MORLEY SEQUENCES IN DEPENDENT THEORIES

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Abstract. We characterize nonforking (Morley) sequences in dependent theories in terms of a generalization of Poizat's special sequences and show that average types of Morley sequences are stationary over their domains. We characterize generically stable types in terms of the structure of the "eventual" type. We then study basic properties of "strict Morley sequences", based on Shelah's notion of strict nonforking. In particular we prove "Kim's lemma" for such sequences, and a weak version of local character.

1. Introduction and Preliminaries

1.1. **Introduction.** This paper is a natural continuation of [Usv], where we proved several useful facts on nonforking sequences in dependent theories which are also indiscernible sets. In particular, it was shown that given a nonforking indiscernible set I over a set A, the global average type $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A (and furthermore, if $A = \mathrm{acl}(A)$, then $\mathrm{Av}(I, \mathfrak{C})$ is the unique nonforking extension of its restriction to A). These properties of nonforking indiscernible were crucial for understanding generically stable types. A natural question was: what if one does not assume that I is an indiscernible set? Are the results in [Usv] a particular case of a general theory of (arbitrary)nonforking sequences in a dependent theory?

This paper provides a complete and satisfactory answer to the question above. We show that a nonforking sequence (which we call a "Morley sequence" here) is in many ways the "correct" object to work with in a dependent theory. Let us explain what we mean.

In a stable theory a type over an algebraically closed set determines a unique global nonforking extension. This is not the case for an arbitrary type in a dependent theory, the simplest example being the type p "at infinity" over the model $\mathbb{Q}$ in the theory of dense linear orders without endpoints. This type has two global nonforking extensions: one is the type of the cut $\mathbb{Q}^+$ which is finitely satisfiable in $\mathbb{Q}$, and another one is the type "at infinity" over the monster model (which is the extension with respect to the natural definition of p). One of the main results of this paper is that whereas a type p does not determine a unique global nonforking extension, such an extension *is* determined by a Morley sequence in p. More precisely, we show that given a Morley sequence I over a

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set A, the type $\mathrm{Av}(I, A \cup I)$ is *stationary over* A. That is, there exists a unique global type extending it which does not fork over A. In the example above, the first global type is determined by a decreasing Morley sequence (which is a co-heir sequence), and the second one - by an increasing one.

Having realized that $\mathrm{Av}(I, A \cup I)$ has a unique extension q which does not fork over A, we proceed to understanding q. The natural conjecture that q is the global average type of I (as was the case in [Usv]) fails immediately: e.g., taking I to be an increasing sequence in the type "at infinity" over $(\mathbb{Q}, <)$ discussed above. Instead, we have to work with the so-called "eventual" type of I, $\mathrm{Ev}(I)$. This notion is essentially due to Poizat [Poi79], although the name was proposed by Adler in [Adl]. Poizat studied eventual types of "special" sequences. We work with a slight generalization of his notion, which we call "Lascar special" or "weakly special" sequences.

Given a special sequence I over a set A (see definitions in section 2), Poizat gave a natural construction of a global type $\mathrm{Ev}(I)$ which extends $\mathrm{Av}(I, A \cup A)$ and does not split over A. Using a similar construction, given a weakly special sequence I over A, we obtain a global type $\mathrm{Ev}(I)$ which extends the average type of I and does not *Lascar split* (equivalently, does not *fork*) over A. Then we show that a sequence I is weakly special over A if and only if it is Morley over A. Summarizing all of the above, we can conclude that:

- A Morley sequence over A determines a unique global type which does not fork (= is Lascar invariant) over A. This type is precisely the eventual type of I, where I is viewed as a weakly special sequence over A.

The reader might ask at this point how the results in [Usv] fall into this picture. From the construction of the eventual type, it is easy to see that if I is a Morley indiscernible set, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. So it follows that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A. We go further and show that this characterizes indiscernible sets: $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A iff $\mathrm{Av}(I, \mathfrak{C}) = \mathrm{Ev}(I)$ iff I is an indiscernible set iff $\mathrm{Av}(I, A)$ is generically stable. This gives us another nice characterization of generically stable types.

Although all Morley sequences are important objects, some of them might be more useful than others. Recall that one of the most important properties of Morley sequences in simple theories is the following fact (which we refer to as "Kim's Lemma"): Suppose the formula $\varphi(x, a)$ divides over a set A; then for every Morley sequence I in $\mathrm{tp}(a/A)$, we have that $\varphi(x, I) = \{\varphi(x, a') \colon a' \in I\}$ is inconsistent. A natural question is: is some version of Kim's Lemma true in dependent theories? In [OU] Alf Onshuus and the author showed (roughly) that it becomes true if one replaces "for all" Morley sequence with "there exists" one. The proof works, in fact, in a more general context of NTP_2 theories which includes both dependent and simple theories. Still, for many applications this result is insufficient. So one could ask: is there a stricter notion of a Morley sequence for which Kim's Lemma is true is stated?

We answer this question positively in section 4 of the paper, using the notion of "strict nonforking" introduced by Shelah in [She]. That is, we show that for strict Morley sequences Kim's Lemma is true. We also investigate the notion of strict nonforking and show that a sequence in a global nonforking heir is a strict Morley sequence. Furthermore, we show existence of such sequences over models by deducing existence of nonforking heirs from recent results of Chernikov and Kaplan. Let us also point out that Kim's Lemma for strict Morley sequences was shown by Chernikov and Kaplan independently in [CK].

We conclude the paper with further properties of strict nonforking. In particular we show weak version of local character ("bounded weight" for strictly nonforking sequences) and discuss different versions of weak orthogonality.

A note on terminology: in [Usv] we restricted the term "Morley sequence" to a sequence in a definable type constructed with respect to a definition. Since then it has become very clear that arbitrary nonforking sequences are central objects, hence deserve a special name. Any nonforking (indiscernible) sequence (not necessarily in a definable type) is called "Morley" in a simple theory, so we decided to adopt the name. Note that in [Usv] we only worked with indiscernible sets, and it was shown there that any such nonforking sequence is also a sequence with respect to a definition. Hence the terminology here does not differ, in fact, from the one in [Usv].

A note on the framework: Several proofs in this paper are carried out in the context of continuous model theory. We do this for several reasons. First, it has recently become clear that continuous model theory might be quite useful for studying dependent theories. Second, the terminology and the notation of continuous model theory are often very convenient. Third, some concepts developed in this paper require a slightly more sophisticated treatment when working in the continuous context, and we prefer to make sure everything works in this generality.

In spite of all this, the main examples, motivations, etc, behind the results in this paper come from classical model theory. Hence the reader can safely assume (having gotten used to the slightly different notation, explained in the next subsection), that everything is happening in the classical (discrete) context. In fact, most of the proofs would not change at all if we decided to eliminate all traces of continuous model theory from the paper. The only subsection which would be significantly simplified is the discussion of the eventual type, subsection 2.2. Still, since we believe that eventual type is a central concept, we decided to develop it in the slightly higher generality.

1.2. **Notations.** In this paper, T will denote a complete theory (sometimes continuous), τ will denote the vocabulary of T, L will denote the language of T. We will assume that everything is happening in the monster model of T which will be denoted by $\mathfrak{C}$. Elements and finite tuples of $\mathfrak{C}$ will be denoted a, b, c, sets (which are all subsets of $\mathfrak{C}$) will be denoted A, B, C, and models of T (which are all elementary submodels of $\mathfrak{C}$) will be denoted by M, N, etc.

Given an order type O and a sequence $\langle a_i : i \in O\rangle$, we often denote $a_{<i} = \langle a_j : j < i\rangle$, similarly for $a_{\leq i}$, $a_{>i}$, etc. We will often identify a tuple a or a sequence $\langle a_i : i \in O\rangle$ with the set which is its union, but it should always be clear from the context what we mean (although sometimes when confusions might arise, we make the distinction, e.g. $\mathrm{Av}(I, \cup I)$ will denote the average type of a sequence I over itself).

By $a \equiv_A b$ we mean $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. By $a \equiv_{\mathrm{Lstp},A} b$ we mean that a and b have the same Lascar strong type over A; we also write $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. By $a \equiv_{ind,A} b$ we mean that there is an A-indiscernible sequence containing both a and b. Obviously $a \equiv_{ind,A} b \Longrightarrow a \equiv_{\mathrm{Lstp},A} b \Longrightarrow a \equiv_A b$.

Since part of the time we are working with a continuous theory, we adopt some notation that seems convenient. For example, we denote the truth value of a formula $\varphi(x, a)$ with respect to a type $p \in S(A)$ with $a \in A$ by $\varphi^p(x, a)$. When no confusion arises, a sentence $\psi(a)$ will denote its truth value in the monster model. So $\varphi^p(x, a) = \psi(b)$ means, when working with a classical first order theory, that for every (some) realization $c \models p$ we have $\mathfrak{C} \models \varphi(c, a) \leftrightarrow \psi(b)$. Given a sequence of truth values $\langle t_i : i < \lambda\rangle$ which is eventually constant, we denote the value which appears co-boundedly many times by $\lim_{i<\lambda} t_i$ (clearly, this is the limit of the sequence in discrete topology).

1.3. **Preliminaries.** Recall that a theory (discrete or continuous) T is called *dependent* if for every indiscernible sequence $I = \langle a_i : i < \lambda\rangle$, a formula $\varphi(x, y)$ and c we have

$$\exists \lim_{i<\lambda} \varphi(a_i, c)$$

Recall that a sequence $I = \langle a_i : i \in O\rangle$ (where O is a linear ordering) is called *indiscernible* over a set A if the type of $a_{i_1}, \ldots, a_{i_k}$ over A depends only on the order between the indices $i_1, \ldots, i_k$ for every k. I is called an indiscernible *set* if the type above depends on k only.

We will call an ω-type $Q(x_0, x_1, \ldots)$ *indiscernible* over a set A if every (equivalently, some) realization of it is an indiscernible sequence over A. Note that x_i can be finite tuples. Clearly by compactness we can speak (slightly abusing the usual terminology) of realizations I of Q where I has any (infinite) order type. In other words, we treat Q as the Ehrenfeucht-Mostowski type of an indiscernible sequence. We will always assume, though, that our indiscernible sequences *do not have a last element.*

Slightly abusing notation, given an A-indiscernible sequence I, we will often say that Q as above is the type of I over A even if I is not of order type ω; similarly for Lascar type. Given two such sequences I and J, we write $I \equiv_A J$ if they are realizations of the same indiscernible (over A) type Q (but possibly not of the same order type); similarly for Lascar type. In other words, we write $I \equiv_A J$ for $\mathrm{EM}(I/A) = \mathrm{EM}(J/A)$.

Definition 1.1. Let I be an A-indiscernible sequence. We say that a sequence J *continues* I over A if $I^\frown J$ is A-indiscernible.

A hyperimaginary element a is said to be *bounded* over a set A if the orbit of a under the action of $\mathrm{Aut}(\mathfrak{C}, A)$ is small, i.e. of cardinality less than $|\mathfrak{C}|$. The *bounded closure of* A, denoted by $\mathrm{bdd}(A)$, is the collection of all elements bounded over A.

1.4. **Global Assumptions.** All theories mentioned here are *assumed to be dependent* unless stated otherwise. For the sake of clarity of presentation we also assume $T = T^{eq}$.

1.5. **Forking and splitting in dependent theories.** The following observations are well-known by now. The proofs can be found e.g. in [Usv], section 2.

Fact 1.2. (i) *Strong splitting implies dividing.*
(ii) *Lascar splitting implies forking.*
(iii) *There are boundedly many global types which do not fork over a given set A.*

Fact 1.3.
Let $I = \langle a_i : i < \lambda \rangle$ be such that

- $\mathrm{tp}(a_i/Aa_{<i})$ *does not fork over A*
- $\mathrm{Lstp}(a_i/Aa_{<i}) = \mathrm{Lstp}(a_j/Aa_{<i})$ *for every $j \geq i$.*

Then I is a Morley (nonforking) sequence over A (that is, it is indiscernible over A).

Fact 1.4. *The following are equivalent for a global type p and a set A:*

- *p forks over A*
- *p divides over A*
- *p splits strongly over A*
- *p Lascar splits over A*

We will use the facts above all the time, sometimes without quoting.

2. Average and Eventual types of indiscernible sequences

2.1. **Average types.** Proofs in this subsection are carried out in the setting of continuous logic, but the reader can easily ignore this and think only in terms of classical model theory.

Definition 2.1. Let $I = \langle a_i : i < \lambda \rangle$ be an indiscernible sequence B a set. We define the *average type* of I over B to be

$$\mathrm{Av}(I, B) = \{\varphi(x, b) = \lim_i \varphi(a_i, b)\}$$

.

Recall

Fact 2.2. *T is dependent if and only if for every I, B as above, we have* $\mathrm{Av}(I, B) \in \mathrm{S}(B)$.

Remark 2.3. Let I be an indiscernible sequence over a set A. Then $a \models \mathrm{Av}(I, A \cup I)$ if and only if $I^\frown\{a\}$ is indiscernible over A.

Observation 2.4. Let $I = \langle a_i : i \in O \rangle$ be an indiscernible sequence over a set A and let p be a global type which extends $\mathrm{Av}(I, A \cup I)$ and does not fork over A. Suppose that $I' = \langle a'_i : i \in O' \rangle$ satisfies $a'_i \models p{\restriction}AIa'_{<i}$. Then $J = I^\frown I'$ is indiscernible over A.

Proof. The mere existence of p implies that I is a nonforking sequence. Clearly I' is nonforking over AI based on A. Hence by Fact 1.3 it is enough to show that $a'_j \equiv_{\mathrm{Lstp},a_{<i}} a_i$ for all $i,j<\omega$. But this is also clear since $a'_j \models \mathrm{Av}(I, Aa_{<i})$, so $I^\frown a'_j$ is indiscernible, hence in fact $a'_j \equiv_{ind,a_{<i}} a_i$. QED$_{2.4}$

Observation 2.5. Let $I=\langle a_i : i\in O\rangle$ be an indiscernible sequence over a set A and let p be a global type which extends $\mathrm{Av}(I, A\cup I)$ and does not fork over A. Suppose that $I'\equiv_{\mathrm{Lstp},A} I$. Then $p\restriction AI' = \mathrm{Av}(I', A\cup I')$.

Proof. As before, existence of p implies that I is a nonforking sequence. Clearly I' is nonforking over A.

Let $c'\models p\restriction AI'$; clearly, it is enough to show that $I'^\frown c'$ is A-indiscernible. Let c be such that $Ic\equiv_{\mathrm{Lstp},A} I'c'$; it is enough to show that $I^\frown c$ is A-indiscernible. Note that p does not fork, hence does not Lascar-split over A. It follows that $c\models p\restriction AI = \mathrm{Av}(I, A\cup I)$, and we are done. QED$_{2.5}$

Remark 2.6. Note that in the previous Observation, p extends both the averages of I and I' over A. Moreover, if in the proof of the Observation we chose $c\models p\restriction AII'$, we would have that both $I^\frown c$ and $I'^\frown c$ are A-indiscernible. This property of a Morley sequence will become very important later.

Observation 2.7.
(i) Let I be an indiscernible sequence over a set A, p a global type extending $\mathrm{Av}(I, A\cup I)$ which does not fork over A. Then for every A-indiscernible sequence I' continuing I, we have $p\restriction AII' = \mathrm{Av}(I', AII')$.
(ii) Same conclusion if p is just a type over AII' which does not split strongly over A.

Proof.
(i) Let $I=\langle a_i : i\in O\rangle$, $I'=\langle a'_i : i\in O'\rangle$. Assume that $\varphi(x, a_{<i}a'_{<i})\in \mathrm{Av}(J, AIJ)$ where $\varphi(x, yy')$ is a formula over A. Then clearly since I' continues I it is also the case that $\varphi(x, a_{<2i})\in \mathrm{Av}(I, AI)\subseteq p$. As p does not fork, hence does not split strongly over A, we have $\varphi^p(x, a_{<i}, a'_{<i}) = \varphi^p(x, a_{<2i})$, as required.
(ii) Same proof.

QED$_{2.7}$

We have mentioned that a type $p\in \mathrm{S}(A)$ in a dependent theory has boundedly many global nonforking extensions. This does not mean, of course, that p is stationary (even if A is a model), unlike in the stable case. The following lemma shows that once given a Morley (nonforking) sequence in p, it completely determines a global nonforking extension.

Lemma 2.8.
(i) *Let I be an indiscernible sequence over a set A and let p, q be global types extending $\mathrm{Av}(I, A\cup I)$, both do not fork over A. Then $p=q$.*
(ii) *Let I be a Morley (nonforking) sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I, A\cup I)$ which does not fork over A. In other words, $\mathrm{Av}(I, A\cup I)$ is stationary over A.*

Proof. (i) Assume towards contradiction that $q \neq p$, so there is $\varphi(x, b)$ such that $\varphi(x, b) \in p$ but $\neg\varphi(x, b) \in q$.

Construct by induction on $\alpha < \omega$ sequence $J_\alpha = \langle a_i^\alpha : i < \omega \rangle$ such that

- $a_i^{2\alpha} \models p \restriction AbIJ_{<\alpha} a_{<i}^{2\alpha}$
- $a_i^{2\alpha+1} \models q \restriction AbIJ_{<\alpha} a_{<i}^{2\alpha+1}$

We claim that $J = J_0^\frown J_1^\frown \cdots$ is an indiscernible sequence. Once we have shown this, it yields an immediate contradiction to dependence, since $\varphi(a_i^\alpha, b) \neq \varphi(a_i^{\alpha+1}, b)$ for all α.

So we show by induction on α that $J^\alpha = I^\frown J_0^\frown \cdots^\frown J_\alpha$ is indiscernible (even over A). For $\alpha = 0$ this is true by Observation 2.4.

Let us take care of $\alpha = 1$ (the continuation is the same). By Observation 2.7 q extends $\mathrm{Av}(J^0, A \cup J^0)$. Now apply Observation 2.4 again.

(ii) This is just a restatement of (i).

$\mathrm{QED}_{2.8}$

Example 2.9. Let p be the type "at infinity" over $(\mathbb{Q}, <)$. Then it has *two* global nonforking extensions: one is the type of the cut $\mathbb{Q}^+$, which is finitely satisfiable in $\mathbb{Q}$, and the other one is the type "at infinity" over the monster model, which is the extension with respect to the definition of p. The first one is determined by a decreasing Morley sequence, whereas the second one - by an increasing one (those are precisely the two possible types of a Morley sequence in p).

2.2. **Eventual types.** So we have shown that a nonforking sequence I over A, it determines uniquely a global type that does not fork over A. It is natural to ask what this global extension looks like. A first guess might be $\mathrm{Av}(I, \mathfrak{C})$ (after all, this is the case in stable theories), but it is easy to see that this does not always work. For example, taking I to be an increasing sequence of rational numbers in the structure $(\mathbb{Q}, <)$, which is clearly nonforking over the empty set, the global nonforking extension is the type "at infinity" and not the global average (which is the type of the cut of I).

The reader might recall from section 2 of the authors previous article [Usv] that when one starts with a "stable-like" (more precisely, generically stable) type (so when I is an indiscernible *set*) then $\mathrm{Av}(I, \mathfrak{C})$ is indeed the unique global nonforking extension of $\mathrm{Av}(I, A \cup I)$. But this case is in a sense "too easy" and does not reflect the subtleties of the general situation. As a matter of fact, we will see later in the paper that the global average is the unique nonforking extension *if and only if* it is generically stable if and only if I is an indiscernible set.

So for the general case we will need to apply a slightly more careful analysis and introduce a different notion of a "limit" type of a nonforking sequence. The ideas behind this notion generalize Poizat [Poi79], Shelah [She] and, more recently, Adler [Adl], who work with co-heir and "special" (see below) sequences, but as we shall see, everything generalizes to an arbitrary Morley sequence quite easily. To the best of our knowledge, Adler was the first to actually give the central notion of this subsection a name. We will follow his terminology and call the "limit" type we are interested in "the eventual type of the sequence". In this subsection we work in the continuous setting. This is the only place in the paper where the proofs would become somewhat simpler if we decided to remain in the classical context. But since the concept of eventual type seems important enough, we decided to make an effort and do everything carefully in the more general framework.

Observation 2.10. For every $\varphi(x, y)$ (maybe with parameters, maybe y is empty) and $\varepsilon > 0$ there exists $k < \omega$ such that there does not exist an infinite indiscernible sequence $\langle b_i : i < \omega\rangle$ and $c \in \mathfrak{C}$, such that for all $i < k$ we have

$$|\varphi(b_i, c) - \varphi(b_{i+1}, c)| > \varepsilon$$

We denote *minimal* such k by $k_{\varphi,\varepsilon}$.

For notational convenience, let us introduce the following notation for the number for ε-alternations (see also Adler [Adl] for a related notion of "alternation rank").

Definition 2.11.
(i) Let $I = \langle a_i : i \in O\rangle$ be an indiscernible sequence, $\varphi(x, b)$ a formula, $\varepsilon > 0$. We denote by $\mathrm{alt}(\varphi(x, b), \varepsilon, I)$ the maximal k such that there exist $i_0 < \ldots < i_k \in O$ such that $|\varphi(a_{i_j}, b) - \varphi(a_{i_{j+1}}, b)| > \varepsilon$ for all $j < k$. Clearly $\mathrm{alt}(\varphi(x, b), \varepsilon, I) \leq k_{\varphi(x,b),\varepsilon}$.
(ii) When we omit I, we mean the maximum over all I, that is, $\mathrm{alt}(\varphi(x, b), \varepsilon) = k_{\varphi(x,b),\varepsilon}$.
(iii) When we replace I with an indiscernible type Q, we mean the maximum over all realizations of Q.

The following notion is due to Poizat [Poi79].

Definition 2.12.
(i) We call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $\mathrm{tp}(I/A)$, there exists c such that $I_1{}^\frown c$ and $I_2{}^\frown c$ are A-indiscernible.
(ii) In this case we call the type $\mathrm{tp}(I/A)$ *special* over A.

It is known in classical model theory (see e.g. [Adl]) that a sequence $I = \langle a_i : i \in O\rangle$ is special over A if and only if there exists a global type p which does not split over A and $a_i \models p{\restriction}Aa_{<i}$ for all $i \in O$. The same proof works in the continuous context (as we shall see), but this is not quite what we are looking for: the assumption that p does not split over A is too strong for us, and we would like to replace it with forking, equivalently, Lascar splitting. This is why we will find it more convenient to work with a Lascar strong type of a sequence instead of a type.

Definition 2.13.
(i) We call an A-indiscernible sequence *weakly special* (or *Lascar special*) if for any two realizations I_1 and I_2 of $\mathrm{Lstp}(I/A)$, there exists c such that $I_1{}^\frown c$ and $I_2{}^\frown c$ are A-indiscernible.
(ii) In this case we say that the type $\mathrm{tp}(I/A)$ is *weakly special* (or *Lascar special*) over A and that Lascar strong type $\mathrm{Lstp}(I/A)$ is *special* over A.

The following follows by induction and compactness:

Observation 2.14. Let Q be a Lascar special type over A. Then for every collection $\langle I_i : i < \lambda \rangle$ of $I_i \models Q$ of the same Lascar strong type over A and for every order type O there exists $J \models Q$ of order type O which continues all the I_i's over A.

Observation 2.15. Let Q be a Lascar special type over A, $\varphi(x, b)$ a formula, $\varepsilon > 0$, $J, J' \models Q$, $J \equiv_{\mathrm{Lstp},A} J'$, such that $\mathrm{alt}(\varphi(x, b), \varepsilon, J) = \mathrm{alt}(\varphi(x, b), \varepsilon, J') = \mathrm{alt}(\varphi(x, b), \varepsilon, Q)$. Denote $q = \mathrm{Av}(J, Ab)$, $q' = \mathrm{Av}(J', Ab)$. Then

$$|\varphi^q(x, b) - \varphi^{q'}(x, b)| \leq 2\varepsilon$$

Proof. Denote $\rho = \varphi^q(x, b)$, $\rho' = \varphi^{q'}(x, b)$. So $\lim_J \varphi(x, b) = \rho$, $\lim_{J'} \varphi(x, b) = \rho'$. Let a be such that both $J^\frown a$ and $J'^\frown a$ are indiscernible. Clearly both satisfy Q, and so by the assumption the value of $\varphi(x, b)$ can not change by more than ε in any sequences extending J or J'. So $|\rho - \varphi(a, b)| \leq \varepsilon$ and $|\rho' - \varphi(a, b)| \leq \varepsilon$, hence $|\rho - \rho'| \leq 2\varepsilon$ as required. $\mathrm{QED}_{2.15}$

Observation 2.16. Let I be a Lascar special sequence over A, $\varphi(x, b)$ a formula, $\varepsilon > 0$. Let $J \equiv_{\mathrm{Lstp},A} I$ be such that $\mathrm{alt}(\varphi(x, b), \varepsilon, J) = \mathrm{alt}(\varphi(x, b), \varepsilon, Q)$. Then I can be extended to I' such that $|\lim_{I'} \varphi(x, b) - \lim_J \varphi(x, b)| \leq \varepsilon$.

Proof. Let $J' \models \mathrm{Lstp}(I/A)$ be an ω-sequence which continues both I and J; clearly,

$$|\lim_{J'} \varphi(x, b) - \lim_J \varphi(x, b)| \leq \varepsilon$$

. $\mathrm{QED}_{2.16}$

Definition 2.17. Let $\varphi(x, b)$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(x, b)$ if $\lim_{J'} \varphi(x, b)$ is constant for all J' continuing J.

Lemma 2.18. *Let Q be a Lascar special type over A, $\varphi(x, b)$ a formula.*

(i) *There exists $J \models Q$ which eventually determines $\varphi(x, b)$.*

(ii) *For every $I, J \models Q$ with $J \equiv_{\mathrm{Lstp},A} I$ which eventually determine $\varphi(x, b)$ we have $\lim_I \varphi(x, b) = \lim_J \varphi(x, b)$, that is, the "eventual value" of $\varphi(x, b)$ depends only on Lascar strong type of J over A, and not on the choice of J. We call this number the* eventual value of $\varphi(x, b)$ with respect to $\mathrm{Lstp}(J/A)$.

(iii) *Every $I \models Q$ can be extended to $J \models Q$ that eventually determines $\varphi(x, b)$. For every J, J' which continue I and eventually determine $\varphi(x, b)$ we have $\lim_J \varphi(x, b) = \lim'_J \varphi(x, b)$.*

Proof. (i) Let $\varepsilon > 0$, and let $J_\varepsilon \models Q$ be such that $\mathrm{alt}(\varphi(x, b), \varepsilon, J) = \mathrm{alt}(\varphi(x, b), \varepsilon, Q)$ and denote $\rho_\varepsilon = \lim_{J_\varepsilon} \varphi(x, b)$.

Let $\zeta < \varepsilon$. As in previous Observation we can extend J_ε to J'_ε such that $|\lim_{J'_\varepsilon} \varphi(x, b) - \lim_{J_\zeta} \varphi(x, b)| \leq \zeta$. Also, clearly $|\lim_{J'_\varepsilon} \varphi(x, b) - \lim_{J_\varepsilon} \varphi(x, b)| \leq \varepsilon$. So $|\rho_\zeta - \rho_\varepsilon| \leq \zeta + \varepsilon$.

Denote $J_n = J_{\frac{1}{n}}$, $\rho_n = \rho_{\frac{1}{n}}$. The inequality above amounts to $|\rho_n - \rho_m| \leq \frac{1}{n} + \frac{1}{m}$. So ρ_n converges; denote $\rho = \lim \rho_n$.

Note that we may assume that if $\zeta < \varepsilon$ then J_ζ extends J_ε. This is because we can replace J_ζ with $J_\varepsilon J'$ where J' is an ω-sequence which continues both J_ζ and J_ε. So working only with $\varepsilon_n = \frac{1}{n}$, we can do the replacement process above for all n by simple induction on the natural numbers, obtaining $J_n \subseteq J_{n+1}$, and for simplicity assume $\mathrm{otp}(J_n) = n\omega$.

By compactness there is J of order type ω^2 which extends all the J_ε's. Clearly, if J' extends J, then the value of $\lim_{J'} \varphi(x, b)$ has to agree with $\lim_{J_n} \varphi(x, b) = \rho_{\frac{1}{n}}$ up to $\frac{1}{n}$, hence has to equal ρ, which completes the proof of the first part of the claim.

(ii) Now suppose J and I both eventually determine $\varphi(x, b)$ and satisfy the same Lascar strong type. We can find J' which continues both; so

$$\lim_I \varphi(x, b) = \lim_{IJ'} \varphi(x, b) = \lim_{J'} \varphi(x, b) = \lim_{JJ'} \varphi(x, b) = \lim_J \varphi(x, b)$$

as required.

(iii) Let $J' \models Q$ eventually determine $\varphi(x, b)$ (see clause (i) above), and let J continue both I and J', clearly J is as required.

For uniqueness, note that $\mathrm{Lstp}(J/A) = \mathrm{Lstp}(I/A)$ for every J continuing I, and use clause (ii).

$\mathrm{QED}_{2.18}$

We are now ready to make the main definition of the subsection.

Definition 2.19. (i) Given a Lascar special sequence I over A and a formula $\varphi(x, b)$ we denote the *eventual value* of $\varphi(x, b)$ with respect to I (see Lemma 2.18 (ii)) by $\varphi^I(x, b)$. If Q is a type which implies $\mathrm{Lstp}(I/A)$ (e.g. $Q = \mathrm{tp}(I/M)$ where M is a model containing A, or just $Q = \mathrm{tp}(I/\mathrm{bdd}^{heq}(A))$), we write $\varphi^Q(x, b)$ for $\varphi^I(x, b)$ (clearly, $\varphi^I(x, b)$ depends only on Q).

(ii) Given a Lascar special sequence I over A and a set C we define the *eventual type* of I over C, $q = \mathrm{Ev}(I, C)$ as follows: given a formula $\varphi(x, b)$ over C, let $\varphi^q(x, b)$ equal $\varphi^I(x, b)$.

Again, if Q implies $\mathrm{Lstp}(I/A)$, we write $\mathrm{Ev}(Q, C)$.

(iii) If we omit C, we mean $C = \mathfrak{C}$ (so we obtain a global type).

Remark 2.20. $\mathrm{Ev}(I, C)$ as above is well-defined and is a complete type over C.

Proof. Compactness and existence + uniqueness of eventual value, that is, Lemma 2.18.

$\mathrm{QED}_{2.20}$

Remark 2.21. Let I be a Lascar special sequence over A. Then $\mathrm{Ev}(I, A \cup I) = \mathrm{Ev}(I){\restriction}AI = \mathrm{Av}(I, A \cup I)$.

Proof. So $a \models \mathrm{Ev}(I, A \cup I)$ if and only if for every formula $\varphi(x, a_{<i})$ over A and for some/every continuation J of I which eventually determines $\varphi(x, a_{<i})$ we have

$$\varphi(a, a_{<i}) = \varphi^J(x, a_{<i}) = \lim_J \varphi(x, a_{<i}) = \lim_I \varphi(x, a_{<i})$$

The last equality is true because J is an indiscernible sequence continuing I. So clearly $a \models \mathrm{Ev}(I, A \cup I)$ if and only if $a \models \mathrm{Av}(I, A \cup I)$. $\text{QED}_{2.21}$

Observation 2.22. Let I be Lascar special over A. Then $\mathrm{Ev}(I)$ does not Lascar-split (equivalently, does not fork) over A.

Proof. Let $b \equiv_{\mathrm{Lstp},A} b'$, $\varphi(x, y)$ a formula. Suppose that the eventual value of $\varphi(x, b)$ is determined by $J \equiv_{\mathrm{Lstp},A} I$. Let J' be such that $J'b' \equiv_{\mathrm{Lstp},A} Jb$. Clearly J' eventually determines $\varphi(x, b')$, $J' \models Q$ and $\lim_J \varphi(x, b) = \lim_{J'} \varphi(x, b')$. By uniqueness of eventual value, J' determines $\varphi^I(x, b')$, so we are done. $\text{QED}_{2.22}$

Discussion 2.23. Note that we could have gone through the same process with special sequences instead of Lascar special, replacing "Lascar strong type" with "type" in all the statements and proofs above. In this case, the eventual type of a special sequence I over A depends only on $Q = \mathrm{tp}(I/A)$, and we denote it by $\mathrm{Ev}(Q)$. In the Observation above we would get then:

Observation 2.24. Let Q be a special type over A. Then $\mathrm{Ev}(Q)$ does not split (and therefore does not fork) over A.

Corollary 2.25.
(i) *Let I be Lascar special over A. Then for every set C the type* $\mathrm{Ev}(I, C)$ *does not fork over A.*
(ii) *Let Q be special over A. Then for every set C the type* $\mathrm{Ev}(Q, C)$ *does not either split or fork over A.*

The following is a known characterization of special sequences mentioned above:

Lemma 2.26. *An A-indiscernible type $Q(x_i)$ is special if and only if there exists a global type $q = q(x)$ which does not split over A such that any realization $I = \langle a_i : i \in O \rangle$ of Q satisfies $a_i \models q{\restriction}Aa_{<i}$.*

Proof. If any realization of Q is a nonsplitting sequence in q over A, then given $I, I' \models Q$ simply choose $c \models q{\restriction}AII'$. By nonsplitting both $I^\frown c$ and $I'^\frown c$ are indiscernible.

On the other hand, assume that Q is special over A and let $q = \mathrm{Ev}(Q)$. Then by Corollary 2.25 q does not split over A and clearly for any realization I of Q and for every $a_i \in I$ we have $a_i \models \mathrm{Ev}(Q){\restriction}Aa_{<i}$. $\text{QED}_{2.26}$

We are more interested in the similar characterization of Lascar special sequences:

Lemma 2.27. *An A-indiscernible sequence I is Lascar special if and only if it is a Morley (nonforking) sequence over A.*

Proof. A nonforking sequence is Lascar special by Remark 2.6.

On the other hand, if I is Lascar special, then by Observation 2.22 $q = \mathrm{Ev}(I)$ does not fork over A, hence $\mathrm{Av}(I, A \cup I) = \mathrm{Ev}(I, A \cup I)$ (see Remark 2.21) does not fork over A; so I is nonforking. $\mathrm{QED}_{2.27}$

Discussion 2.28. Comparing the two lemmas above, the reader can notice that unlike special sequences, which are characterized in terms of existence of a certain *global* type, Lascar special types have an *internal* characterization (it can be seen from the type whether or not it is nonforking). This is part of the reason we believe that Lascar special (equivalently, Morley) sequences are a better notion.

Note that simply replacing "nonforking" with "nonsplitting" in the characterization of Lascar special sequences will *not* lead to a characterization of special sequences. This is because nonsplitting does not satisfy the extension axiom; in fact, locally nonsplitting does not imply nonforking (unless one works over slightly saturated models). See section 6 of [Usv] for examples of forking nonsplitting sequences.

Proposition 2.29. (i) *Let I be a Lascar special sequence over A. Then* $\mathrm{Ev}(I)$ *is the unique global extension of* $\mathrm{Av}(I, A \cup I)$ *that does not split over A.*
(ii) *Let Q be a special type over A, $I \models Q$. Then* $\mathrm{Ev}(Q)$ *is the unique global extension of* $\mathrm{Av}(I, A \cup I)$ *that does not split over A.*

Proof. We have already shown in Observation 2.22 that $\mathrm{Ev}(Q)$ does not split over A. By Remark 2.21 $\mathrm{Ev}(Q)$ extends $\mathrm{Av}(I, A \cup I)$. Uniqueness follows now from Lemma 2.8. $\mathrm{QED}_{2.29}$

3. Indiscernible sets and generic stability

In this section we will characterize generically stable types in terms of the eventual type of their Morley sequences. The proofs are carried out in the continuous context.

3.1. **Indiscernible sets.** Having achieved in the previous section a pretty good understanding of an arbitrary nonforking sequence I, we will try to draw more conclusions assuming that I is an indiscernible set.

Recall

Definition 3.1. Let $I = \langle b_i : i \in O \rangle$ be an infinite indiscernible sequence. We say that a formula $\varphi(x, y)$ is *stable* for I if for every $c \in \mathfrak{C}$ and $\varepsilon > 0$, either the set $\{i \in I \colon \varphi(b_i, c) < \varepsilon\}$ or the set $\{i \in I \colon \varphi(b_i, c) > \varepsilon\}$ is finite.

Restating Observation 2.10 for indiscernible sets we get

Observation 3.2. If $\langle b_i : i \in I \rangle$ is an infinite indiscernible set, then every $\varphi(x, y)$ is stable for $\langle b_i \rangle$. Moreover, for every $\varphi(x, y)$ and $\varepsilon > 0$ there exists $k = k_{\varphi,\varepsilon} < \omega$ such that for every $c \in \mathfrak{C}$, for all but k-many $i < \lambda$ we have

$$|\varphi(b_{i_1}, c) - \varphi(b_{i_2}, c)| \leq \varepsilon$$

Observation 3.3. Let I_1, I_2 be equivalent indiscernible sets, that is, there is an indiscernible sequence (set) J continuing both. Then $\mathrm{Av}(I_1, \mathfrak{C}) = \mathrm{Av}(I_2, \mathfrak{C})$.

Proof. Let $\varphi(x, b)$ be a formula. By Observation 3.2 it is easy to see that

$$\lim_{I_1} \varphi(x, b) = \lim_{I_1 J} \varphi(x, b) = \lim_{J} \varphi(x, b)$$

and similarly for I_2. $\mathrm{QED}_{3.3}$

Observation 3.4. (i) Let Q be a special type of an indiscernible set. Then $\mathrm{Ev}(Q) = \mathrm{Av}(I, \mathfrak{C})$ for any realization I of Q.

(ii) Let I be a nonforking sequence over A which is also an indiscernible set. Then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.

Proof. For (i), let $I \models Q$, $\varphi(x, b)$ a formula, J continuing I eventually determines $\varphi(x, b)$. By the previous observation $\lim_I \varphi(x, b) = \lim_J \varphi(x, b)$, as required.

(ii) Similar. $\mathrm{QED}_{3.4}$

As a consequence we can conclude the main result of section 2 of [Usv]:

Corollary 3.5. *Let $I = \langle b_i : i < \omega \rangle$ be a nonforking sequence over A which is also an indiscernible set. Denote $p = \mathrm{Av}(I, A \cup I)$. Then p has a unique nonsplitting extension over $\mathfrak{C}$, which equals $\mathrm{Av}(I, \mathfrak{C})$. In particular, $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A.*

3.2. Generically stable types.

We recall the main equivalences and facts from [Usv].

Theorem 3.6. *The following are equivalent for a type $p \in \mathrm{S}(A)$:*

(i) *There is a nonforking sequence in p which is an indiscernible set.*

(ii) *p is extensible and every nonforking sequence in it is an indiscernible set.*

(iii) *p is definable over $\mathrm{acl}(A)$ and some/every Morley sequence with respect to a definition of p is an indiscernible set.*

(iv) *Some/every global nonforking extension of p is both definable over and finitely satisfiable in a model of density character $|A| + |T|$ containing A.*

In case one/all of the equivalences above hold/s, we call p *generically stable*.

Theorem 3.7. *Let $p \in \mathrm{S}(A)$ be a generically stable type definable over A (e.g. $A = \mathrm{acl}(A)$, or A is a Morley sequence in p, or just p is finitely satisfiable in A, see section 5 of* [Usv]*). Then p is stationary.*

We now connect eventual types to generic stability.

Proposition 3.8. *Let I be a Morley sequence over A, and assume that $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. Then $\mathrm{Av}(I, A)$ is generically stable, hence so is $\mathrm{Av}(I, \mathfrak{C}) = \mathrm{Ev}(I)$, and I is an indiscernible set.*

Proof. First, note that $q = \mathrm{Av}(I, \mathfrak{C})$ does not split over A (since it equals $\mathrm{Ev}(Q)$), so it is enough to show that I is an indiscernible set. Without loss of generality we may assume $\mathrm{otp}(I) = \omega$.

Let $I = \langle a_i : i < \omega \rangle$. Construct $I' = \langle a_{\omega+i} : i < \omega \rangle$ by $a_i \models q \restriction A a_{<i}$. Clearly II' is an A-indiscernible sequence, and it is enough to show that I' is an indiscernible set.

Assume $\varphi(a_{<\omega+i}, a_{\omega+i}, a_{\omega+i+1}, a_{\omega+i+2}) = 0$ and let us show $\varphi(a_{<\omega+i}, a_{\omega+i+1}, a_{\omega+i}, a_{\omega+i+2}) = 0$. The general case will work in the same way. Let $\varepsilon > 0$.

- By indiscernibility $\varphi(a_{<i}, a_{\omega+i}, a_{\omega+i+1}, a_{\omega+i+2}) = 0$.
- $\varphi^q(a_{<i}, a_{\omega+i}, a_{\omega+i+1}, x) = 0$ (by the choice of $a_{\omega+i+2}$)
- $q = \mathrm{Av}(I, \mathfrak{C})$, so $\lim_I \varphi(a_{<i}, a_{\omega+i}, a_{\omega+i+1}, x) = 0$.
- For all $k < \omega$ big enough we have $\varphi(a_{<i}, a_{\omega+i}, a_{\omega+i+1}, a_k) \leq \varepsilon$.
- Similarly, $\varphi(a_{<i}, a_{\omega+i}, x, a_k) \in \mathrm{Av}(I, \mathfrak{C})$, hence for all $k < \ell < \omega$ big enough we have $\varphi(a_{<i}, a_{\omega+i}, a_\ell, a_k) \leq 2\varepsilon$.
- By indiscernibility $\varphi(a_{<i}, a_{\omega+i+1}, a_\ell, a_k) \leq 2\varepsilon$.
- By indiscernibility again (recall that $\ell > k$) we have $\varphi(a_{<i}, a_{\omega+i+1}, a_{\omega+i}, a_k) \leq 2\varepsilon$ for all $k < \omega$ big enough. So $\varphi^q(a_{<i}, a_{\omega+i+1}, a_{\omega+i}, x) \leq 2\varepsilon$.
- $\varphi^q(a_{<i}, a_{\omega+i+1}, a_{\omega+i}, a_{\omega+i+2}) \leq 2\varepsilon$.

Since ε was arbitrary, we are done. $\mathrm{QED}_{3.8}$

So we have obtained a new characterization of a generically stable type. Note that clauses (iv) and (v) in the Theorem below with "sequence" replaced with "set" appears already in [Usv]. It is interesting to find out that the requirement of I being an indiscernible set is not needed and follows from the fact that the average type does not fork over A.

Theorem 3.9. *Let $p \in \mathrm{S}(A)$. Then the following are equivalent:*

(i) *p is generically stable.*
(ii) *For some Morley sequence I in p we have $Ev(I) = \mathrm{Av}(I, \mathfrak{C})$.*
(iii) *p is extensible and for all Morley sequences I in p we have $Ev(I) = \mathrm{Av}(I, \mathfrak{C})$.*
(iv) *For some indiscernible sequence I in p, $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A.*
(v) *p is extensible and for every Morley sequence I in p, $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A.*

Proof. (i) $\Rightarrow$(iii) By Corollary 3.5.
(iii) $\Rightarrow$(ii) Clear.
(ii) $\Rightarrow$(i) By Proposition 3.8.
(ii) $\Rightarrow$(iv) Clear by now.
(iv) $\Rightarrow$(ii) Clearly $\mathrm{Av}(I, \mathfrak{C})$ extends $\mathrm{Av}(I, A \cup I)$; since it also does not fork over A, we immediately get that I is a nonforking sequence, and by stationarity of $\mathrm{Av}(I, A \cup I)$ over A (see Corollary 2.25) $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
(iv) $\Longleftrightarrow$(v) is easy by stationarity. $\mathrm{QED}_{3.9}$

4. On strictly free extensions

In this section we will investigate a strong notion of a Morley sequence in a dependent theory, which turns out to have some very good properties.

The following definitions are equivalent to the ones given in section 5 of [She].

Definition 4.1. (i) Let $A \subseteq B$. We say that a type $p \in S(B)$ is a strictly nondividing extension of $p \restriction A$ if for every $a \models p$
- $tp(a/B)$ does not divide over A
- $tp(B/Aa)$ does not divide over A.

(ii) Let $A \subseteq B$. We say that a type $p \in S(B)$ is a *strictly free* (or strictly nonforking) extension of $p \restriction A$ if there exists a global type q extending p which is a strictly nondividing extension of $p \restriction A$. We also say that p is *strictly free* over A. If $a \models p$, we write $a \downarrow^{st}_A B$.

Remark 4.2. We will say that a type $p \in S(B)$ *co-divides* over a set A if there is $a \models p$ such that $tp(B/Aa)$ divides over A. In other words, p co-divides over A if there exist $a \models p$, a formula $\varphi(x, b) \in p$ such that $\varphi(a, y)$ divides over A.

Clearly $p \in S(B)$ is a strictly non-dividing extension over A if and only if it does not divide and does not co-divide over A. We will make several observations about co-dividing.

Observation 4.3. (T any theory) Let $A \subseteq B \subseteq C$, $p \in S(C)$ is a heir of $p \restriction B$ and $p \restriction B$ does not co-divide over A. Then p does not co-divide over A.

Proof. Assume towards contradiction that $a \models p$, $\varphi(x, c) \in p$, $\varphi(a, y)$ divides over A. Let $\varphi(x, b) \in p \restriction B$ (recall that p is a heir over B). Clearly $a \models p \restriction B$ and $\varphi(a, y)$ exemplify co-dividing of $p \restriction B$, a contradiction. QED$_{4.3}$

Observation 4.4. (T any theory) Let A be a set, N an $(|A| + |T|)^+$-saturated model containing A, $p \in S(N)$ does not split over A. Let q be the unique global extension of p which does not split over A (see e.g. [Usv], Lemma 2.23). Then q is a heir of p.

Proof. Let $\varphi(x, c) \in q$, and choose $b \in N$ of the same type as c over A. By nonsplitting, clearly $\varphi(x, b) \in q \restriction N = p$. QED$_{4.4}$

Corollary 4.5. *Let N be saturated enough over a model M, $p \in S(N)$ is a strongly nondividing extension of $p \restriction M$ (in particular p does not split over M). Let q be the unique global extension of p which does not fork/split over M. Then q is a strongly nondividing extension of $p \restriction M$.*

Proof. Clearly q does not divide over M, so we only need to show that it also does not co-divide, which follows from the observations above. QED$_{4.5}$

In the arguments above nonsplitting can be replaced with Lascar-nonsplitting, and a model M with a set A. We can conclude the following quite desirable characterization of strict nonforking:

Corollary 4.6. *Let N be saturated enough over A. Then*

(i) *A type $p \in \mathrm{S}(N)$ is strictly free over A if and only if for every $a \models p$*
- $a \perp_A N$
- $\mathrm{tp}(N/Aa)$ *does not divide over A.*

(ii) *If $p \in \mathrm{S}(N)$ is a heir of $p \restriction A$ and does not fork over A, it is strictly free over A. In particular this is the case if p is both a heir and a co-heir of $p \restriction A$.*

Proof. (i) Follows directly from the previous Corollary. For (ii), note that if p is a heir of $p \restriction A$, then for every $a \models p$ the type $\mathrm{tp}(N/Aa)$ is finitely satisfiable in A (hence does not divide over A). $\mathrm{QED}_{4.6}$

Definition 4.7. (i) Let O a linear order, A a set. We call a sequence $I = \langle a_i : i \in O \rangle$ a *strict Morley sequence over B based on A* if it is an indiscernible sequence over B and $\mathrm{tp}(a_i/Ba_{<i})$ is strictly free over A for all $i \in O$.
(ii) In the previous definition, we omit "based on A" if $A = B$.
(iii) Let $p \in \mathrm{S}(B)$ be a type. We call a sequence I a *strict Morley sequence in p* if it is a strict Morley sequence over B of realizations of p.

Definition 4.8. (i) We call a type $p \in \mathrm{S}(A)$ *strictly extendible* if there exists a global type extending p which is strictly free over A.
(ii) We call a set A a *strict extension base* if every type over A is strictly extendible.

In [OU] Alf Onshuus and the author show a weak version of Kim's Lemma, that is: if $\varphi(x, a)$ divides over a set A and witnessed by a sequence J and $\mathrm{tp}(J/A)$ is extendible, then there is a Morley sequence I in $\mathrm{tp}(a/A)$ which witnessed dividing, that is, the set $\varphi(x, I)$ is inconsistent. Having one such Morley sequence is often insufficient for applications, though, and we were wondering whether for some stronger notion of a Morley sequence "there is" above can be replaced with "for all". It turns out that strict nonforking provides us with exactly what we need.

Proposition 4.9. *Suppose that $\varphi(x, a)$ divides over a set A witnessed by an A-indiscernible sequence J such that $\mathrm{tp}(J/A)$ is extendible. Assume furthermore that $\mathrm{tp}(a/A)$ is strictly extendible. Then every strict Morley sequence I in $\mathrm{tp}(a/A)$ witnesses dividing, that is, the set*

$$\varphi(x, I) = \{\varphi(x, a') \colon a' \in I\}$$

is inconsistent.

Remark 4.10. Note that since $\mathrm{tp}(a/A)$ is strictly extendible, there exist strict Morley sequences in this type.

Proof. Let p be a global type extending $\mathrm{tp}(a/A)$ which is strictly free over A. Let M be an $(|A| + |T|)^+$-saturated model containing A. Let $a' \models p \restriction M$. Without loss of generality (by applying an automorphism), $a' = a$ and J starts with a. Denote $J = \langle a_\alpha : \alpha < \omega \rangle$.

Since p is strictly free over A, $\mathrm{tp}(M/Aa)$ does not divide over A. Since J is indiscernible over A, it follows that there exists $J' \equiv_{Aa} J$ which is indiscernible over N.

Let $Q = \mathrm{tp}(J/M) = Q(x_0, x_1, x_2, \ldots)$. Construct a sequence $\langle J_i : i < \omega\rangle$ in M by $J_i \models Q{\restriction}A \cup J_{<i}$. Denote $J_i = \langle a_{i,\alpha} \colon \alpha < \omega\rangle$.

Note that

(*) There is a unique type of "an infinite sequence in p over A". In other words, let (for $\ell = 1, 2$) $I_\ell = \langle b_{\ell,\beta\colon \beta<\omega}\rangle$ be such that $b_{\ell,\beta} \models p{\restriction}Ab_{\ell,<\beta}$. Then $\mathrm{tp}(I_1/A) = \mathrm{tp}(I_2/A)$. Let us call this type P.

(**) For any $I_1, I_2 \models P$, the set $\varphi(x, I_1)$ is consistent if and only if $\varphi(x, I_1)$ is.

(***) Let $\eta \colon \omega \to \omega$. Since J is an indiscernible sequence over M of realizations of $p{\restriction}M$, one easily sees that the sequence $I_\eta = \langle a_{i,\eta(i)} \colon i < \omega\rangle$ realizes P.

So we can conclude

⧫ For any $I \models P$, the set $\varphi(x, I)$ is inconsistent.

If not, by (**), (***) above we see that

∘ $\varphi(x, J_i)$ is k-inconsistent for all i

∘∘ For any $\eta \colon \omega \to \omega$, the set $\varphi(x, I_\eta)$ is consistent.

This is an easy contradiction to dependence. The most straightforward argument would be that this gives the tree property of the second kind, TP_2, which implies the independence property. Since we haven't defined TP_2 here, let us give some details. By compactness and an Erdös-Rado argument, we may assume that the sequences J_i are mutually A-indiscernible, and $\mathrm{otp}(J_i) = \mathbb{Q}$. We now claim that for every $\eta \colon \omega \to \omega$ the set

$$\{\varphi(x, b_{i,\eta(i)})\} \cup \{\neg\varphi(x, b_{i,\alpha})\}_{\alpha \neq \eta(i)}$$

is consistent. Indeed, if $d \models \varphi(x, I_\eta)$, then since $\varphi(x, J_i)$ is k-inconsistent, we have that $\neg\varphi(x, b_{i,\alpha})$ for almost all α. By mutual indiscernibility (and choosing an infinite subsequence of J_i), the consistence of the set above follows. This contradicts dependence.

So we have shown ⧫ above, which states that if p is a strictly free global extension of $\mathrm{tp}(a/A)$, and P is constructed as in (*) above, then any $I \models P$ exemplifies dividing of $\varphi(x, a)$.

Now let I be *any* strict Morley sequence in $\mathrm{tp}(a/A)$, and let p be a global extension of $\mathrm{Av}(I, A \cup I)$ which is strictly free over A. Constructing P as in (*) above, clearly $I \models P$. So I exemplifies dividing of $\varphi(x, a)$, as required.

$\mathrm{QED}_{4.10}$

We have assumed in the Proposition above that a certain type is strictly extendible. It would be nice to know that there are "enough" strictly extendible types. We will show that any type over a model is such. In fact, we show more: we prove that any type p over a model M has a global extension q which is a nonforking heir, that is, q is a heir of p which does not fork over M. The proof relies on the following result due to Chernikov and Kaplan [CK]:

Fact 4.11. *Let M be a model, $p \in \mathrm{S}(M)$. Assume that*

$$p \vdash \bigvee_{i<k} \varphi_i(x, b_i) \vee \bigvee_{j<n} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over M, $\varphi_i(x, b_i)$ does not divide over M for all i, and $\psi_j(x, c_j)$ divides over M for all j. Then there are $m < \omega$ and automorphisms $\sigma_0, \dots, \sigma_{m-1}$ over M such that

$$p \vdash \bigvee_{i<k} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))$$

Corollary 4.12. *Let M be a model, $p \in \mathrm{S}(M)$. Then there exists a global heir of p which does not fork over M.*

Proof. Assume not, then

$$p \vdash \bigvee_{i} \varphi_i(x, b_i) \vee \bigvee_{j} \psi_j(x, c_j)$$

where $\varphi_i(x, y_i), \psi_i(x, z_j)$ are over M, $\neg\varphi_i(x, m) \in p$ for every $m \in M$ for all i, and $\psi_j(x, c_j)$ divides over M for all j. By Fact 4.11, there is are automorphisms $\sigma_0, \dots, \sigma_{m-1}$ over M such that

$$p \vdash \bigvee_{i} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))$$

Continuing as in the well-known proof of existence of heirs, we get for some $\theta(x) \in p$

$$\models \exists \bar{y} \forall x [\theta(x) \to \bigvee_{i} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))]$$

hence

$$M \models \exists \bar{y} \forall x [\theta(x) \to \bigvee_{i} \bigvee_{\ell<m} \varphi_i(x, \sigma_\ell(b_i))]$$

which is clearly impossible. $\text{QED}_{4.12}$

Remark 4.13. We have just shown that any model is a strong extension base.

An easy conclusion is a particular case of Chernikov and Kaplan's theorem [CK]: in a dependent theory, dividing and forking coincide over a model.

Corollary 4.14. *(Chernikov,Kaplan,* [CK]*) Let $\varphi(x, a)$ be a formula, M a model. Then $\varphi(x, a)$ divides over M if and only if $\varphi(x, a)$ forks over M.*

Proof. Suppose $\varphi(x, a)$ forks over M; so

$$\varphi(x, a) \vdash \bigvee_{i<k} \psi_i(x, a_i)$$

where each $\psi_i(x, a_i)$ divides over M. Assume without loss of generality that $\varphi(x, a) = \bigvee_{i<k} \psi_i(x, a_i)$.

Let $I = \langle a^\alpha_{<k} \colon \alpha < \omega\rangle$ be a strict Morley sequence in $\mathrm{tp}(a_{<k}/M)$ (exists since M is a strict extension base). It is enough to show that $\varphi(x, I)$ is inconsistent. Suppose not, and let $b \models \varphi(x, I)$. Then for some $i < k$, we have $b \models \psi_i(x, a^\alpha_i)$ for infinitely many $\alpha < \omega$. But the sequence $I_i = \langle a^\alpha_i \colon \alpha < \omega\rangle$ is a strict Morley sequence over M starting with a_i, hence by "Kim's Lemma", Proposition 4.9, it has to exemplify dividing of $\psi_i(x, a_i)$, hence $\psi_i(x, I_i)$ is k_i-inconsistent for some $k_i < \omega$. This gives the desired contradiction. QED$_{4.14}$

5. Bounded weight and orthogonality

In this section we will point out another important property of strictly nonforking sequences. So we do not study Morley sequences, but rather nonforking sequences and sets, not necessarily indiscernible. In order to develop some of their properties, we will need to understand collections of indiscernible sequences whose first elements are sufficiently independent.

We begin with the following definition:

Definition 5.1. Let A be a set and $\langle I_i : i < \alpha\rangle$ a sequence of sequences. We say that sequences $\langle I_i : i < \alpha\rangle$ are *half-mutually A-indiscernible* if I_i is indiscernible over $AI_{<i}a_{>i}$.

Fact 5.2. *(Shelah) Let $\langle a_i : i < \alpha\rangle$ be a strictly nonforking sequence over A, that is, $a_i \mathop{\smile\!\!\!\!\!\!|}^{st}_A a_{<i}$, and let I_i be an A-indiscernible sequence starting with a_i. Then there exist $I'_i \equiv_{Aa_i} I_i$ such that I'_i is indiscernible over $AI'_{<i}a_{>i}$ (so $\langle I'_i : i < \alpha\rangle$ are half-mutually A-indiscernible).*

Proof. This is included in [She], Claim 5.13, but let us still sketch the proof. We prove this by induction on α. It is enough to take care of the case $\alpha < \omega$.

Suppose $\langle a_i : i < \alpha + 1\rangle$, $\langle I_i : i < \alpha + 1\rangle$ are given. By the induction hypothesis we may assume that $\langle I_i : i < \alpha\rangle$ are half-mutually A-indiscernible.

Since $a_\alpha \mathop{\smile\!\!\!\!\!\!|}^{st}_A a_{<\alpha}$, we may assume without loss of generality that $a_\alpha \mathop{\smile\!\!\!\!\!\!|}^{st}_A I_{<\alpha}$. In particular, $a_\alpha \mathop{\smile\!\!\!\!\!\!|}_A I_{<\alpha}$, hence for every $j < \alpha$ we have $a_\alpha \mathop{\smile\!\!\!\!\!\!|}_{AI_{<j}a_{>j}} I_j$. By preservation of indiscernibility, e.g. Observation 8.9 of [Usv], this implies that for every $j < \alpha$ we have I_j are indiscernible over $AI_{<j}a_{\leq\alpha,\neq j}$.

Since $a_\alpha \mathop{\smile\!\!\!\!\!\!|}^{st}_A I_{<\alpha}$, it also the case that the type $\mathrm{tp}(I_{<\alpha}/Aa_\alpha)$ does not divide over A. Since I_α starts with a_α, there is I'_α such that

- $I'_\alpha \equiv_{Aa_\alpha} I_\alpha$
- I'_α is indiscernible over $AI_{<\alpha}$

This completes the induction step.

QED$_{5.2}$

The following Corollary is somewhat close to [She], Claim 5.19, but since we do not think the proof there works as written (and it is not quite clear whether the claim is true as stated), we decided to include a precise statement and a proof.

Corollary 5.3. *(Weak Local Character)*

(i) *Let* $I = \langle a_i : i < |T|^+\rangle$ *be a strictly nonforking sequence over* A *(that is,* $a_i \mathop{\smile\!\!\!\!\!\!\,|}^{st}_A a_{<i}$*),* b *a finite tuple (or even of cardinality* $\leq |T|$*). Then for almost all* $i < |T|^+$ *(that is, except* $|T|$*-many) we have that*
   - $\mathrm{tp}(b/Aa_i)$ *does not divide over* A

(ii) *If* T *is strongly dependent,* I *is an infinite strictly nonforking sequence and* b *is a finite tuple, then for almost all (all but finitely many)* $a_i \in I$ *we have that* $\mathrm{tp}(b/Aa_i)$ *does not divide over* A.

Proof. (i) Suppose not. So without loss of generality $tp(b/Aa_i)$ divides over A for all $i < |T|^+$, and let I_i be an A-indiscernible sequence starting with a_i witnessing this dividing. More precisely, there are formulae $\varphi_i(x, y_i)$ such that

- $\varphi_i(b, a_i)$ holds
- The set $\varphi(x, I_i) = \{\varphi(x, a') : a' \in I_i\}$ is inconsistent, and even k_i-inconsistent for some k_i.

Without loss of generality, $\varphi_i(x, y_i) = \varphi(x, y)$ and $k_i = k$ for all $i < |T|^+$.

By Fact 5.2 there are I_i' such that

- $I_i' \equiv_{Aa_i} I_i$
- I_i' are half-mutually A-indiscernible.

Note that $\varphi(x, I_i)$ are k-inconsistent for all i. Let $I_i = \langle a_{i,j} : j < \omega\rangle$. We are going to show the following:

- For every $\eta \colon |T|^+ \to \omega$ the set $\{\varphi(x, a_{i,\eta(i)}) : i < |T|^+\}$ is consistent.

This will contradict dependence as in the proof of "Kim's Lemma" (Proposition 4.9). In other words, this gives the tree property of the second kind, TP_2, which implies the independence property.

In fact, we will show that

- For every $\eta \colon |T|^+ \to \omega$ we have

$$\langle a_{i,\eta(i)} : i < |T|^+\rangle \equiv_A \langle a_i : i < |T|^+\rangle$$

This will certainly suffice, because the set $\{\varphi(x, a_i) : i < |T|^+\}$ is consistent (witnessed by b).

We prove by induction on $\alpha < |T|^+$ that

$$\langle a_{i,\eta(i)} : i < \alpha\rangle^\frown\langle a_i : \alpha \leq i < |T|^+\rangle \equiv_A \langle a_i : i < |T|^+\rangle$$

The case $\alpha = 0$ is trivial, and for limit stages use compactness. So let us take care of a successor stage. Assume $\langle a_{i,\eta(i)} : i < \alpha\rangle^\frown\langle a_i : \alpha \leq i < |T|^+\rangle \equiv_A \langle a_i : i < |T|^+\rangle$. Since I_α is indiscernible over $AI_{<\alpha}a_{>\alpha}$, we have

$$\langle a_{i,\eta(i)} : i < \alpha\rangle^\frown\langle a_i : \alpha \leq i < |T|^+\rangle \equiv_A \langle a_{i,\eta(i)} : i < \alpha\rangle^\frown\langle a_{\alpha,\eta(\alpha)}\rangle \frown \langle a_i : \alpha < i < |T|^+\rangle$$

which finishes the proof.

(ii) The proof is very similar. That is, using the same arguments we arrive at the following situation: for $i < \omega$, there are formulae $\varphi_i(x, a_i)$ which divide over A as exemplified by the sequences I_i, whereas for every $\eta \colon \omega \to \omega$ the set

$\{\varphi(x, a_{i,\eta(i)}) : i < |T|^+\}$ is consistent. As in the proof of Proposition 4.9, we may assume by compactness that I_i are of order type $\mathbb{Q}$, so $I_i = \langle a_{i,q} : q \in \mathbb{Q} \rangle$, with $a_i = a_{i,0}$. It is now easy to see (by taking infinite subsequences of I_i, since $\varphi(x, I_i)$ is k_i-inconsistent for some k_i) that the following set is consistent for every $\eta : \omega \to \omega$:

$$\{\varphi_i(x, b_{i,\eta(i)}) : i \in \mathbb{Q}\} \cup \{\neg\varphi_i(x, b_{i,\alpha}) : i \in \mathbb{Q}, \alpha \neq \eta(i)\}$$

And this is precisely the definition of lack of strong dependence.

$\text{QED}_{5.3}$

Note that one can regard this "weak local character" as a kind of "bounded pre-weight", or "rudimentarily finite pre-weight" in the case of strongly dependent theories (developing further some concepts introduced by Alf Onshuus and the author in [OU]). Indeed, we have shown that given an "independent enough" sequence $\langle a_i : i < |T|^+ \rangle$ and a tuple b, it is the case that b can only divide with a few a_i's (when working over a model or any set which is a strong extension base, one can replace dividing with forking by [CK], see Corollary 4.14).

There are several natural questions that arise in this context. For example, given an ordinal α, one can define $p = tp(b/A)$ to have *forking pre-weight* at least α if there are $\{a_i : i < \alpha\}$ forking independent over A (that is, $a_i \perp_A a_{\neq i}$) and $\text{tp}(b/Aa_i)$ divides over A for all i. On the other hand, we can define p to have *strict forking pre-weight* at least α if there is $\langle a_i : i < \alpha \rangle$ a strictly forking independent sequence over A (that is, $a_i \perp^{st}_A a_{<i}$) and $\text{tp}(b/Aa_i)$ divides over A for all i. Let us say that p has *rudimentarily finite* pre-weight (forking, strict forking, etc) if it is not the case that the pre-weight of p is at least ω.

Corollary 5.4. (i) *If T is strongly dependent, then every type has rudimentarily finite strict forking pre-weight. In fact, if a type p is strongly dependent (as defined in* [OU]*, Definition 2.6), then p has rudimentarily finite strict forking pre-weight.*

(ii) *If every type in T has rudimentarily finite forking pre-weight, then T is strongly dependent. In fact, if a type p has rudimentarily finite forking pre-weight, then p is strongly dependent.*

Proof. (i) By the previous Corollary.

(ii) This is a trivial consequence of [OU], Theorem 2.12(ii).

$\text{QED}_{5.4}$

We see that: T has rudimentarily finite forking weight $\Longrightarrow$ T is strongly dependent $\Longleftrightarrow$ T has rudimentarily finite weight in the sense of [OU], Definition 2.3 $\Longrightarrow$ T has rudimentarily finite strict forking weight. So it is natural to wonder

Question 5.5. Are any of the above implications reversible?

Reading the proofs of the results in this section carefully, one sees that the main issue has to do with appropriate notions of *weak orthogonality*. Let us give several possible definitions and point out some connections between them.

Definition 5.6. (i) (Shelah, e.g. [She], Definition 5.32) We call two types $p, q \in \mathrm{S}(A)$ *weakly orthogonal* or if $p(x) \cup q(y)$ is a complete type over A. We write $p \perp_w q$. If a, b realize p, q respectively, then we write $a/A \perp_w b/A$ or $a \perp_w b$ when A is fixed and clear from the context.

(ii) We call $\mathrm{tp}(a/A), \mathrm{tp}(b/A)$ *weakly orthogonal*^{1} if whenever I, J are A-indiscernible sequences starting with a, b respectively, there are I', J' mutually A-indiscernible such that $I \equiv_{Aa} I'$ and $J \equiv_{Ab} J'$. We write $a/A \perp_w^1 b/A$ or $a \perp_w^1 b$ when A is fixed and clear from the context.

(iii) We call $\mathrm{tp}(a/A), \mathrm{tp}(b/A)$ *weakly orthogonal*$^{\frac{1}{2}}$ if whenever I, J are A-indiscernible sequences starting with a, b respectively, there are I', J' half-mutually A-indiscernible such that $I \equiv_{Aa} I'$ and $J \equiv_{Ab} J'$. We write $a/A \perp_w^{\frac{1}{2}} b/A$ or $a \perp_w^{\frac{1}{2}} b$.

(iv) We call $\mathrm{tp}(a/A), \mathrm{tp}(b/A)$ *weakly orthogonal*st if whenever $a \models p$ and $b \models q$, we have $a \downarrow_A^{st} b$ and $b \downarrow_A^{st} a$. We write $a/A \perp_w^{st} b/A$ or $a \perp_w^{st} b$.

(v) We call $\mathrm{tp}(a/A), \mathrm{tp}(b/A)$ *weakly orthogonal*^{fk} if whenever $a \models p$ and $b \models q$, we have $a \downarrow_A b$ and $b \downarrow_A a$. We write $a/A \perp_w^{fk} b/A$ or $a \perp_w^{fk} b$.

Observation 5.7. Let A be a set, $p, q \in \mathrm{S}(A)$.

(i) If A is an extension base (or just p, q do not fork over A) and $p \perp_w q$, then $p \perp_w^{fk} q$.
(ii) If A is a strict extension base (or just p, q are strictly free over A) and $p \perp_w q$, then $p \perp_w^{st} q$.
(iii) If $p \perp_w^{st} q$ then $p \perp_w^{fk} q$.
(iv) If $p \perp_w^{st} q$ then $p \perp_w^{\frac{1}{2}} q$.
(v) If $p \perp_w^1 q$ then $p \perp_w^{\frac{1}{2}} q$.

Proof. Easy, e.g. (iv) is Fact 5.2. $\mathrm{QED}_{5.7}$

Lemma 5.8. *Let A be an extension base(e.g. a model), $p, q \in \mathrm{S}(A)$. If $p \perp_w q$, then $p \perp_w^1 q$.*

Proof. Let I, J be A-indiscernible starting with a, b respectively where $a \models p$ and $b \models q$. Let $I = \langle a_i : i < \omega \rangle$ and $J = \langle b_i : i < \omega \rangle$.

By an Erdös-Rado argument, there is $I' \equiv_A I$ which is indiscernible over AJ. Similarly, there is $J' \equiv_A J$ which is indiscernible over AI', moreover, denoting $J' = \langle b'_i : i < \omega \rangle$, we have that

(♦) for every $k < \omega$ there are $j_1, \dots j_k$ such that $\langle b'_i : i < k \rangle \equiv_{AI'} \langle b_{j_i} : i < k \rangle$.

Denote $I' = \langle a'_i : i < \omega \rangle$. We claim that it is still the case that I' is indiscernible over AJ'. Suppose not; so there is a formula $\varphi(\bar{x}, \bar{b}')$ over AJ' such that $\varphi(a'_{i_1}, \ldots, a'_{i_k}, \bar{b}') \wedge \neg\varphi(a'_{j_1}, \ldots, a'_{j_k}, \bar{b}')$ for some $i_1 < \ldots < i_1 < \omega$ and $j_1 < \ldots < j_1 < \omega$. But by ($\blacklozenge$) above, there is a tuple $\bar{b}$ of elements of J satisfying the same formula, that is, $\varphi(a'_{i_1}, \ldots, a'_{i_k}, \bar{b}) \wedge \neg\varphi(a'_{j_1}, \ldots, a'_{j_k}, \bar{b})$, which implies that I' is not indiscernible over AJ, a contradiction.

Finally, let us note that I', J' have the same type over A as I, J respectively. Hence in particular $a'_0 \models p$ and $b'_0 \models q$. By the assumption $p \perp_w q$, we have $a'_0 b'_0 \equiv_A ab$. So without loss of generality $a'_0 b'_0 = ab$, and I', J' are as required in the definition of $p \perp^1_w q$.

$\text{QED}_{5.8}$

Example 5.9. Let $(\mathbb{Q}, <, P)$ be the theory of $(\mathbb{Q}, <)$ with a dense co-dense predicate P. Let $p, q \in \mathrm{S}(\mathbb{Q})$ be the types over the prime model $\mathbb{Q}$ such that if a, b realize p, q respectively, then $a, b > \mathbb{Q}$, $P(a), \neg P(b)$.

Clearly $p \not\perp_w q$ since p, q do not determine whether $a < b$ or $b < a$. On the other hand, it is easy to see that $p \perp^1_w q$ and $p \perp^{st}_w q$.

Hence the implications in the Lemma above, as well as Observation 5.7(i),(ii) are not reversible.

So we obtain:

Corollary 5.10. *Let A be a strict extension base (e.g. a model), $p, q \in \mathrm{S}(A)$. Then*

$$p \perp_w q \implies p \perp^{st} q \implies p \perp^{\frac{1}{2}} q$$
$$p \perp_w q \implies p \perp^{1} q \implies p \perp^{\frac{1}{2}} q$$
$$p \perp_w q \implies p \perp^{st} q \implies p \perp^{fk}$$

Again, there are many natural questions.

Question 5.11.
(i) Which of the implications above are reversible? We know that the first one in each row is *not*.
(ii) What is the relation between $\perp^1_w$ and $\perp^{st}_w$?
(iii) What is the relation between $\perp^{\frac{1}{2}}_w$ and $\perp^{fk}_w$?
(iv) More specifically, is it the case that whenever $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_A a$ (and A is sufficiently nice), then $a \mathop{\smile\!\!\!\!\!\!\mid}^{st} b$? The reverse is clearly true.

The following is a small step in the direction of (possibly) answering Question 5.11 (iii) or (iv):

Lemma 5.12. *Let A be a set, I_i (for $i < k$) be a sequence starting with the element a_i such that I_i is indiscernible over A. Assume furthermore that $\{a_i : i < k\}$ is forking independent over A. Then without loss of generality I_i is indiscernible over $Aa_{\neq i}$. In other words, there are I'_i such that*

- $I'_i \equiv_{Aa_i} I_i$

- *I'_i is indiscernible over $Aa_{\neq i}$*

Proof. We prove this by induction on k, the case $k = 1$ being trivial.

So let $k > 1$ and assume that I_i is indiscernible over $Aa_{<k,\neq i}$ for all $i < k$.

Recall that $a_{<k} \mathop{\smile\!\!\!\!\!\!\,|}_A a_k$. Let $a'_{<k} \equiv_{Aa_k} a_{<k}$ be such that $a'_{<k} \mathop{\smile\!\!\!\!\!\!\,|}_A I_k$. Let $\sigma \in \mathrm{Aut}(\mathfrak{C}/Aa_k)$ take $a'_{<k}$ to $a_{<k}$, and denote $I'_k = \sigma(I_k)$. Clearly

- $I'_k \equiv_{Aa_k} I_k$
- $a_{<k} \mathop{\smile\!\!\!\!\!\!\,|}_A I'_k$

Hence by preservation of indiscernibility (e.g. Observation 8.9 in [Usv]), I'_k is indiscernible over $Aa_{<k}$.

Now since $a_k \mathop{\smile\!\!\!\!\!\!\,|}_A a_{<k}$, we can find (as before) I'_i for $i < k$ satisfying

- $I'_{<i} \equiv_{Aa_{<k}} I_i$
- $a_k \mathop{\smile\!\!\!\!\!\!\,|}_A I'_{<k}$

In particular, we obtain for every $i < k$:

- $a_k \mathop{\smile\!\!\!\!\!\!\,|}_A a_{<k} I_i$, hence $a_k \mathop{\smile\!\!\!\!\!\!\,|}_{Aa_{<k,\neq i}} I'_i$. By the induction hypothesis and preservation of indiscernibility this implies that I'_i is indiscernible over $Aa_{\neq i}$.

Recall that I'_k is indiscernible over $Aa_{<k}$. So we are done.

$\mathrm{QED}_{5.12}$

One of the earlier versions of [She] contained the statement that the conclusion of the Lemma above can be obtained when starting from a weaker assumption: the sequence $\langle a_i : i < k\rangle$ is nonforking. The following example shows that this is not always possible:

Example 5.13. Consider the theory of $(\mathbb{Q}, <)$, and let $b = \langle 0, 2\rangle, a = 1$. Then $b \mathop{\smile\!\!\!\!\!\!\,|} a$, but if $I = \langle\langle 0, 2\rangle, \langle 3, 5\rangle, \langle 6, 8\rangle, \ldots\rangle$ and $J = \langle 1, 4, 7, \ldots\rangle$, then clearly there are no I', J' of the same type as I, J respectively, starting with a, b such that J' is indiscernible over a.

We constructed the example with $A = \varnothing$, but it is as easy to modify it such that A is any set, in particular a model.

Let us conclude this section with the following Lemma, which is a slight generalization of Proposition 4.13 in [OU], and ideas behind the proof are very similar. We include it because it might also become useful for questions related to issues discussed above (although right now we do not see any concrete applications).

Lemma 5.14. *Let A be a set, I_i (for $i < k$) be a sequence starting with the element a_i such that:*

($\blacklozenge$) $I_i \mathop{\smile\!\!\!\!\!\!\,|}_A a_{<i}$

($\blacklozenge\blacklozenge$) *$I_i$ is indiscernible over $Aa_{<i}$*

Then without loss of generality I_i is indiscernible over $AI_{\neq i}$. Moreover, there are I'_i such that

- $I'_i \equiv_{Aa_i} I_i$
- *I'_i is indiscernible over $AI'_{\neq i}$*

- $I'_i \perp_A I'_{<i}$

Proof. We prove the lemma by induction on k, the case $k = 1$ being trivial.

Let $a_{\leq k}$, $I_{\leq k}$ be as in the assumptions of the lemma. By the induction hypothesis we can assume that for $i < k$ we have sequences I''_i satisfying the conclusion, that is

(i) $I''_i \equiv_{Aa_i} I_i$
(ii) I''_i is indiscernible over $AI'_{\neq i}$
(iii) $I''_i \perp_A I''_{<i}$

So in particular each I''_i (for $i < k$) starts with the element a_i.

Recall that by the assumptions on I_k we also have

(iv) I_k is indiscernible over $Aa_{<k}$
(v) $I_k \perp_A a_{<k}$

By (v) above and the existence of nonforking extensions, there are $I^*_{<k}$ such that

- $I^*_{<k} \equiv_{Aa_{<k}} I''_{<k}$
- $I_k \perp_A I^*_{<k}$

So without loss of generality we may assume in addition that

(vi) $I_k \perp_A I''_{<k}$

Since we can make I_k as long as we wish, applying Erdös-Rado, there exists I''_k such that

(*) I''_k is indiscernible over $AI''_{<k}$
(**) Every n-type of I''_k over $AI''_{<k}$ "appears" in I_k

Note:

($\diamondsuit$) $I''_k \equiv_{Aa_{<k}} I_k$ [since I_k satisfies (iv) above and I''_k satisfies (*),(**)]
($\diamondsuit\diamondsuit$) $I''_k \perp_A I''_{<k}$ [by (vi) and (**) above]

Let $\sigma \in \mathrm{Aut}(\mathfrak{C}/Aa_{<k})$ be such that $\sigma(I''_k) = I_k$. Define $I'_i = \sigma(I''_i)$ for $i < k$ and $I'_k = I_k$. We claim that $I'_{\leq k}$ satisfy the conclusion of the lemma, which completes the induction step. Indeed,

- $I'_i \equiv_{Aa_i} I_i$ for all $i \leq k$:
  - $i < k$. By the induction hypothesis + σ being over $a_{<k}$.
  - $i = k$. Clear since $I'_k = I_k$.
- For every $i \leq k$ we have that I_i is indiscernible over $AI_{\neq i}$:
  - $i < k$. By ($\diamondsuit\diamondsuit$) above we have $I'_k \perp_A I'_{<k}$, hence $I'_k \perp_B I'_i$ where $B = AI'_{<k,\neq i}$. By the induction hypothesis, I'_i is indiscernible over B, hence by preservation of indiscernibility, I'_i is indiscernible over $AI'_{\neq i}$, as required.
  - $i = k$. Follows immediately from the choice of I''_k.
- $I'_i \perp_A I'_{<i}$ for all i:
  - $i < k$. By the induction hypothesis.
  - $i = k$. By ($\diamondsuit\diamondsuit$) above.

QED$_{5.14}$

Corollary 5.15. *Let A be a set, I_i (for $i < k$) be a sequence starting with the element a_i such that:*

($\blacklozenge$) *I_i is a Morley sequence over $Aa_{<i}$ based on A*

Then there are I'_i such that

- $I'_i \equiv_{Aa_i} I_i$
- *I'_i is indiscernible over $AI'_{\neq i}$*
- $I'_i \downarrow_A I'_{<i}$

Proof. Follows immediately from the previous lemma (and transitivity of nonforking on the left). $\text{QED}_{5.15}$

Some of the questions in this section will be addressed and partially answered in a subsequent work of Itay Kaplan and the author [KU].

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