_0^{-1}X_0$. Let $\tilde{G}$ be the subgroup generated by X (so $\tilde{G}$ is $\bigvee$-definable). Then there exists a countable model M and a type-definable over M subgroup S of G. S is of bounded index in $\tilde{G}$, and normal in $\tilde{G}$, $S \subseteq X^2$, and X can be covered by finitely many cosets of S.*

In the talk I will explain why this theorem is called the "Stabilizer Theorem" and how it is related to previous work on stable, simple and dependent theories.

For those who have looked at the paper, note that Theorem 2 follows from Udi's Theorem 3.4: By Lemma 2.14, we can find a countable model M and a global type p which is finitely satisfiable in M, such that if $a \models p{\restriction}M$ and $b \models p{\restriction}Ma$, then $\mathrm{tp}(a/Mb)$ is μ-wide. Since X_0 has positive measure, we can make sure (e.g. modifying the proof of Lemma 2.14 a bit) that $X_0 \supseteq p$. Let $q = p{\restriction}M$. So q is μ-wide, and given a, b above, clearly $b \mathop{\smile\!\!\!\!\!\!|}_M a$ (by finite satisfiability) and $a \mathop{\smile\!\!\!\!\!\!|}_M b$ (since $\mathrm{tp}(a/Mb)$ is wide). This takes care of the assumptions of Theorem 3.4. Applying the theorem, we get S as required; note that $S = (q^{-1}q)^2$, so $S \subseteq X^2$.

Let us establish some very basic facts needed in the proof below.

Claim 1. Let p be a complete type over a model M, $p \in \mathrm{S}^k(M)$. Then there exists a global type q extending p, which is finitely satisfiable in M.

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Proof. Note that p can be viewed as a filter F on subsets of M^k: $X \in F$ if and only if $X = \varphi^M(x)$ for some $\varphi(x) \in p$. Let U be an ultrafilter on M^k extending F. Now define a global type q as follows: $\varphi(x,c) \in q$ if and only if $\varphi^{\mathfrak{C}}(x,c) \cap M \in U$. It is easy to verify that this is indeed a (complete) global type extending p, f.s. i n M. QED$_2$

Now suppose we are in the situation described in the Stabilizer Theorem, and suppose $a \equiv_M b$. We would like to deduce that ab^{-1} is in S. In fact, if $a \equiv_M b$, then a and b have the same Lascar type over M, so, more precisely:

Claim 2. Let $a, b \in \tilde{G}$, $a \equiv_M b$, and let $E(x,y)$ be an $\mathrm{Aut}(\mathfrak{C}/M)$-invariant equivalence relation with boundedly many classes in $\tilde{G}$. Then $E(a,b)$.

Proof. Suppose not. Let $p = \mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, and let q be as in Claim 1. Define a sequence $I = \langle c_i : i < \lambda \rangle$ by $c_i \models q{\restriction}Mabc_{<i}$. Since q is $\mathrm{Aut}(\mathfrak{C}/M)$-invariant, it is easy to see that both sequences $a^\frown I$ and $b^\frown I$ are indiscernible. By automorphism-invariance of E, since $\neg E(a,b)$, clearly all the c_i's are in different E-classes. But they are all in $\tilde{G}$ (since they all satisfy p, for example), and λ was an arbitrary cardinal. A contradiction. QED$_2$

We are now ready to prove Corollary 1.2.

Proof. (Proof of Corollary 1.2) Let L_0 be the basic language that contains the language of groups and a predicate for X_0. We work in the expanded language L, in which the counting measures with respect to X on finite structures and all their finite powers are definable. In other words, we define $\mu(Y) = \frac{|Y|}{|X|^\ell}$ for all sets $Y \subseteq M^\ell$ (could be infinite if Y is infinite; since we will only care about subsets of powers of X, this will not create problems for us).

If the conclusion fails, then by compactness there is a a group G, $X_0 \subseteq G$, $X = X_0^{-1}X_0$, definable Keisler measures μ_ℓ on definable sets in ℓ variables (satisfying Fubini), such that:

- $\mu_1(X) = 1$, $\mu_1(X_0X) \le k \cdot \mu_1(X_0) < \infty$.
- Denote $Q = \{\bar{a} \in X^\ell \colon \mu_\ell(\bar{a}^X) \ge \frac{1}{m}\}$. Then Q is type-definable and $\mu_\ell(Q) = \mu_\ell(X^\ell) = 1$.
- There is no definable (possibly with parameters; see $\diamondsuit$ below) subgroup $S \subseteq X^2$ such that finitely many cosets of S cover X.

In order to apply compactness, we note that for any rational $p < 1$ and K there is a group $G_{p,K}$, finite X_0 and X with $|X_0X| < k|X_0|$, Q (as defined above) has measure at least p, and for every formula with one variable $\varphi(x)$ *with parameters*, the following statement holds:

$\clubsuit$ If $\varphi(x)$ is a subgroup and $\varphi(x) \subseteq X^2$, then X is not covered by K cosets of $\varphi(x)$

Now let $\hat{G}$ be an ultraproduct of $G_{p,K}$ with respect to some non-principal ultrafilter; so it is $\aleph_1$-saturated. We think of it as an elementary submodel of the monster model $\mathfrak{C}$.

We will write μ for μ_1. Let M, S be as in Theorem 2. Note that by saturation of $\hat{G}$, without loss of generality $M \subseteq \hat{G}$. By the construction, since in $\clubsuit$ we took all the formulas with parameters in $G_{p,K}$, we have the following:

$\diamondsuit$ There is no M-definable subgroup $S \subseteq X^2$ such that finitely many cosets of S cover X.

Recall that if a and b have the same type over a model M, then they have the same Lascar type, that is, they agree on all automorphism-invariant bounded (say, in $\tilde{G}$) equivalence relations, see Claim 2. In particular (since S is of bounded index in $\tilde{G}$), if $a \equiv_M b$, then $a/S = b/S$.

In our case M is countable, so we can write the equivalence relation $\equiv_M$ as an intersection of countably many nested *definable* equivalence relations E_α. For example, $E_\alpha(x,y)$ can say $\varphi_\alpha(x) \iff \varphi_\alpha(y)$, where φ_α is some enumeration of all formulas over M, $\varphi_{\alpha+1} \models \varphi_\alpha$. Since E_α are definable and bounded, they are in fact finite.

For each E_α, let F_α be a class with $\mu(X_0 \cap F_\alpha) > 0$. Clearly such a class exists: X_0 is covered by finitely many classes of E_α, and $\mu(X_0) > 0$. Denote $C_\alpha = F_\alpha^{-1}F_\alpha \cap X$; so $\mu(C_\alpha) > 0$, hence $\mu_\ell(C_\alpha^\ell) > 0$ (here ℓ is not an index, but rather indicates taking the cartesian product). Because $\mu_\ell(Q) = \mu_\ell(X^\ell))$, the intersection $Q \cap C_\alpha^\ell$ has positive measure. So for every α we have $\bar{a}^\alpha = (a_1^\alpha, \ldots, a_\ell^\alpha) \in Q$ such that $a_i^\alpha \in C_\alpha$ for all i. By compactness (recall that Q is type-definable and E_α are definable and they refine each other), we have $\bar{a} = (a_i) \in Q$ such that for each i there are b_i, c_i such that

- $a_i = b_i^{-1}c_i$
- $E_\alpha(b_i, c_i)$ for all α (so $b_i \equiv_M c_i$).

It follows that $a_i \in S$ for all i. Since S is normal in $\tilde{G}$, $a_i^X \subseteq S$ for all i. Recall that $\mu(\bar{a}^X) > 0$ (because $\bar{a} \in Q$), so $\mu(S) > 0$. Hence finitely many cosets of S cover X^2. S is type-definable, and the complement of S in X^2 is a finite union of cosets of S (which is type-definable), intersected with a definable set X^2. Hence the complement of S is type-definable as well.

We conclude that $S \subseteq X^2$ is an M-definable subgroup, finitely many cosets of which cover X, which contradicts $\diamondsuit$. This finishes the proof.

QED$_2$

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