Generic Stability

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Let's get started

PART I - Motivations

Three motivations

- Stable domination and stable independence in unstable contexts
- Stable types in theories interpretable in o-minimal structures
- Stability in Banach spaces and ℓ^p spaces

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Different kinds of stability

- A partial type $\pi(\bar{x})$ is called *Lascar-Poizat stable* (*LP-stable*) if every extension of it to a global type is definable.
- A type $p \in S(A)$ is called *stably dominated* if there exists a collection of stable sets $\bar{D} = \langle D_i : i < \alpha \rangle$ and definable functions $f_i \colon p^{\mathfrak{C}} \to D_i$ such that for every set $B \supseteq A$ and $\bar{a} \models p$, if $f_i(\bar{a}) \mathop{\smile\!\!\!\!\!\!|}^{st}_{A} B$ for all i (which in this context just means that $\mathrm{tp}(f_i(a)/B)$ is definable over A), then (denoting $\bar{f} = \langle f_i : i < \alpha \rangle$) $\mathrm{tp}(f(\bar{a})/B) \vdash \mathrm{tp}(\bar{a}/B)$.
- In both cases above, the set of realizations admits a stable independence relation (the stably dominated case is due to Haskell, Hrushovski and Macpherson).

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Stable types in Banach spaces (Shelah, U.)

Let T be a continuous theory of a Banach space.

- We call a type r *wide* if $r^{\mathfrak{C}}$ contains (the unit sphere of) an infinite-dimensional Banach space.
- Let r be a stable wide type. Then it can be extended to a *minimal* stable wide type q. That is, every forking extension of q is not wide. We call such types *minimal*.
- Let q be minimal. Then q is "generically" ℓ^p. That is, every Morley sequence in q (which is obtained simply by taking the unique wide extension at each stage) is isometric to ℓ^p for some $1 \leq p < \infty$

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Main Definition and Main Properties

Part II - Generically stable types in dependent theories

- Strongly motivated by Shelah's "stable" types
- The name "generically stable" is due to Hrushovski and Pillay who investigated some of their properties independently and simultaneously.

Dependence

- Recall that a theory T is called *dependent* if there does not exist a formula which exemplifies the independence property. We are mostly going to use the following equivalent definition:
- T is dependent if and only if there do not exist an indiscernible sequence $I = \langle \bar{a}_i : i < \lambda \rangle$, a formula $\varphi(\bar{x}, \bar{y})$ and $\bar{c}$ such that both $\{i \colon \models \varphi(\bar{a}_i, \bar{c})\}$ and $\{i \colon \models \neg\varphi(\bar{a}_i, \bar{c})\}$ are unbounded in λ.
- If $\langle \bar{b}_i : i \in I \rangle$ is an infinite indiscernible *set*, then for every $\varphi(\bar{x}, \bar{y})$ there exists $k = k_\varphi < \omega$ such that for every $\bar{c} \in \mathfrak{C}$, either $|\{i \in I \colon \varphi(\bar{b}_i, \bar{c})\}| < k$ or $|\{i \in I \colon \neg\varphi(\bar{b}_i, \bar{c})\}| < k$.

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Continuous Dependence

- A continuous theory T is dependent if for every indiscernible sequence $I = \langle \bar{a}_i : i < \lambda \rangle$, a formula $\varphi(\bar{x}, \bar{y})$ and $\bar{c}$ we have
$$\exists \lim_{i<\lambda} \varphi(\bar{a}_i, \bar{c})$$

- If $\langle \bar{b}_i : i \in I \rangle$ is an infinite indiscernible *set*, then for every $\varphi(\bar{x}, \bar{y})$ and $\varepsilon > 0$ there exists $k = k_{\varphi,\varepsilon} < \omega$ such that for every $\bar{c} \in \mathfrak{C}$, for all but k-many $i < \lambda$ we have
$$|\varphi(\bar{b}_{i_1}, \bar{c}) - \varphi(\bar{b}_{i_2}, \bar{c})| \leq \varepsilon$$

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Stable types in dependent theories

Let $p \in S(A)$. The Following Are Equivalent for a type in a dependent theory:

1. p is LP-stable.
2. For every $B \supseteq A$, p has at most $|B|^{\aleph_0}$ extensions in $S(B)$.
3. Every extension of p is generically stable.
4. Every indiscernible sequence in p is an indiscernible set.
5. There is no "order property" on p (with or without external parameters)
6. On the set of realizations of p there is no definable (with or without external parameters) partial order with infinite chains.

Averages and limits (T dependent)

- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be an indiscernible sequence, B a set. We define the *average type* of I over B, $\mathrm{Av}(I, B)$ as the set of all formulae $\varphi(\bar{x}, \bar{b})$ such that $\{i \colon \neg\varphi(\bar{a}_i, \bar{b})\}$ is bounded in λ.
- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be an indiscernible sequence in a continuous theory, B a set. We define the *average type* or the *limit type* of I over B to be

$$\mathrm{Av}(I, B) = \lim(I, B) = \{\varphi(\bar{x}, \bar{b}) = \lim_i \varphi(\bar{a}_i, \bar{b})\}$$

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Properties of Averages (T dependent)

- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be an indiscernible set. Then $\mathrm{Av}(I, \mathfrak{C})$ is definable over $\cup I$.
- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be an indiscernible set. If $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A, then $\mathrm{Av}(I, \mathfrak{C})$ is definable over $\mathrm{acl}(A)$.
- The reason: *in a dependent theory any type has boundedly many global nonforking extensions*. Not true in general (for example, simple unstable theories).

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Main Definition and Main Properties

- We call a type $p \in \mathrm{S}(A)$ *generically stable* if there exists a nonforking sequence $\langle \bar{b}_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.
- p is definable almost over A.
- If $A = \mathrm{acl}(A)$ then p is stationary and its unique global extension p' which does not fork (equivalently, split) over A is a free extension with respect to some/any good definition over A.

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Equivalent Definitions

Theorem

Let $p \in S(A)$. The Following Are Equivalent:

1. *p is generically stable, that is, there exists a nonforking sequence in p which is an indiscernible set.*
2. *Every nonforking sequence in p is an indiscernible set.*
3. *p is definable over $\mathrm{acl}(A)$ and some Morley sequence in p is an indiscernible set.*
4. *p is definable over $\mathrm{acl}(A)$ and every Morley sequence in p is an indiscernible set.*
5. *There is a nonforking extension of p to a $(|A| + |T|)^+$-saturated model M which is both definable over and finitely satisfiable in some countable indiscernible set contained in M.*

More properties

- If p is finitely satisfiable in A then p is stationary, and its unique global extension which does not split strongly/fork over A, is both its coheir and a free extension with respect to some/any good definition over A.
- Any nonforking extension of p is generically stable. Moreover, any q which is parallel to p is generically stable.
- Any nonforking sequence in p is an indiscernible set and a Morley sequence. Any two nonforking sequences in p have the same type.

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Forking Independence

- For $\bar{a} \models p$ we say that it is *stably forking independent* (or just *forking independent*) of B over A, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$, if $\operatorname{tp}(\bar{a}/AB)$ does not fork over A.
- Let $\operatorname{tp}(\bar{a}/A)$ be generically stable, then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B$ if and only if $a \models p|^d B$ with respect to one of the (finitely many) definitions of p over $\operatorname{acl}(A)$.

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Properties of Independence

Let $p, q \in S(A)$ be generically stable, $\bar{a}, \bar{b}$ realize p, q respectively, and let $\bar{c}, \bar{d}$ be any tuples (maybe infinite). Then:

- *Irreflexivity* $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{a}$ if and only if p is algebraic
- *Monotonicity* If $a \mathop{\smile\hskip-0.9em ^| \ }_A \bar{b}\bar{c}\bar{d}$, then $a \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}\bar{b}$.
- *Symmetry* $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{b}$ if and only if $\bar{b} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{a}$
- *Transitivity* $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}\bar{d}$ if and only if $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_{A\bar{c}} \bar{d}$ and $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}$
- *Existence* Let $B \supseteq A$, then there exists $\bar{a}' \equiv_A \bar{a}$ such that $\mathrm{tp}(\bar{a}'/B)$ is generically stable and $\bar{a}' \mathop{\smile\hskip-0.9em ^| \ }_A B$.
- *Uniqueness* If $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}$, $\bar{a}' \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}$ and $\bar{a}' \equiv_{\mathrm{acl}(A)} \bar{a}$, then $\bar{a} \equiv_{A\bar{c}} \bar{a}'$
- *Local Character* If $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_A \bar{c}$, then for some subset A_0 of A of cardinality $|T|$, $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_{A_0} \bar{c}$. If $A = M$ is an $\aleph_1$-saturated model, there exists a countable $A_0 \subseteq M$ such that $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_{A_0} \bar{b}$.

Splitting

- A type $p \in S(B)$ *does not split* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
- A type $p \in S(B)$ *does not split strongly* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same Lascar strong type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
- If $A = \mathrm{bdd}(A)$, the two notions agree.

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Forking

- Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with strong splitting.
- *Easy but very important:* In a dependent theory strong splitting implies dividing.
- There are boundedly many global types which do not fork over a given set A.

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Thank you

I like tea! Are there any cooki... questions?