

# ON DP-MINIMALITY, STRONG DEPENDENCE AND WEIGHT

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ABSTRACT. We study dp-minimal and strongly dependent theories and investigate connections between these notions and weight.

## 1. INTRODUCTION AND PRELIMINARIES

**1.1. Introduction.** The original goal of this paper was to introduce a notion of minimality for dependent theories, which generalizes all (or at least most) known notions of minimality. We believe that a natural candidate for such a notions should be related to strong dependence, introduced by Shelah in [Sh783] and [Sh863]. In fact, in [Sh863] Shelah defines a family of “minimality notions” ( $\text{dp}^\ell$ -minimality for different  $\ell$ ) based on ranks  $\text{dp-rk}^\ell$  related to strong dependence. Most of these notions imply a very natural property, which is very easy to state, and can be phrased as “as strongly dependent as possible”. In this article, we call this property simply “dp-minimality” and work with it. In fact, dp-minimality turns out to be equivalent to Shelah’s  $\text{dp}^\ell$ -minimality for some  $\ell$ , but we will not concentrate on this here, since we do not intend to define Shelah’s dp-ranks. We hope that the reader will be convinced that this is a natural notion just by looking at the definition (and if not, after having seen the characterization of dp-minimality in terms of weight in section 3).

So the class of dp-minimal theories is a subclass of dependent theories (theories with no independence property). It includes all classical “minimal” classes such as strongly minimal,  $U$ -rank 1 super stable, o-minimal theories, and more. But there are also other examples; for instance, there are stable non super stable dp-minimal theories (see Observation 3.4). So the first question we investigated was: what does it mean for a stable theory to be dp-minimal? The characterization turns out to be quite nice: a stable theory is dp-minimal if and only if every 1-type has weight 1 if and only if no 1-type admits “criss-crossed” forking (see section 3 for the definitions).

Having realized that dp-minimality is strongly related to weight, we tried to develop a theory of weight in the more general context of dependent theories. Recall that a type  $p \in S(A)$  in a stable theory  $T$  has pre-weight  $\geq \gamma$  if for every  $\alpha < \gamma$  there exist  $\bar{a} \models p$  and an  $A$ -independent set  $\{\bar{b}_i : i \leq \alpha\}$  such that  $\bar{a} \not\perp_A \bar{b}_i$  for all  $i \leq \alpha$ . It seems that in a dependent theory this definition is not strong enough, because (even if  $A$  is a model) the type  $\text{tp}(\bar{b}_i/A)$  does not carry enough information. On the other hand, by [Us2] in a dependent theory a *Morley sequence* in the type over  $A$  determines a parallelism class. This gave us the motivation to define weight using *mutually Morley sequences* in  $\text{tp}(\bar{b}_i/A)$ .

We show that this notion of weight generalizes the classical weight for stable theories, every type in a dependent theory has bounded weight, and a dependent theory is strongly dependent

if and only if every type has rudimentarily finite weight. We deduce that dp-minimality is equivalent (for an arbitrary dependent theory) to every 1-type having weight 1.

We also investigate the meaning of our concepts and results in a more general context of rosy theories with forking replaced with thorn-forking. In particular, we study a related notion of criss-crossed  $\mathfrak{b}$ -forking/dividing. Our hope is that these results will assist in understanding the connection between dp-minimality and weight in rosy dependent theories.

After reading the first draft of our paper, Hans Adler developed the notion of “burden” strongly related to weight. His definition makes sense in any theory, and he calls a theory “strong” if every type has rudimentarily finite burden. For dependent theories, strong coincides with strongly dependent, but Adler investigates this notion more generally. Several definitions in the current version of our paper were motivated by Adler’s work, and several proofs were simplified using his ideas.

We would also like to mention the recent work of John Goodrick [\[Gd\]](#), who studies dp-minimal densely ordered groups, and shows that in many ways they are similar to weakly o-minimal ones. This further confirms our conjecture that dp-minimality is a good minimality notion in the context of dependent theories.

**1.2. Notations and Preliminaries.** Given a theory  $T$ , we will work inside its monster model denoted by  $\mathfrak{C}$ . By “monster” we mean that all cardinals we mention are “small” (i.e. smaller than saturation of  $\mathfrak{C}$ ), all sets are small subsets of  $\mathfrak{C}$ , all models are small elementary submodels of  $\mathfrak{C}$ , and truth values of all formulae and all types are calculated in  $\mathfrak{C}$ . We denote tuples (finite unless said otherwise) by  $\bar{a}, \bar{b}, \bar{c}$  etc, elements of  $\mathfrak{C}$  by  $a, b, c$  etc, sets by  $A, B, C$  etc, models by  $M, N$  etc.

By  $\bar{a} \equiv_A \bar{b}$  we mean  $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$ . Recall that this is equivalent to having  $\sigma \in \text{Aut}(\mathfrak{C}/A)$  satisfying  $\sigma(\bar{a}) = \bar{b}$ .

Given an order type  $O$ , a sequence  $I = \langle \bar{a}_i : i \in O \rangle$  and  $j \in O$ , we often denote the set  $\{\bar{a}_i : i < j\}$  by  $\bar{a}_{<j}$ . Similarly for  $\bar{a}_{\leq j}, \bar{a}_{>j}$  etc. We also often identify the sequence  $I$  with the set  $\cup I$ ; that is, when no confusion should arise we write  $\text{tp}(\bar{a}/I)$  etc.

We will write  $\bar{a} \downarrow_A B$  for “ $\text{tp}(\bar{a}/AB)$  does not fork over  $A$ ” even if  $T$  is not simple. Although nonforking is generally not an independence relation, we still find this notation convenient.

For simplicity we assume  $T = T^{eq}$  for all theories  $T$  mentioned in this paper, unless said otherwise.

The preliminaries include some stability theory and some known facts about rosy theories. We will state all of the prerequisites in the following subsections.

**1.3. Stability.** We will assume that the reader is familiar with the basic concepts of stability theory such as forking and dividing. There are some facts we will need, most of which are quite well known and we will only state them. For the definitions and proofs the reader can refer to [\[P:book\]](#).

We define a sequence  $\langle \bar{a}_i \rangle$  to be  $A$ -indiscernible if for any  $i_1 < i_2 < \dots < i_n$  and  $j_1 < j_2 < \dots < j_n$  we have

$$\text{tp}(\bar{a}_{i_1} \dots \bar{a}_{i_n}/A) = \text{tp}(\bar{a}_{j_1} \dots \bar{a}_{j_n}/A).$$

A set  $\{\bar{a}_i\}_{i \in I}$  is  $A$ -indiscernible if the above conclusion holds given any  $i_1 \dots i_n$  and  $j_1 \dots j_n$  in  $I$ .

indiscernible

**Fact 1.1.** *In a model of a stable theory, any  $A$ -indiscernible sequence is an  $A$ -indiscernible set.*

We will say that a sequence  $\langle a_i \rangle$  is  $A$ -independent if for any  $i$  we have

$$\bar{a}_i \downarrow_A \{\bar{a}_j\}_{j < i}.$$

**Definition 1.1.** (i) *We define a Morley sequence in  $\text{tp}(\bar{a}/A)$  to be an  $A$ -independent and  $A$ -indiscernible sequence of elements realizing  $\text{tp}(\bar{a}/A)$ .*

*We say that a sequence  $\langle \bar{a}_i \rangle$  is a Morley sequence over  $A$  if it is a Morley sequence in  $\text{tp}(\bar{a}_0/A)$ .*

- (ii) *We say that a Morley sequence over  $A$  is based on  $A_0 \subseteq A$  if it is an independent set over  $A_0$  (so clearly it is also a Morley sequence over  $A_0$ ).*
- (iii) *A type  $p$  is defined to split over  $A$  if there are  $\bar{a}, \bar{b}$  and a formula  $\phi(x, y)$  such that  $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$  and  $\phi(x, \bar{a}) \in p$  and  $\neg\phi(x, \bar{b}) \in p$ .*

forking

**Fact 1.2.** *The following results are well known facts from stability theory and can be found in either [P:book] or [Sh:c].*

Let  $T$  be stable.

- (i) *A formula  $\phi(\bar{x}, \bar{a})$  forks over  $A$  if and only if for every Morley sequence  $\langle \bar{a}_i \rangle$  in the type of  $\bar{a}$  over  $A$  the set*

$$\{\phi(\bar{x}, \bar{a}_i)\}$$

*is  $k$ -inconsistent for some  $k \in \mathbb{N}$ .*

- (ii) *If  $\bar{a} \downarrow_A \bar{c}$ , then any Morley sequence  $\langle \bar{a}_i \rangle$  in  $\text{tp}(\bar{a}/A\bar{c})$  is also a Morley sequence in  $\text{tp}(\bar{a}/A)$  (so it is based on  $A$ ).*
- (iii) *If  $A = \text{acl}(A)$ ,  $\bar{a} \downarrow_A \bar{c}$  and  $\langle \bar{a}_i \rangle$  is an independent sequence in  $\text{tp}(\bar{a}/A\bar{c})$  then  $\langle \bar{a}_i \rangle$  is a Morley sequence in  $\text{tp}(\bar{a}/A\bar{c})$ .*
- (iv) *Given any type  $p$  set  $B$  with  $A = \text{acl}(A) \subseteq B$ , if  $p$  does not fork over  $A$ , then  $p$  does not split over  $A$ .*
- (v) *If  $\bar{a} \downarrow_A \bar{c}$ ,  $\langle \bar{a}_i \rangle$  is an independent sequence in  $\text{tp}(\bar{a}/A\bar{c})$  and indiscernible over  $A$ , then  $\langle \bar{a}_i \rangle$  is a Morley sequence in  $\text{tp}(\bar{a}/A\bar{c})$ .*

Let us also recall several basic definitions:

**Definition 1.2.** *Let  $T$  be stable.*

- (i) *Two stationary types  $p(x)$  and  $q(x)$  are orthogonal if there are extensions  $p'(x)$  and  $q'(x)$  of  $p$  and  $q$  such that  $p', q' \in S(B)$  and for every tuple  $a \models p'$  and  $b \models q'$  we have  $a \downarrow_B b$ .*
- (ii) *A type  $p(x)$  over  $A$  is regular if for any stationary forking extension  $q(x)$  of  $p(x)$ ,  $q(x)$  is orthogonal to  $p(x)$ .*
- (iii) *A type  $p(x)$  over some set  $B$  has pre-weight 1 if there are no  $B$ -independent elements  $b_1, b_2$  and no realization  $a$  of  $p$  such that  $a \not\downarrow_B b_1$  and  $a \not\downarrow_B b_2$ .*
- (iv) *More generally, the pre-weight of a type  $p \in S(A)$ ,  $\text{pwt}(p)$  is defined as follows ( $\alpha$  is an ordinal):  $\text{pwt}(p) \geq \alpha$  iff for every  $\beta < \alpha$  there exist  $\bar{b} \models p$  and an independent set  $\{\bar{a}_i : i < \beta\}$  such that  $\bar{b} \not\downarrow_A \bar{a}_i$  for all  $i < \beta$ .*

(v) *The weight of  $p$  is the supremum over all nonforking extensions  $q$  of  $p$  of  $\text{pwt}(q)$ .*

The following lemma is easy and probably well-known, but we include the proof.

**toca** **Lemma 1.3.** *( $T$  stable) Let  $\bar{a}$  and  $\bar{b}$  be any two tuples such that  $\bar{a} \perp_A \bar{b}$ . Then there are sequences  $\langle \bar{a}_i \rangle_{i \in \omega}$  and  $\langle \bar{b}_i \rangle_{i \in \omega}$  such that  $\bar{a}_0 = \bar{a}$ ,  $\bar{b}_0 = \bar{b}$ ,  $\langle \bar{a}_i \rangle_{i \in \omega}$  is a Morley sequence in  $\text{tp}(\bar{a}/A \langle \bar{b}_i \rangle_{i \in \omega})$  based on  $A$  and  $\langle \bar{b}_i \rangle_{i \in \omega}$  is a Morley sequence in  $\text{tp}(\bar{b}/M \langle \bar{a}_i \rangle_{i \in \omega})$  based on  $A$ . In particular,*

*$\langle \bar{a}_i \rangle_{i \in \omega}$  and  $\langle \bar{b}_i \rangle_{i \in \omega}$  are mutually  $A$ -indiscernible sequences.*

*Proof.* We will prove that the usual construction of the Morley sequence actually gives the desired result.

Let  $\langle \bar{a}_i \rangle$  be a Morley sequence in  $\text{tp}(\bar{a}/A \bar{b})$  with  $\bar{a} = \bar{a}_0$ . By definition  $\bar{b} \perp_A \langle \bar{a}_i \rangle$ ; let  $\langle \bar{b}_i \rangle$  be a Morley sequence in  $\text{tp}(\bar{b}/A \langle \bar{a}_i \rangle)$  with  $\bar{b} = \bar{b}_0$ .

Note that as  $\bar{b} \perp_A \bar{a}$  and  $\langle \bar{a}_i \rangle$  is a Morley sequence over  $A \bar{b}$ , it follows that  $\bar{b} \perp_A \langle \bar{a}_i \rangle$ , so by Fact 1.2(ii), both  $\langle \bar{a}_i \rangle$  and  $\langle \bar{b}_j \rangle$  are based on  $A$ .

So we only need to prove that  $\langle \bar{a}_i \rangle$  is a Morley sequence over  $A \langle \bar{b}_j \rangle$ . By Fact 1.2(v) it is enough to prove that the sequence is  $A \langle \bar{b}_j \rangle$ -independent; i.e., that for any element  $\bar{a}_n$  we have  $\bar{a}_n \perp_{A \langle \bar{b}_j \rangle} \langle \bar{a}_i \rangle_{i < n}$ . Clearly, if we show that  $\bar{a}_n \perp_A \langle \bar{a}_i \rangle_{i < n} \langle \bar{b}_j \rangle$ , we're done (by transitivity). So it is enough to prove that for any  $k$ ,

$$\bar{a}_n \perp_A \langle \bar{a}_i \rangle_{i < n} \langle \bar{b}_j \rangle_{j < k}.$$

As  $\langle \bar{b}_j \rangle$  is a Morley sequence over  $A \langle \bar{a}_i \rangle$ , we have  $\bar{b}_k \perp_{A \langle \bar{a}_i \rangle} \langle \bar{b}_j \rangle_{j < k}$ . Recall that  $\bar{b} \perp_A \langle \bar{a}_i \rangle$  and, since as  $\text{tp}(\bar{b}_j/A \langle \bar{a}_i \rangle) = \text{tp}(\bar{b}/A \langle \bar{a}_i \rangle)$ , we get that  $\bar{b}_j \perp_A \langle \bar{a}_i \rangle$  for all  $j$ . Now it is easy to see that  $\bar{b}_k \perp_A \langle \bar{b}_j \rangle_{j < k} \langle \bar{a}_i \rangle$ . So in particular  $\bar{b}_k \perp_A \langle \bar{b}_j \rangle_{j < k} \langle \bar{a}_i \rangle_{i < n} \bar{a}_n$ . As  $\langle \bar{a}_i \rangle$  is based on  $A$ , clearly  $\bar{a}_n \perp_A \langle \bar{a}_i \rangle_{i < n} \langle \bar{b}_j \rangle_{j < k}$ .  $\square$

1.4. **p-Forking.** The following definitions and facts can be found in [\[Onshuus\]](#).

**Definition 1.3.** *Let  $\phi(\bar{x}, \bar{y})$  be a formula,  $\bar{b}$  be a tuple and  $C$  be any set. Then we define the following.*

- $\phi(\bar{x}, \bar{b})$  strongly divides over  $D$  if  $\bar{b}$  is non algebraic over  $D$  and the set

$$\{\phi(\bar{x}, \bar{b}')\}_{\bar{b}' \models \text{tp}(\bar{b}/D)}$$

*is  $k$ -inconsistent for some  $k \in \mathbb{N}$ .*

- $\phi(\bar{x}, \bar{b})$   $\mathfrak{p}$ -divides over  $C$  if there is some  $D \supset C$  such that  $\phi(\bar{x}, \bar{b})$   $\mathfrak{p}$ -divides over  $D$ .
- $\phi(\bar{x}, \bar{b})$   $\mathfrak{p}$ -forks over  $C$  if there are finitely many formulas  $\psi_1(\bar{x}, \bar{b}_1), \dots, \psi_n(\bar{x}, \bar{b}_n)$  such that  $\phi(\bar{x}, \bar{b}) \Rightarrow \bigvee_i \psi_i(\bar{x}, \bar{b}_i)$  and  $\psi_i(\bar{x}, \bar{b}_i)$   $\mathfrak{p}$ -divides over  $C$  for  $1 \leq i \leq n$ .

**algebraic**

**Remark 1.4.** If  $\bar{a} \models \phi(\bar{x}, \bar{b})$  and  $\phi(\bar{x}, \bar{b})$  strongly divides over  $C$  then  $\bar{b} \in \text{acl}(\bar{a}, C)$ .

*Proof.* Notice that if  $S$  is the set of elements in the orbit of  $\text{tp}(\bar{b}/\bar{a}C)$  then

$$\bar{a} \models \bigwedge_{\bar{b}' \in S} \phi(\bar{x}, \bar{b}').$$

By inconsistency, there can only be finitely many such  $\bar{b}'$ 's.  $\square$

th-forking

**Fact 1.5.** *If  $T$  is stable then  $\phi(\bar{x}, \bar{a})$  forks over  $C$  if and only if  $\phi(\bar{x}, \bar{a})$   $\mathfrak{p}$ -forks over  $C$ .*

As is the case between  $\mathfrak{p}$ -dividing/ $\mathfrak{p}$ -forking and dividing/forking, standard forking terminology is used for  $\mathfrak{p}$ -forking. As such,  $U^{\mathfrak{p}}$ -rank is defined to be the foundation rank of the partial order (defined on complete types)  $p <_{\mathfrak{p}} q$  defines as “ $p$  is a  $\mathfrak{p}$ -forking extension of  $q$ ”. A theory  $T$  is “super rosy” if and only if the  $U^{\mathfrak{p}}$ -ranks of all types in all models of  $T$  are ordinal valued.

lascar

**Fact 1.6.** *Let  $T$  be a super rosy theory and let  $a, b, A$  be subsets of a model  $M$  of  $T$ .*

*Then*

$$U^{\mathfrak{p}}(tp(b/aA)) + U^{\mathfrak{p}}(tp(a/A)) < U^{\mathfrak{p}}(tp(ab/A)) < U^{\mathfrak{p}}(tp(b/aA)) \oplus U^{\mathfrak{p}}(tp(a/A)).$$

*Proof.* Theorem 4.1.10 in [\[Onshuus\]](#). □

**1.5. Dependent theories.** Recall that a theory  $T$  is called *dependent* if there does not exist a formula which exemplifies the independence property. We are mostly going to use the following equivalent definition:

fct:depend

**Fact 1.7.**  *$T$  is dependent if and only if there do not exist an indiscernible sequence  $I = \langle \bar{a}_i : i < \lambda \rangle$ , a formula  $\varphi(\bar{x}, \bar{y})$  and  $\bar{b}$  such that both*

$$\{i : \models \varphi(\bar{a}_i, \bar{b})\}$$

*and*

$$\{i : \models \neg\varphi(\bar{a}_i, \bar{b})\}$$

*are unbounded in  $\lambda$ .*

The following are easy (but important) consequence of dependence (originally due to Shelah). Proofs can be found e.g. in [\[U1\]](#).

act:splitfork

**Fact 1.8.** *( $T$  dependent) Strong splitting implies dividing (and therefore forking).*

We will refer to the following as “transitivity of forking on the left”:

transitivity

**Fact 1.9.** *( $T$  dependent) Let  $A, B$  be sets and assume that  $I = \langle a_i : i < \lambda \rangle$  is a nonforking sequence based on  $A$ , that is,  $a_i \perp_A Ba_{<i}$  for all  $i < \lambda$ . Then  $I \perp_A B$ , that is,  $tp(I/AB)$  does not fork over  $A$ .*

*Proof.* This is Claim 5.16 in [\[Sh783\]](#). □

*Remark 1.10.* Transitivity of (non-)forking on the left is, in fact, true in any theory and follows immediately from transitivity of non-dividing on the left, which is well-known. But we are not going to make use of this.

cor:indep set

**Corollary 1.11.** *Let  $\{A_i : i < \lambda\}$  be a nonforking (independent) set over  $A$ , that is,  $A_i \perp_A A_{\neq i}$  for all  $i$ . Then for every  $W, U \subseteq \lambda$  disjoint we have  $A_{\in W} \perp_A A_{\in U}$ .*

*Proof.* Monotonicity and transitivity on the left. □

disc preserve

**Observation 1.12.** *( $T$  dependent) Suppose  $I$  is an indiscernible sequence over  $A$  and  $B \perp_A I$ . Then  $I$  is indiscernible over  $AB$ .*

*Proof.* By Fact 1.8  $tp(B/AI)$  does not split strongly over  $A$ . Recall that this implies that for every  $\bar{a}_1, \bar{a}_2 \in I$  which are on the same  $A$ -indiscernible sequence we have  $B\bar{a}_1 \equiv_A B\bar{a}_2$ , which is precisely what we want. □

## 2. STRONG DEPENDENCE, STRONG STABILITY AND WEIGHT

In this section we propose a notion of weight in the context of dependent theories. It generalizes weight in stable theories as well as stable weight for generically stable types in dependent theories defined in [U1]. We show that strongly dependent theories are exactly those in which every finitary type has rudimentary finite weight.

Let  $T$  be a dependent theory. We will use the following terminology:

dfn:basic

**Definition 2.1.** (i) Let  $I$  be a linear order. A sequence  $\langle \bar{a}_i : i \in I \rangle$  is called a Morley sequence in the type  $p \in S(B)$  over a set  $A \subseteq B$  if

- $\langle \bar{a}_i : i \in I \rangle$  is indiscernible over  $B$
  - $\bar{a}_i \models p$  for all  $i \in I$
  - $\text{tp}(\bar{a}_{i_1} \bar{a}_{i_2} \dots \bar{a}_{i_k} / B \bar{a}_{<i})$  does not fork over  $A$  for every  $k < \omega$  and  $i_1 < \dots < i_k \in I$
- (ii) If  $A = B$  in the definition above, we often omit  $A$  and simply say “Morley sequence in  $p$ ”.
- (iii) We say that a sequence  $\langle \bar{a}_i : i \in I \rangle$  is based on  $(A, B)$  if it is a Morley sequence in some type  $p \in S(B)$  over  $A \subseteq B$ . If  $A = B$  we simply say “Morley sequence based on  $A$ ” or “Morley sequence over  $A$ ”.
- (iv) We call a set of tuples  $\mathcal{B}$  independent over a set  $A$  if for every  $\bar{b}_1, \dots, \bar{b}_n \in \mathcal{B}$ ,  $\text{tp}(\bar{b}_1, \dots, \bar{b}_n / A \cup \mathcal{B} \setminus \{\bar{b}_1, \dots, \bar{b}_n\})$  does not fork over  $A$ .

**Observation 2.1.** (i) In the definition of a Morley sequence in the type  $p \in S(B)$  over a set  $A \subseteq B$  it is enough to assume  $\text{tp}(\bar{a}_i / B \bar{a}_{<i})$  does not fork over  $A$  for all  $i$ .

- (ii) In the definition of an independent set  $\mathcal{B}$  over  $A$  it is enough to assume that  $\text{tp}(\bar{b} / A \cup \mathcal{B} \setminus \{\bar{b}\})$  does not fork over  $A$  for all  $\bar{b} \in \mathcal{B}$ .

*Proof.* By Fact 1.9 and Corollary 1.11. □

Before giving the next definition, we would like to introduce some notation. Let  $\mathbf{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$  be an array of tuples from the monster model. We will think of  $\mathbf{a}$  as a collection of  $\kappa$  sequences of length  $\lambda$  :  $\mathbf{a} = \langle (\bar{a}_{<\lambda})^\alpha : \alpha < \kappa \rangle$ . For simplicity, we will denote the  $\alpha^{\text{th}}$  sequence  $\langle \bar{a}_i^\alpha : i < \lambda \rangle$  by  $\bar{a}_{<\lambda}^\alpha$ . We also denote the union of all sequences except the  $\alpha^{\text{th}}$  one by  $\bar{a}_{<\lambda}^{\neq\alpha}$ . That is,

$$\bar{a}_{<\lambda}^{\neq\alpha} = \{\bar{a}_{<\lambda}^\beta : \beta \neq \alpha\}.$$

**Definition 2.2.** (i) We will call an array  $\mathbf{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$  indiscernible over a set  $A$  if for a fixed  $\alpha < \kappa$ , the sequence  $\bar{a}_{<\lambda}^\alpha$  is indiscernible over  $A \cup \bar{a}_{<\lambda}^{\neq\alpha}$ . That is,  $\mathbf{a}$  is a collection of sequences which are indiscernible over each other (and over  $A$ ).

- (ii) We will call an array  $\mathbf{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$  Morley over a set  $A$  if for a fixed  $\alpha < \kappa$ , the sequence  $\bar{a}_{<\lambda}^\alpha$  is based on  $(A, A \cup \bar{a}_{<\lambda}^{\neq\alpha})$ . That is,  $\mathbf{a}$  is a collection of sequences which are Morley over each other (based on  $A$ ).

dfn:morleyarray

**Lemma 2.1.** Let  $\mathbf{a}$  be indiscernible over a set  $A$ . Then there exists  $B \supseteq A$  such that  $\mathbf{a}$  is Morley over  $B$ .

*Proof.* The following easy argument was inspired by a conversation with Hans Adler. Let  $\langle \bar{a}_i^\alpha : i \in \omega^* \rangle$  ( $\omega^*$  is  $\omega$  with reversed order) be such that the array  $\langle \bar{a}_i^\alpha : \alpha < \kappa, i \in \lambda \frown \omega^* \rangle$  is still indiscernible, and let  $B = A \cup \{\bar{a}_i^\alpha : i \in \omega^*\}$ . Then for every  $\alpha < \kappa$ ,  $k < \omega$  and  $i_1 < \dots < i_k < \lambda$ ,

the type  $\text{tp}(\bar{a}_{i_1} \dots \bar{a}_{i_k} / B \bar{a}_{<\lambda}^{\neq \alpha} \bar{a}_{<i}^{\alpha})$  is finitely satisfiable in  $B$ , and therefore does not fork over  $B$ , as required.  $\square$

The following definition was motivated by Shelah's notion of "ind-pattern":

**dfn:divsys** **Definition 2.3.** (i) A dividing system  $\mathfrak{D}$  for a type  $p(\bar{x}) \in S(A)$  consists of

- an array  $\mathbf{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$  (where  $\lambda, \kappa$  are ordinals,  $\lambda$  is infinite)
- a sequence of formulae  $\Phi = \langle \varphi_\alpha(\bar{x}, \bar{y}) : \alpha < \kappa \rangle$

such that

- (a)  $\mathbf{a}$  is indiscernible over  $A$ .
- (b)  $p \cup \{\varphi(\bar{x}, \bar{a}_0^\alpha) : \alpha < \kappa\}$  is consistent
- (c) For every  $\alpha < \kappa$ , the set

$$\Sigma_{\mathfrak{D}, \alpha} = \{\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha) : i < \lambda\}$$

is inconsistent.

- (ii) We call  $\kappa$  in the definition above the depth of  $\mathfrak{D}$ , and  $\lambda$  the length of  $\mathfrak{D}$ .
- (iii) Let  $k$  be a natural number. A  $k$ -dividing system  $\mathfrak{D}$  for a type  $p(\bar{x}) \in S(A)$  is a dividing system such that in clause (c) of the definition (i) above,  $\Sigma_{\mathfrak{D}, \alpha}$  is inconsistent and  $k$ -inconsistent with  $p$ .
- (iv) A dividing system  $\mathfrak{D} = (\mathbf{a}, \Phi)$  for  $p$  is called Morley if  $\mathbf{a}$  is Morley.
- (v) We say that the pre-weight of a type  $p$  is at least  $\mu$  (where  $\mu$  is an ordinal) if for every  $\kappa < \mu$  there exists a Morley dividing system  $\mathfrak{D}$  for  $p$  of depth  $\kappa$ . The pre-weight of a type  $p$ ,  $\text{pwt}(p)$  is the supremum (if exists) of the depths of Morley dividing systems for  $p$ . If the supremum does not exist, we say that  $p$  has unbounded pre-weight.
- (vi) The weight of a type  $p$ ,  $\text{wt}(p)$  is the supremum over all nonforking extensions  $q$  of  $p$  of  $\text{pwt}(q)$  (could be unbounded).
- (vii) We say that a type  $p$  has rudimentarily finite pre-weight if there is no Morley dividing system for  $p$  of depth  $\omega$ . We say that a type  $p$  has rudimentarily finite weight if every nonforking extension of it has rudimentarily finite pre-weight.

**rem:eta**

**Remark 2.2.** Note that by indiscernibility of  $\mathbf{a}$  it follows from the definition of a dividing system that for any  $\eta \in {}^\kappa \lambda$ , the set  $p \cup \{\varphi_\alpha(\bar{x}, \bar{a}_{\eta(\alpha)}^\alpha)\}$  is consistent.

Recall that for a type  $p \in S(A)$  in a stable theory  $T$ ,  $\text{pwt}(p) \geq \mu$  iff for every  $\kappa < \mu$  there exist  $\bar{b} \models p$  and an independent set  $\{\bar{a}_\alpha : \alpha < \kappa\}$  such that  $\bar{b} \not\perp_A \bar{a}_\alpha$  for all  $\alpha < \kappa$ . So the following is clear:

**ight coincide** **Lemma 2.3.** For a stable theory  $T$ , the notions of pre-weight and weight defined in Definition 2.3 coincide with the classical ones.

*Proof.* One can easily generalize Lemma 1.3 to the situation where one has an  $A$ -independent set  $\{\bar{a}_\alpha : \alpha < \kappa\}$  instead of just two elements  $a, b$ . This gives us a Morley array. Now use Fact 1.2(i) in order to show that it is dividing.  $\square$

Our next goal is to connect weight and strong dependence (in a dependent theory), for which we recall the notion of a randomness pattern from [U1], which is simply a witness for the lack of strong dependence (and is based on Shelah's "ict-patterns"). Some of the terminology in the definition below is very similar to that in the definition of dividing systems, which will be justified by Lemma 2.11.

**Definition 2.4.** (i) A randomness pattern  $\mathfrak{X}$  for a type  $p(\bar{x}) \in S(A)$  consists of

- an array  $\mathbf{a} = \langle \bar{a}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$  (where  $\lambda, \kappa$  are ordinals,  $\lambda$  is infinite)
- a sequence of formulae  $\Phi = \langle \varphi_\alpha(\bar{x}, \bar{y}) : \alpha < \kappa \rangle$

such that

- (a)  $\mathbf{a}$  is indiscernible over  $A$ .
- (b) For every  $\eta \in {}^\kappa \lambda$ , the set

$$\Sigma_{\mathbf{a}, \eta} = \{\varphi_\alpha(\bar{x}, \bar{a}_{\eta(\alpha)}^\alpha) : \alpha < \kappa\} \cup \{\neg \varphi_\alpha(\bar{x}, \bar{a}_i^\alpha) : \alpha < \kappa, i \neq \eta(\alpha)\}$$

is consistent with  $p(\bar{x})$ .

- (ii) We call  $\kappa$  in the definition above the depth of  $\mathfrak{X}$ , and  $\lambda$  the length of  $\mathfrak{X}$ .
- (iii) A randomness pattern  $\mathfrak{X} = (\mathbf{a}, \Phi)$  for  $p$  is called Morley if  $\mathbf{a}$  is Morley.
- (iv) A randomness pattern  $\mathfrak{X} = (\mathbf{a}, \Phi)$  is called dividing if for every  $\alpha < \kappa, i < \lambda$ , the sequence  $\bar{a}_{<\lambda}^\alpha$  exemplifies that  $\varphi(\bar{x}, \bar{a}_i^\alpha)$  divides over  $A \cup \bar{a}_{<\lambda}^{\neq \alpha}$ .

The notion of strong dependence was defined in [Sh783] and investigated more in [Sh863]. We give an obviously equivalent definition (as usual, it is easier to define the negation).

**Definition 2.5.** (i) A theory  $T$  is said to be not strongly dependent if there exists a model  $M$  of  $T$ , an array  $\mathbf{a}$  in  $M$  of infinite depth and a sequence of formulae  $\Phi$  such that  $\mathfrak{X} = (\mathbf{a}, \Phi)$  is a randomness pattern.

(ii) A theory is said to be strongly stable if it is strongly dependent and stable.

onevariable *Remark 2.4.* Shelah shows in [Sh863] Observation 1.7 that if there exists a type  $p(\bar{x})$  which is not strongly dependent, then there exists such a type  $p'(x)$  with  $x$  being a singleton. Therefore in the definition above it would be enough to restrict ourselves to randomness patterns in one variable, that is,  $\varphi_\alpha = \varphi_\alpha(x, \bar{y}_\alpha)$  where  $x$  is a singleton.

*Remark 2.5.* It is easy to see that every super-stable theory is strongly stable. Also, o-minimal, c-minimal, p-minimal theories are strongly dependent. In fact, they turn out to be even dp-minimal, as we will discuss in the next section.

Our definition of a strong dependent theory motivates the following:

**Definition 2.6.** Let  $T$  be a theory.

- (i) A type  $p \in S(A)$  is called strongly dependent if there does not exist a randomness pattern of infinite depth for  $p$ .
- (ii) A type  $p \in S(A)$  is called strongly generically stable if it is strongly dependent and generically stable (see [Us] for the definition of generic stability).

One can characterize dependence in terms of randomness patterns. This characterization explains why strong dependence is a “uniform” or “almost finite” version of dependence. The proof is easy and left to the reader.

fact:depeq **Fact 2.6.** The following are equivalent:

- (i)  $T$  is dependent
- (ii) The depth of a possible randomness pattern is bounded by some ordinal. In other words, there are no unboundedly deep randomness patterns.
- (iii) There is no randomness pattern of depth  $|T|^+$ .

- (iv) *There is no randomness pattern  $\mathfrak{X} = (\mathbf{a}, \Phi)$  of depth  $\kappa = \omega$  such that  $\varphi_\alpha = \varphi$  for all  $\alpha < \omega$  and  $\varphi_\alpha \in \Phi$ .*

Note that based on the Observation above, one can define a notion of a *dependent type* in any theory  $T$ :

**Definition 2.7.** *A type  $p \in S(A)$  is called dependent if there is no randomness patterns for  $p$  of unbounded depths.*

The characterization above remains true when working locally, that is:

obs:depeq

**Observation 2.7.** *The following are equivalent:*

- (i)  *$p$  is dependent*
- (ii) *The depth of a possible randomness pattern for  $p$  is bounded by some ordinal. In other words, there are no unboundedly deep randomness patterns for  $p$ .*
- (iii) *There is no randomness pattern for  $p$  of depth  $|T|^+$ .*
- (iv) *There is no randomness pattern  $\mathfrak{X} = (\mathbf{a}, \Phi)$  for  $p$  of depth  $\kappa = \omega$  such that  $\varphi_\alpha = \varphi$  for all  $\alpha < \omega$  and  $\varphi_\alpha \in \Phi$ .*

And clearly

**Corollary 2.8.** *A theory is dependent if and only if every type is dependent.*

obs:divrand

**Observation 2.9.** *Let  $T$  be dependent. If there exists a (Morley) randomness pattern  $\mathfrak{X} = (\mathbf{a}, \Phi)$  for a type  $p$ , then there exists a (Morley) dividing randomness pattern  $\mathfrak{X}' = (\mathbf{a}', \Phi')$  for  $p$ .*

*Proof.* Take  $\Phi' = \langle \varphi'_\alpha(\bar{x}, \bar{y}_1^\alpha \frown \bar{y}_2^\alpha) : \alpha < \kappa \rangle$  where  $\varphi'_\alpha(\bar{x}, \bar{y}_1^\alpha \frown \bar{y}_2^\alpha) = \varphi_\alpha(\bar{x}, \bar{y}_1^\alpha) \wedge \neg \varphi_\alpha(\bar{x}, \bar{y}_2^\alpha)$  and let  $\mathbf{a}' = \{\bar{a}_{2i}^\alpha \bar{a}_{2i+1}^\alpha : \alpha < \kappa, i < \lambda\}$ . It is easy to check that this is still a randomness pattern. It is dividing since  $T$  is dependent, and therefore the set

$$\{\varphi(\bar{x}, \bar{a}_i^\alpha)^{\text{parity}(i)} : i < \lambda\}$$

can not be consistent for any  $\alpha$ . Clearly, if the original pattern was Morley, so is the new one.  $\square$

In order to obtain the converse, we need to solve a basic exercise in logic:

n of dividing

**Lemma 2.10.** (i) *Let  $p(x)$  be a type over a set  $A$ , let  $I = \langle b_i \rangle_{i \in \mathcal{O}}$  be a sequence indiscernible over  $A$ , and let  $\varphi(x, y)$  be a formula such that  $p(x) \cup \varphi(x, b_i)$  is consistent for some (all)  $i$  and  $\{\varphi(x, b_i)\}_{i \in \mathcal{O}}$  is  $k$ -inconsistent for some  $k \in \mathbb{N}$ . Then*

$$p(x) \cup \{\varphi(x, b_l)\} \cup \{\neg \varphi(x, b_i)\}_{i \neq l}$$

*is consistent for all  $l$ .*

- (ii) *Let  $p(x)$  be a type over a set  $A$ ,  $n < \omega$  and let  $\langle b_i^\alpha : \alpha < n, i < \omega \rangle$ ,  $\{\varphi_\alpha(x, y_\alpha) : \alpha < n\}$  be a dividing pattern for  $p$  over  $A$  of depth  $n$ . Then there exists a randomness pattern for  $p$  over  $A$  of depth  $n$ ; in fact, the randomness pattern is given by the same array and collection of formulae.*
- (iii) *Clause (ii) holds also when the depth  $n < \omega$  is replaced with any cardinal  $\kappa$ .*

*Proof.* (i) Without loss of generality let us assume that  $O = \mathbb{Q}$  and  $l = 0$ . Assume also that  $k$  is *minimal* such that the set  $\Delta = \{\varphi(x, b_i) : i \in \mathbb{Q}\}$  is  $k$ -inconsistent. By the assumptions  $k > 1$ .

By indiscernibility it is enough to show that the set

$$\{\varphi(x, b_0)\} \cup \{\neg\varphi(x, b_i) : i \in \mathbb{Z}, i \neq 0\}$$

is consistent. Since  $\Delta$  is  $(k - 1)$ -consistent, the set

$$\{\varphi(x, b_0)\} \cup \{\varphi(x, b_{\frac{1}{i+1}}) : 1 < i < k\}$$

is consistent, realized by some  $d$ . But  $\Delta$  is  $k$ -inconsistent, so clearly

$$d \models \{\varphi(x, b_0)\} \cup \{\varphi(x, b_{\frac{1}{i+1}}) : 1 < i < k\} \cup \{\neg\varphi(x, b_i) : i \in \mathbb{Z}, i \neq 0\}$$

and we are done.

- (ii) A very similar proof (working with  $\bigwedge_\alpha \varphi_\alpha(x, b_0^\alpha)$  instead of  $\varphi(x, b_0)$ ) is left to the reader.
- (iii) By clause (ii) and compactness. □

We summarize the connections between randomness patterns and dividing systems:

**lem:divrand**

- Lemma 2.11.** (i) *A (Morley) dividing randomness pattern for a type  $p$  is a (Morley) dividing system for  $p$ .*
- (ii) *A dividing (Morley) system for a type  $p$  is a dividing (Morley) randomness pattern for  $p$ .*
- (iii) *There exists a (Morley) dividing system for  $p$  of depth  $\kappa$  iff there exists a (Morley) dividing randomness pattern for  $p$  of depth  $\kappa$ .*
- (iv) *If  $T$  is dependent, then there exists a (Morley) dividing system for  $p$  of depth  $\kappa$  iff there exists a (Morley) randomness pattern for  $p$  of depth  $\kappa$ .*
- (v) *If  $T$  is dependent, then  $\text{pwt}(p)$  is the supremum of the depths of Morley randomness patterns for  $p$ .*

*Proof.* Clause (i) is obvious. Clause (ii) is Lemma 2.10(ii). Clause (iii) is now clear. Clause (iv) holds by Observation 2.9, and clause (v) follows immediately. Note that since we are not changing the array  $\mathbf{a}$  in either direction, one is Morley iff the other one is. □

So we can conclude:

**weightdep**

**Theorem 2.12.** *Let  $T$  be dependent.*

- (i) *Every type has bounded weight.*
- (ii)  *$T$  is strongly dependent if and only if every type has rudimentarily finite pre-weight if and only if every type has rudimentarily finite weight.*
- (iii) *Let  $p$  be a type, then  $\text{pwt}(p)$  is finite if and only if the depth of a Morley randomness pattern for  $p$  is bounded (by a finite number).*

*Proof.* (i) Similar to (ii) (using Fact 2.6).

- (ii) If  $T$  is strongly dependent then by the definition there is no randomness pattern of infinite depth, which by Lemma 2.11(iv) implies no dividing systems of infinite depth, hence any type has rudimentarily finite pre-weight. On the other hand, if  $T$  is not strongly dependent, then there exists a not strongly dependent type  $p$ , that is, there

is a randomness pattern in  $p$  of infinite depth and therefore a dividing system in  $p$  of infinite depth (again, by Lemma 2.11(iv)); applying Lemma 2.1 we obtain a type  $p'$  and a *Morley* dividing system in  $p'$  which yields a type of infinite pre-weight.

(iii) Clear. □

**Corollary 2.13.** *If  $T$  is stable, then  $T$  is strongly stable if and only if every type has finite weight.*

*Proof.* It is well known that in a stable theory a type of rudimentary finite weight is domination equivalent to a finite free product of weight-1 types, and hence has finite weight. □

This corollary appears already in [Us1] and was observed independently by Adler in [Ad].

### 3. DP-MINIMALITY, STABILITY AND WEIGHT

In this section we investigate dp-minimality. Let us start with the main definition and some examples.

We give a simplified version of the definition which appears in [Sh863].

**Definition 3.1.** *A theory is not dp-minimal if there are mutually indiscernible sequences  $\langle \bar{a}_i \rangle_{i \in \omega}$  and  $\langle \bar{b}_j \rangle_{j \in \omega}$  over  $M$  and formulas  $\phi(x, \bar{y})$  and  $\psi(x, \bar{z})$  where  $x$  is a single variable in the home sort (not an imaginary) and for any  $n, m \in \omega$  the type*

$$\phi(x, \bar{a}_m) \wedge \bigwedge_{i \neq m} \neg \phi(x, \bar{a}_i) \wedge \psi(x, \bar{c}_n) \wedge \bigwedge_{j \neq n} \neg \psi(x, \bar{c}_j)$$

*is consistent.*

Clearly, dp-minimality is in some sense the “strongest” version of strong dependence, that is:

**Observation 3.1.** *A theory is dp-minimal if and only if there does not exist a randomness pattern  $\mathfrak{X}$  of depth  $\kappa = 2$  for the type  $x = x$ , where  $x$  is a single variable. In particular, a dp-minimal theory is strongly dependent, hence dependent.*

*Proof.* One only needs to note that if there is a randomness pattern of infinite depth, there is such a pattern for  $x = x$  where  $x$  is a singleton (that is, lack of strong dependence can be witnessed by a pattern in one variable), see Remark 2.4. □

Just like with dependent and strongly dependent, we can define dp-minimal types. One may hope that this (or a similar) notion will play the role of a minimal type in the strongly dependent context (although our guess is that the related class of weight-1 types has a better chance to fulfill this hope).

**Definition 3.2.** (i) *A type  $p$  is called dp-minimal if there is no randomness pattern for  $p$  of depth 2.*  
(ii) *We will call a randomness pattern  $\mathfrak{X}$  of depth 2 for a type  $p$  a witness of non dp-minimality of  $p$ . Such a witness is called Morley if  $\mathfrak{X}$  is Morley.*

It is quite easy to verify that given any infinite model  $M$  the theory of  $M^2$  is never dp-minimal. Also, dp-minimality generalizes many known and well-studied notions of minimality:

**Fact 3.2.**

- (i) Any strongly minimal theory is dp-minimal.
- (ii) Any o-minimal theory is dp-minimal.
- (iii) Any super-stable theory of U-rank 1 is dp-minimal.

*Proof.* This follows straight from the definitions. □

*Remark 3.3.* In [Sh863] Shelah proved that any p-adically closed field is strongly dependent. We believe that his proof in fact shows that all p-adically closed fields are dp-minimal. In fact, it can probably be generalized to all p-minimal structures, thus widening the class of examples.

There are new minimal theories with respect to this notion, though. The following is one of our motivating examples:

table example

**Observation 3.4.** *The theory of  $\aleph_0$  nested equivalence relations with infinitely many infinite classes  $\{E_n : n < \omega\}$  such that  $E_{n+1}$  refines each class of  $E_n$  into infinitely many classes is a stable non super-stable theory which is dp-minimal.*

*Proof.* Straightforward. □

In order to understand why the example above is dp-minimal, let us apply the techniques developed in the previous section for strong dependence to the more restrictive notion of dp-minimality and characterize it in more familiar terms.

dpweight

**Theorem 3.5.** *Let  $T$  be dependent.*

- (i) A dp-minimal type has pre-weight 1.
- (ii)  $T$  is dp-minimal if and only if every 1-type has pre-weight 1 if and only if every 1-type has weight 1.
- (iii) A stable theory is dp-minimal if and only if every 1-type has pre-weight 1 if and only if every 1-type has weight 1 (in the usual sense of stable theories).

*Proof.* Although the proofs are very similar to what was done in the previous section, we will still sketch them.

- (i) If  $p$  has pre-weight at least 2, there is a dividing system for  $p$  of depth 2, hence a randomness pattern for  $p$  of depth 2 by Lemma 2.11.
- (ii) The “only if” direction follows from (i). On the one hand, if  $T$  is not dp-minimal, then there is a randomness pattern of depth 2 for  $x = x$ , hence a dividing system of depth 2 (Lemma 2.11 again) for  $x = x$ , hence a Morley dividing system of depth 2 for some  $p'$  (which extends of  $x = x$ , in particular still is a 1-type) by Lemma 2.1. Hence  $p'$  has pre-weight at least 2.
- (iii) Use Lemma 2.3. □

*Remark 3.6.* One can easily check that in the example of nested equivalence relations above not only do all 1-types have weight 1, but they are all in fact *regular*. This does not need to be the case, of course, even for a super-stable theory. For instance, consider the following well-known example of weight 1 non-regular type:

$T = \text{Th}(G, +)$  where  $G = (\mathbb{Z}_4)^\omega$ . Then  $T$  is  $\omega$ -stable of Morley rank 2, totally categorical, and the generic type of  $G$  is a weight 1 non-regular type (although it is domination equivalent

to the generic type of  $2G$ , which is strongly minimal). Still, every 1-type has weight 1 and so  $T$  is dp-minimal.

Recall that if  $T$  is dependent and  $M$  is a model, one can form the *Shelah expansion*  $M^*$  of  $M$  by enriching it with the traces of all externally definable subsets of it (see section 1 of [Sh783] for a formal definition). Recall also that the main theorem of section 1 of [Sh783] states that  $\text{Th}(M^*)$  eliminates quantifiers. We would like to point out that this fact gives us many additional examples of theories we are interested in:

**Observation 3.7.** *Shelah expansion of a model of a strongly dependent/dp-minimal theory is strongly dependent/dp-minimal.*

*Proof.* Let  $L(T)$  be the language of  $T$ ,  $\mathfrak{C}$  the monster model of  $T$ . Similarly,  $L(T^*)$  is the language of  $T^* = \text{Th}(M^*)$ , and  $\mathfrak{C}^*$  is the expanded monster of  $T^*$ .

Let us prove the Observation for dp-minimal theories. Suppose  $T^*$  is not dp-minimal. Then there is a randomness pattern  $(\mathfrak{a}, \Phi)$  of depth 2 in  $x = x$ , so let  $\Phi = (\varphi(x, \bar{y}), \psi(x, \bar{z}))$ . By quantifier elimination for  $T^*$ ,  $\varphi(x, \bar{y})$  is equivalent to a basic formula, which is itself equivalent to a formula  $\varphi'(x, \bar{y}, \bar{c})$  where  $\varphi' \in L(T)$  and  $\bar{c} \in \mathfrak{C}$ . Same is true for  $\psi(x, \bar{y})$ . So we obtain a randomness pattern of depth 2 for  $x = x$  in  $\mathfrak{C}^*$  defined by  $L(T)$ -formulas with additional parameters. Hence this is also a randomness pattern in  $\mathfrak{C}$ , a contradiction.  $\square$

**Corollary 3.8.** *A weakly o-minimal theory is dp-minimal.*

We would like to try to develop an analogous theory for a rosy theory, replacing forking with  $\mathfrak{b}$ -forking. From now on we assume that the theory  $T$  is rosy.

**cc** **Definition 3.3.**

- (i) We say that a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  and a set  $A$  witness crisscrossed forking (*cc-forking*) if  $\models \exists x \phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ , both  $\phi(x, \bar{a})$  and  $\psi(x, \bar{b})$  fork over  $A$ ,  $\bar{a} \perp_A \bar{b}$  and  $\bar{b} \perp_A \bar{a}$ .
- (ii) We say that a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  and a set  $A$  witness crisscrossed strong dividing (*cc-strong-dividing*) if  $\models \exists x \phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ , both  $\phi(x, \bar{a})$  and  $\psi(x, \bar{b})$  strongly divide over  $A$  and  $\bar{a} \perp_A^{\mathfrak{b}} \bar{b}$ .
- (iii) We say that a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  and a set  $A$  witness crisscrossed  $\mathfrak{b}$ -dividing (*cc- $\mathfrak{b}$ -dividing*) if  $\models \exists x \phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ , both  $\phi(x, \bar{a})$  and  $\psi(x, \bar{b})$   $\mathfrak{b}$ -divide over  $A$  and  $\bar{a} \perp_A^{\mathfrak{b}} \bar{b}$ .
- (iv) We say that a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  and a set  $A$  witness crisscrossed  $\mathfrak{b}$ -forking (*cc- $\mathfrak{b}$ -forking*) if  $\models \exists x \phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ , both  $\phi(x, \bar{a})$  and  $\psi(x, \bar{b})$   $\mathfrak{b}$ -fork over  $A$  and  $\bar{a} \perp_A^{\mathfrak{b}} \bar{b}$ .
- (v) We say that  $T$  admits *cc-forking* (or *strong dividing* or  *$\mathfrak{b}$ -dividing* or  *$\mathfrak{b}$ -forking*) if there exists a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  witnessing *cc-forking* (or *strong dividing* or  *$\mathfrak{b}$ -dividing* or  *$\mathfrak{b}$ -forking*) over  $A$ .
- (vi) Let  $p$  be a 1-type over a set  $A$ . We say that a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  witnesses crisscrossed forking (or *strong dividing* or  *$\mathfrak{b}$ -forking*,  *$\mathfrak{b}$ -dividing*) in  $p$  if  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$ ,  $A$  witness *cc-forking* (or *strong dividing* or  *$\mathfrak{b}$ -forking*,  *$\mathfrak{b}$ -dividing*) and the pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  is consistent with  $p$ .
- (vii) We say that a type  $p \in S_1(A)$  admits *cc-forking* (or *strong dividing* or  *$\mathfrak{b}$ -dividing* or  *$\mathfrak{b}$ -forking*) if there exists a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  witnessing *cc-forking* (or *strong dividing* or  *$\mathfrak{b}$ -dividing* or  *$\mathfrak{b}$ -forking*) in  $p$ .

**ccweight**

*Remark 3.9.* (i)  $T$  admits  $cc$ -forking (strong dividing,  $\mathfrak{b}$ -forking,  $\mathfrak{b}$ -dividing) iff there exists a set  $A$ , a type  $p \in S(A)$  and a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  which witnesses  $cc$ -forking (strong dividing,  $\mathfrak{b}$ -forking,  $\mathfrak{b}$ -dividing) in  $p$ .

(ii) Let  $T$  be stable. Then a type  $p \in S_1(A)$  does not admit  $cc$ -forking if and only if it has pre-weight 1.

(iii) So a stable  $T$  does not admit  $cc$ -forking iff every 1-type has pre-weight 1 iff every 1-type has weight 1.

**witness**

*Remark 3.10.* In a stable theory, a pair  $(\phi(x, \bar{a}), \psi(x, \bar{b}))$  and a set  $A$  witness  $cc$ -forking if and only if it witnesses  $cc$ - $\mathfrak{b}$ -forking.

*Proof.* By Fact 1.5 we know that any instance of  $\mathfrak{b}$ -forking is an instance of forking and viceversa.  $\square$

The following theorem shows that in a rosy theory one can pass from a  $cc$ - $\mathfrak{b}$ -forking witness to a much stronger version. Recalling that strong dividing of  $\varphi(x, \bar{a})$  means that *every* indiscernible sequence in  $\text{tp}(\bar{a})$  exemplifies dividing, this suggests that starting with  $cc$ - $\mathfrak{b}$ -forking, one should be able to construct a dividing system (hence obtain an instance of weight at least 2 in the sense of the previous section). We will come back to this in a subsequent work [OU].

**equivalent**

**Theorem 3.11.** *The following are equivalent for any  $p \in S_1(A)$ .*

- (i)  $p$  admits  $cc$ - $\mathfrak{b}$ -forking.
- (ii)  $p$  admits  $cc$ - $\mathfrak{b}$ -dividing.
- (iii) *There is an extension  $p(x, B)$  of  $p(x)$  such that  $p(x, B)$  admits  $cc$ -strong dividing.*

*Proof.* Any witness for  $cc$ -strong dividing is a witness of  $cc$ - $\mathfrak{b}$ -dividing and any witness for  $cc$ - $\mathfrak{b}$ -dividing is a witness of  $cc$ - $\mathfrak{b}$ -forking.

(i)  $\Rightarrow$  (ii). Let

$$\{\varphi_i(x, \bar{a}), \psi(x, \bar{b})\}, A$$

be a  $cc$ - $\mathfrak{b}$ -forking witness for  $p$ . By definition there are finitely many formulas  $\varphi_i(x, \bar{a}'_i)$ ,  $\psi_j(x, \bar{b}'_j)$  and tuples  $\bar{c}', \bar{d}'$  such that  $\varphi(x, \bar{a}) \vdash \bigvee_{i=1}^{k_a} \varphi_i(x, \bar{a}'_i)$ ,  $\psi(x, \bar{b}) \vdash \bigvee_{j=1}^{k_b} \psi_j(x, \bar{b}'_j)$  and  $\varphi_i(x, \bar{a}'_i)$  strongly divides over  $A\bar{c}'$  and  $\psi_j(x, \bar{b}'_j)$  strongly divides over  $A\bar{d}'$ .

By hypothesis  $\bar{a} \downarrow_A^{\mathfrak{b}} \bar{b}$  so by extension of  $\mathfrak{b}$ -independence we can find  $\bar{a}'' \models \text{tp}(\bar{a}/A\bar{b})$  such that  $\bar{a}'' \downarrow_A \langle \bar{b}'_j \rangle \bar{d}' \bar{b}$ . Let  $\langle \bar{a}''_i \rangle, \bar{c}''$  be images of  $\langle \bar{a}'_i \rangle, \bar{c}'$  under an automorphism that fixes  $A, \bar{b}$  and sends  $\bar{a}$  to  $\bar{a}''$ .

Using extension on the other side there are  $\bar{b}'' \langle \bar{b}''_j \rangle \bar{d}'' \models \text{tp}(\bar{b} \langle \bar{b}'_j \rangle \bar{d}' / A\bar{a}'')$  such that

$$\langle \bar{a}''_i \rangle \bar{c}'' \bar{a}'' \downarrow_A^{\mathfrak{b}} \bar{b}'' \langle \bar{b}''_j \rangle \bar{d}''.$$

But  $\text{tp}(\bar{a}'' \bar{b}'' / A) = \text{tp}(\bar{a}'' \bar{b} / A) = \text{tp}(\bar{a} \bar{b} / A)$  so by applying an automorphism over  $A$ , we can find  $\langle \bar{a}_i \rangle, \bar{c}, \langle \bar{b}_j \rangle, \bar{d}$  such that  $\langle \bar{a}_i \rangle \bar{c} \bar{a} \downarrow_A \langle \bar{b}_j \rangle \bar{d} \bar{b}$ .

So in particular we have  $\text{tp}(\langle \bar{a}_i \rangle \bar{c} / A\bar{a}) = \text{tp}(\langle \bar{a}'_i \rangle \bar{c}' / A\bar{a})$  and  $\text{tp}(\langle \bar{b}_i \rangle \bar{d} / A\bar{b}) = \text{tp}(\langle \bar{b}'_i \rangle \bar{d}' / A\bar{b})$ .

Therefore

**equ:1**

$$(1) \quad \varphi(x, \bar{a}) \models \bigvee_{i=1}^{k_a} \varphi_i(x, \bar{a}_i), \quad \psi(x, \bar{b}) \models \bigvee_{i=1}^{k_b} \psi_i(x, \bar{b}_i),$$

and

$$\boxed{\text{equ:2}} \quad (2) \quad \begin{array}{l} \varphi(x, \bar{a}_i) \text{ strongly divides over } A\bar{c} \text{ for all } i, \\ \psi(x, \bar{b}_j) \text{ strongly divides over } A\bar{d} \text{ for all } j \end{array}$$

Since  $\varphi(x, \bar{a}) \wedge \psi(x, \bar{b})$  is consistent with  $p$ , it is clear from (1) that the conjunction  $\varphi_i(x, \bar{a}_i) \wedge \psi_j(x, \bar{b}_j)$  is consistent with  $p$  for some  $i, j$ . By monotonicity of  $\mathfrak{b}$ -forking independence we know that  $\bar{a}_i \perp_A \bar{b}_j$  so (2) implies that  $(\varphi(x, \bar{a}_i), \psi(x, \bar{b}_j))$ ,  $A$  is a witness for  $cc$ - $\mathfrak{b}$ -dividing.

(ii)  $\Rightarrow$  (iii). Once again we will prove the case  $n = 2$ .

Let

$$\{\varphi_i(x, \bar{a}), \psi(x, \bar{b})\}, A$$

be a  $cc$ - $\mathfrak{b}$ -dividing witness for  $p$ . Let  $D'$  and  $E'$  be supersets of  $A$  such that  $\varphi(x, \bar{a})$  strong divides over  $D'$  and  $\psi(x, \bar{b})$  strong divides over  $E'$ . Since  $\bar{a} \perp_A \bar{b}$  we can, by extension (as in the proof of (i)  $\Rightarrow$  (ii)), find  $D, E$  satisfy types  $tp(D'/A\bar{a})$  and  $tp(E'/A\bar{b})$  respectively and such that  $\bar{a}D \perp_A \bar{b}E$ ; so in particular  $\varphi(x, \bar{a})$  strong divides over  $D$  and  $\psi(x, \bar{b})$  strong divides over  $E$ ,  $\bar{a} \perp_D \bar{b}$ ,  $\bar{a} \perp_E \bar{b}$ , and  $\bar{a} \perp_{D \cup E} \bar{b}$ .

Since by definition  $\bar{a} \notin \text{acl}(D)$  and  $\bar{b} \notin \text{acl}(E)$  we get that  $\bar{a}, \bar{b} \notin \text{acl}(ED)$ : e.g.,  $\bar{a} \notin \text{acl}(D)$ , but  $\bar{a} \perp_E D$ , so  $\bar{a} \notin \text{acl}(ED)$ .

So

$$\boxed{\text{equ:3}} \quad (3) \quad \begin{array}{l} \varphi(x, \bar{a}) \text{ strongly divides over } E \cup D, \\ \psi(x, \bar{b}) \text{ strongly divides over } E \cup D, \\ \bar{a} \perp_{D \cup E} \bar{b}. \end{array}$$

Let  $B := D \cup E$  and let  $p(x, B, \bar{a}, \bar{b})$  be a non  $\mathfrak{b}$ -forking extension of  $p(x) \cup \{\varphi(x, \bar{a}) \cup \psi(x, \bar{b})\}$  and let  $p(x, B)$  be the restriction of  $p(x, B, \bar{a}, \bar{b})$  to  $B$ . All the conditions in the definition of  $cc$ -strong dividing are satisfied which completes the proof of the theorem.  $\square$

**mainstable**

**Corollary 3.12.** *The following are equivalent for a stable theory  $T$ .*

- (i)  $T$  admits  $cc$ -forking.
- (ii)  $T$  admits  $cc$ -strong-dividing.
- (iii)  $T$  is not  $dp$ -minimal.
- (iv) There exists a 1-type  $p$  with weight bigger than 1.

*Proof.* This is just putting together the propositions we have proved until now.

(i)  $\iff$  (ii): Theorem 3.11.

(iii)  $\iff$  (iv): Theorem 3.5.

(i)  $\iff$  (iv): Remark 3.9.  $\square$

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