

STABLE DOMINATION AND WEIGHT

ALF ONSHUUS AND ALEXANDER USVYATSOV

ABSTRACT. We develop the theory of domination by stable types and stable weight in an arbitrary theory.

1. INTRODUCTION

It has lately become clear in model theory that understanding the stable parts of certain theories is quite useful for the analysis of such theories (for example, [2]). In this paper we study the general theory of types which have a nontrivial stable part. In particular, we develop the theory of domination by stable types and stable weight in an arbitrary theory. We also connect these notions to stable domination, as defined in [2], and, more generally, [3] and [12].

We begin with studying the basic properties of stable types with respect to forking and canonical bases. We believe that many of those facts must be known, but we did not find them in the literature. In a sense what we observe is that when restricting ourselves to realizations of stable types, everything behaves very much like in a stable theory. Moreover, we get a kind of “stable embeddedness” in the following sense: whenever we speak about (non) forking, the base can be restricted to the stable world; whenever we speak about lack of domination, the witnesses can be assumed to realize stable types, etc. This is the content of section 3 (all except the very last statement, which appears in the beginning of section 5).

In section 4 we study basic properties of (forking) domination and weight, partially restricting ourselves to stable types. In particular, we show that in a strong theory, any type has rudimentarily finite stable weight, Lemma 4.7. Then we examine the notion of forking domination in the special context of dependent theories and prove that in a dependent theory, a type dominated by a generically stable type, is itself generically stable, Theorem 4.9. We find this result of interest on its own, but it also plays a major role in the proof of the main theorem of section 5.

Section 5 contains perhaps the most interesting results of the paper. We show that domination by a stable type is witnessed in the definable closure of the dominated tuple, Theorem 5.3. It follows, for example, that a type dominated by a stable type is domination equivalent to its “stable part”. Restricting ourselves to dependent theories, we conclude that a type dominated by a stable type (in a sense of forking) is stably dominated, Theorem 5.4. The main ingredient of the proof is that such a type must be stationary, which is given by Theorem 4.9 and basic properties of generically stable types.

Section 6 is devoted to stable weight. First we observe that Hyttinen’s results [11] work in the context of stable types. We conclude that in a strong theory, any type, which is dominated by a stable stable type, is domination equivalent to a free finite product of weight-1 stable types,

Date: November 4, 2009.

The second author was partially supported by FCT grant SFRH / BPD / 34893 / 2007.

Theorem 6.5. For the some of the proofs, we refer the reader to [\[OnUs1\]](#), where such a decomposition result was proven in a rather general context.

In the second part of section 6, we use stable weight in order to determine the “stable part” of an arbitrary type. We observe that the stable weight of a type of a tuple a is witnessed by a stable tuple in the definable closure of a . Then we show that “morally”, a type is dominated by a stable type if and only if its weight equals to its stable weight. There are certain technical issues with the precise statement. Theorem 6.14 is the most general result. It can be improved under the assumption that T is dependent, Corollary 6.18. Note that this essentially shows that in a strongly dependent theory, a type is stably dominated if and only if it can be “decomposed” into a finite free product of weight-1 stable types.

Appendix A contains the proof of “descent” for stably dominated types (which is a slightly generalized and simplified version of the proof given in [\[U1\]](#)).

1.1. Notations and Assumptions. Given a theory T , we will work inside its monster model denoted by \mathfrak{C} . By “monster” we mean that all cardinals we mention are “small” (i.e. smaller than saturation of \mathfrak{C}), all sets are small subsets of \mathfrak{C} , all models are small elementary submodels of \mathfrak{C} , and truth values of all formulae and all types are calculated in \mathfrak{C} . We denote tuples (finite unless said otherwise) by lower case letters a, b, c, \dots , sets by A, B, C, \dots , models by M, N, \dots , etc.

By $a \equiv_A b$ we mean $\text{tp}(a/A) = \text{tp}(b/A)$. Recall that this is equivalent to having $\sigma \in \text{Aut}(\mathfrak{C}/A)$ satisfying $\sigma(a) = b$.

Given an order type O , a sequence $I = \langle a_i : i \in O \rangle$ and $j \in O$, we often denote the set $\{a_i : i < j\}$ by $a_{<j}$; similarly for $a_{\leq j}, a_{>j}$. We also often identify the sequence I with the set $\cup I$; that is, when no confusion should arise we write $\text{tp}(a/I)$.

We will write $a \downarrow_A B$ for “ $\text{tp}(a/AB)$ does not fork over A ” (see Definition 2.1 below) even if T is not simple. Although nonforking is generally not an independence relation, we still find this notation convenient.

For simplicity we assume $T = T^{eq}$ for all theories T mentioned in this paper.

2. PRELIMINARIES

In this section we remind the reader several basic definitions and facts that will be used in the paper.

2.1. Forking and Morley sequences. The following definitions are standard in stability theory and can be found in [\[Shelahbook\]](#).

Definition 2.1.

- A formula $\varphi(x, a)$ is said to divide over A if there is an infinite indiscernible sequence I containing a such that the set $\{\varphi(x, a') : a' \in I\}$ is inconsistent.
- A (partial) type divides over A if it contains a dividing formula.
- A (partial) type is said to fork over A if every global extension of it divides over A . Equivalently, a type forks over A if and only if it implies a finite disjunction of formulas which divide over A .
- A (partial) type p is said to split over A if there are $\varphi(x, a), \neg\varphi(x, b) \in p$ with $a \equiv_A b$. A type p splits strongly over A if in addition there is an A -indiscernible sequence containing both a and b .

dfn:forking

Definition 2.2. Let $A \subseteq C$ be sets, O an order type. We call a sequence $I = \langle a_i : i \in O \rangle$ a Morley sequence in $\text{tp}(a/C)$ over C based on A if

- I is an indiscernible sequence in $\text{tp}(a/C)$
- $\text{tp}(a_i/Ca_{<i})$ does not fork over A for all $i \in O$

We omit “based on A ” if $A = C$, and sometimes we omit “over C ” when it is stated explicitly that the sequence is in a type over C .

Definition 2.3.

- We call a sequence $\langle A_i : i < O \rangle$ (where O is an order type) nonforking over A if $\text{tp}(A_i/AA_{<i})$ does not fork over A for all $i \in O$.
- We call a set $\{A_i : i < \lambda\}$ a nonforking or an independent set over A if $\text{tp}(A_i/AA_{\neq i})$ does not fork over A for all $i < \lambda$.

2.2. Stable types. Recall that a formula $\varphi(x, y)$ is called *stable* if it does not have the order property (see [15]); this is, if there are no infinite sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ such that $\models \varphi(a_i, b_j)$ if and only if $i \leq j$. A theory is stable if no formula has the order property.

In this subsection, we will define and give the basic results of stable types. This study will be retaken in Section 3 where we will generalize even more results of stability theory to realizations of stable types in a general context.

Recall that in a stable theory every type is definable.

Definition 2.4. We call a type *stable* if every extension of it is definable.

Stable types were introduced by Lascar and Poizat in [6] and have been studied since. The following are very important facts about stable types and in some way they give evidence that this is the right “localization” of stability.

stable types

Fact 2.1. Let $p(x) \in S(M)$. Then the following are equivalent.

- (i) $p(x)$ is stable.
- (ii) For every $N \succ M$ the type p has at most $|N|^{\aleph_0}$ extensions to N .
- (iii) There are no infinite indiscernible sequences $\langle a_i \rangle_{i \in I}$ in p and $\langle b_i \rangle_{i \in I}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.

Proof. The equivalence of (i) and (ii) is Theorem 4.4 in [6]. The equivalence of (ii) and (iii) follows exactly as in the stable case (see for example [15]-II.) □

ing in stable

Fact 2.2. Let p be a stable type over B and let $A \subset B$. Then the following are equivalent.

- (i) $p(x)$ does not fork over A .
- (ii) $p(x)$ is definable over $\text{acl}(A)$.

Proof. This is well known and is proved in [6]. □

We will also need the following.

ing in stable

Theorem 2.3. Let $p = \text{tp}(a/A)$ be a stable type and let $A \subset C$. Then the following hold.

- (i) There is a smallest definably closed set $A_0 \subseteq \text{acl}(A)$ over which p is defined (does not fork). This means that $A_0 = \text{dcl}(A_0)$, p is definable over A_0 and p does not fork over $B \subseteq A$ iff $A_0 \subseteq \text{acl}(B)$. A_0 is called a canonical base of p . It is also in the definable closure of a finite Morley sequence in p over A .

- (ii) If p is stationary (e.g. $A = \text{acl}(A)$), then A_0 as in clause (i) is unique, and we call it the canonical base of p , $\text{Cb}(p)$. In this case $\text{Cb}(p) \subseteq \text{dcl}(A)$. Moreover, any automorphism of \mathfrak{C} fixes $p^{\mathfrak{C}}$ setwise if and only if it fixes $\text{Cb}(p)$ pointwise.
- (iii) Forking has local character for stable types. That is, given a formula $\varphi(x, y)$, there does not exist an increasing chain of sets $\langle A_i : i < \omega \rangle$ and a type p over $A = \cup_{i < \omega} A_i$ such that for every i there exist $a_i, \varphi(x, a_i) \in p \upharpoonright A_{i+1}$, $\varphi(x, a_i)$ forks over A_i .
- (iv) $\text{tp}(a/A)$ does not fork over A .
- (v) If b is such that $\text{tp}(b/A)$ is also both, then $\text{tp}(ab/A)$ is stable.
- (vi) $\text{tp}(b/A)$ is stable for any $b \in \text{dcl}(Aa)$.
- (vii) For any $B \supset A$ there is some $a' \equiv_A a$ such that $a' \downarrow_A B$.

Proof. Properties (i) through (iv) and (vii) are proven just like in stable theories (e.g. [\[U4\]](#) [pillaybook](#)) using definability of stable types, the definition of forking, Fact 2.1 and Fact 2.2. Clauses (v) and (vi) are easy. \square

Notice that (v) above implies, by an easy induction, that any sequence of realizations of stable types is stable.

2.3. Dependent theories and generically stable types. The standard definition of dependence states that a theory T is called *dependent* if there does not exist a formula which exemplifies the independence property. However, we are mostly going to use the following equivalent definition:

Definition 2.5. T is dependent if and only if there do not exist an indiscernible sequence $I = \langle a_i : i < \lambda \rangle$, a formula $\varphi(x, y)$ and \bar{b} such that both

$$\{i : \models \varphi(a_i, \bar{b})\}$$

and

$$\{i : \models \neg \varphi(a_i, \bar{b})\}$$

are unbounded in λ .

Fact 2.4. (Shelah, [\[Sh783\]](#) [\[U0\]](#)) Strong splitting implies dividing (and therefore forking).

We will now define and give the basic properties of generically stable types. For more details and proofs, see [\[U2\]](#).

Definition 2.6. Let T be dependent. We call a type $p \in S(A)$ generically stable if every Morley sequence in it is an indiscernible set.

Remark 2.1. If a type is stable, then it is generically stable (see e.g. [\[U2\]](#) Section 6).

Fact 2.5. (T dependent)

- (i) $p \in S(A)$ is generically stable if and only if some Morley sequence in p is an indiscernible set.
- (ii) If $p \in S(A)$, then it is definable over $\text{acl}(A)$. If p is definable over A (e.g. $A = \text{acl}(A)$), then p is stationary.
- (iii) Let p be generically stable. Nonforking defines on the set of realizations of p a stable independence relation (that is, a relation satisfying all the axioms of a stable independence relation).

Proof. See [U_s12]. □

Note that generic stability is not necessarily closed under extensions. In fact, it is quite easy to see (shown in [U_s12]) that p is stable if and only if every extension of it is generically stable. So whenever one works with stable-like types (stable types, generically stable types or stably dominated types – which will be defined later) one is always trying to understand some specific parts of a structure, and see how much this types can shed light on the full structure.

2.4. Strong dependence and strongness. Since some of the consequences of strong dependence (and more generally, strongness) will be explored in this paper, we remind the reader the basic definitions.

In the definitions below we denote tuples by \bar{x}, \bar{a} (in order to stress the difference between singletons and finite tuples of arbitrary length).

Definition 2.7.

- A (partial) type p over a set A is called strongly dependent if there do not exist formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \omega$ and sequences $\langle \bar{b}_i^\alpha : i < \omega \rangle$ for $\alpha < \omega$ mutually indiscernible over A such that for every $\eta \in {}^\omega \omega$, the set

$$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) : \alpha < \omega\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}_i^\alpha) : \alpha < \omega, i \neq \eta(\alpha)\}$$

is consistent with p .

- A theory is called strongly dependent if the partial type $x = x$ is (here x is a singleton).
- Let T be dependent. A type p is called strongly stable if it is strongly dependent and stable.

Remark 2.2. Note that in [Shelah863] Observation 1.7 Shelah basically shows that if there exists a type $p(\bar{x})$ which is not strongly dependent, then there exists such a type $p'(x)$ with x being a *singleton*. This implies that if there exists a non-strongly dependent type, then T itself is not strongly dependent so that the above definitions are in fact different aspects of the same phenomenon.

A related notion, which will be convenient for us to consider, was investigated by Adler in [Adl]. We are going to use a slightly different terminology (some of it comes from [U_s8]).

Definition 2.8.

- A dividing pattern of depth κ for a (partial) type p over a set A is an array $\langle \bar{b}_i^\alpha : \alpha < \kappa, i < \omega \rangle$ and formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \kappa$ such that
 - The sequences $I^\alpha = \langle \bar{b}_i^\alpha : i < \omega \rangle$ are mutually indiscernible over A . That is, I^α is indiscernible over $AI^{\neq\alpha}$.
 - $\text{len}(\bar{b}_i^\alpha) = \text{len}(\bar{y}_\alpha)$.
 - For every $\eta \in {}^\kappa \omega$, the set

$$\{\varphi_\alpha(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$$

is consistent with p .

- For every $\alpha < \kappa$ there exists $k_\alpha < \omega$ such that the set

$$\{\varphi_\alpha(\bar{x}, \bar{b}_i^\alpha) : i < \omega\}$$

is k_α -inconsistent with p .

dfn:strong

- A (partial) type p over a set A is called strong if there does not exist a dividing pattern for p of depth $\kappa = \omega$.
- A theory is called strong if every finitary type is strong.

rmk:pattern0 Remark 2.6. Note that by mutual indiscernibility in clause (c) of the definition of a dividing pattern it is enough to demand that the set

$$\{\varphi_\alpha(\bar{x}, \bar{b}_0^\alpha) : \alpha < \kappa\}$$

is consistent with p .

It was shown in [Ad] and [OnUs] that in a dependent theory a type is strongly dependent if and only if it is strong. Hence a dependent theory T is strongly dependent if and only if it is strong.

3. STABLE TYPES IN UNSTABLE CONTEXTS

stable types Theorem 2.2 gives a good account of facts about stable types, which can be found in the literature. These results, however, are not enough for what we need. This section is devoted to studying stable types mostly without any assumptions on the theory.

We will start by proving the following easy properties of stable types; some are probably known, but we did not find them in the existing literature.

automorphisms **Observation 3.1.** Let $p(x)$ be a stable type over A , let $B \supset A$ and let $q(x)$ be an extension of $p(x)$ over B . Then $q(x)$ is a non forking extension if and only if for any saturated model $N \supset B$ there is an extension q' of q to N such that the number of different images of q' under automorphisms in $\text{Gal}(N/A)$ is bounded by $|\text{acl}(A)|$ (or $|T|$ if $|A| < |T|$).

Proof. It is well known that whenever $q \in S(\text{acl}(B))$ is a non forking extension of $p \in S(\text{acl}(A))$ then q is definable over $\text{acl}(A)$ so it is fixed under any automorphism which fixes $\text{acl}(A)$. It follows that the number of images of q under $\text{Gal}(N/A)$ is exactly the same as the number of extensions of p to $\text{acl}(A)$ which is, of course, bounded by $|A| + |T|$.

To prove the converse, suppose q is a non forking extension of p , let N be any saturated enough model and let q' be any extension of p to q' . It is clear that q' cannot be an heir of p so by Proposition 4.8 in [6] there are at least $|N|$ many $\text{Gal}(N/A)$ -automorphic copies of q' extending p to types over N . \square

gttransl&br&et3 **Theorem 3.2.** (Left Transitivity) Let $\text{tp}(a/A)$ and $\text{tp}(b/A)$ be stable types. Then if $\text{tp}(b/A)$ is also stable, then $\text{tp}(ab/C)$ does not fork over A if and only if $\text{tp}(a/Cb)$ does not fork over Ab and $\text{tp}(b/C)$ does not fork over A .

Proof. Let N be any saturated enough model extending C . The first item follows immediately from Observation 3.1 in the following way. Let $q(x, y)$ be an extension of $\text{tp}(ab/C)$ to N realized by tuples a', b' . Then notice that any N -automorphism σ fixes $q(x, y)$ if and only if σ fixes both the restriction $q(y)$ of $q(x, y)$ to the y variable (so that b' is a realization of $\sigma(q(y))$) and the type $q(x, b')$. \square

One of the great advantages of working with realizations of stable types, is that symmetry of forking independence holds whenever *one* of the sides of the independence relation satisfies stable types (over the base). This means that not only can we manipulate forking independence with stable types, but we can also use stable types as parameters with almost equal ease. In order to prove this we need the following.

im's symmetry

Fact 3.3. Let $p(x) := \text{tp}(a/A)$ be a type such that all the D -ranks are finite (see [\[Kimthesis\]](#) for a definition of D -ranks). Let $\text{tp}(b/A)$ be a type which does not fork over A . Then the following hold.

- (i) Let $I = \langle b_i \rangle$ be a Morley sequence of $\text{tp}(b/A)$. Suppose that $\langle a_0^j, a_1^j, \dots, a_n^j \rangle_{j \in \omega}$ is an A -indiscernible sequence of $\text{tp}(a_0, a_1, \dots, a_n/A)$. Then there is a sequence J , an A -automorphic copy of I , such that $\langle a_0^j, a_1^j, \dots, a_n^j \rangle^J$ is an A -indiscernible sequence for each $j < \omega$.
- (ii) The following are equivalent.
 - The set $\{\phi(x, b') \mid b' \in I\}$ is inconsistent for any Morley sequence I of $\text{tp}(b/A)$.
 - The set $\{\phi(x, b') \mid b' \in I\}$ is inconsistent for some Morley sequence I of $\text{tp}(b/A)$.
 - $\phi(x, b)$ witnesses that $\text{tp}(a/Ab)$ divides over A .
 - $\phi(x, b)$ witnesses that $\text{tp}(a/Ab)$ forks over A , in the sense that $\phi(x, b)$ implies a finite disjunction of formulas $\vee \psi_i(x, b_i)$ such that $p(x) \cup \psi_i(x, b_i)$ divides over A for each i .
- (iii) If $\text{tp}(b/Aa)$ does not fork over A then $\text{tp}(a/Ab)$ does not fork over A .

Proof. All of the items follow as in the proof of symmetry for simple theories in [\[Kimthesis\]](#).

- (i) See the proof Proposition 3.4 in [\[Kimthesis\]](#).
- (ii) See the proof Proposition 3.6 in [\[Kimthesis\]](#).
- (iii) See the proof Theorem 3.9 in [\[Kimthesis\]](#).

□

With this we can prove the following theorem.

symmetry

Theorem 3.4. (*Symmetry*) Let a, b be tuples and A a set such that $\text{tp}(a/A)$ is a stable type and such that $\text{tp}(b/A)$ does not fork over A . Then $a \perp_A b$ if and only if $b \perp_A a$.

Proof. By Fact 3.3 we know that if $\text{tp}(b/Aa)$ does not fork over A then neither does $\text{tp}(a/Ab)$.

For the other direction, assume that $a \perp_A b$, but $b \not\perp_A a$. So there is a formula $\varphi(y, x)$ such that $\varphi(b, a)$ holds and $\varphi(y, a)$ forks over A . Construct sequences $\langle a_i : i < \omega \rangle$ and $\langle b_i : i < \omega \rangle$ as follows:

- $a_0 b_0 = ab$
- $b_{i+1} \equiv_{\text{acl}(A)} b$, $b_{i+1} \perp_A a_{\leq i} b_{\leq i}$ (This is possible since $\text{tp}(b/A)$ does not fork over A).
- $a_{i+1} \equiv_{Ab} a$, $a_{i+1} \perp_A a_{\leq i} b_{\leq i+1}$ (This is possible since $a \perp_A b$, by stability of $\text{tp}(a/A)$ and transitivity of nonforking for stable types).

Now note:

- Since $a_i \equiv_A a$ for all i , we have that $\varphi(y, a_i)$ forks over A for all i .
- For $i < j$ we have $b_j \perp_A a_i$, hence $\neg \varphi(b_j, a_i)$.
- Let $i \geq j$. Then $\varphi(b, a_i)$ holds. Recall that $a_i \perp_A b b_j$ so the type $\text{tp}(a_i/Ab b_j)$ is definable over $\text{acl}(A)$ (by stability) and it therefore does not split over $\text{acl}(A)$. Since $b \equiv_{\text{acl}(A)} b_j$, $\varphi(b_j, a_i)$ holds.

We have shown that the order property is exemplified on the set of realizations of $\text{tp}(a/A)$ (with external parameters), contradicting Fact 2.1. □

Discussion 3.5. Note that the results above, combined with Theorem 2.2 imply that all properties of stable independence calculus hold for nonforking whenever one studies realizations of stable types. We will use this constantly throughout the paper, most of the time without providing exact references.

We will also extensively use the existence of *canonical bases*, see Theorem 2.2.

It will be important to be able to guarantee that we can restrict ourselves to stable types. In particular, it will be key in many cases to be able to restrict the bases to the stable part of the theory.

canonical base

Remark 3.6. Let a/A be a stable type and let b be any tuple and let $b_0 := Cb(a/Ab)$. Since b_0 can be defined from a Morley sequence in $\text{tp}(a/Ab)$ which is in particular a sequence of realizations of a/A , we have that $b_0 \in \text{dcl}(\bar{a}A)$ such that $\text{tp}(\bar{a}/A)$ is stable. So it is clear that $\text{tp}(b_0/A)$ is stable by Theorem 2.3 (v) and (vi).

stable base

Lemma 3.7. *Let e/A be stable and $A \subseteq B$.*

- (i) *There exists $B_0 \subseteq \text{dcl}(B)$ such that B_0/A is stable and for every tuple a we have*

$$e \downarrow_B a \implies e \downarrow_{B_0} a$$

- (ii) *If a is such that $\text{tp}(a/A)$ is a stable type and $e \not\downarrow_B a$, then there is some $B_0 \subset B$ such that $\text{tp}(B_0/A)$ is stable and $e \not\downarrow_{B_0} a$.*

Proof. To prove (i), let $B'_0 := Cb(e/B)$. By definition $e \downarrow_{B'_0} B$ and by transitivity, if $e \downarrow_B a$ then $e \downarrow_{B'_0} a$. In order to get B_0 we need to revise the construction of the canonical base. Recall (see for example [14]) that the canonical base B'_0 is the tuple of all canonical parameters b'_φ of the formulas defining the φ -definitions of $\text{tp}(e/B)$. Each such b'_φ is in the algebraic closure of B , which means that its orbit over B is finite. Let b_φ be the set of B -conjugates of b'_φ (in \mathfrak{C}^{ea}) and let B_0 be the set of all b_φ 's.

It is clear by definition that $b'_\varphi \in \text{acl}(b_\varphi)$ (so $B'_0 \subseteq \text{acl}(B_0)$) and that $B_0 \subseteq \text{dcl}(B)$. So clearly for any a we have

$$e \downarrow_B a \implies e \downarrow_{B_0} a$$

Finally, by Remark 3.6 we know that $\text{tp}(B'_0/A)$ is stable so that B_0/A must also be stable, completing the proof of (i).

Now, assume $\text{tp}(a/A)$ is stable and let $B_0 := Cb(ea/B)$. By definition $ea \downarrow_{B_0} B$ and $\text{tp}(ea/B)$ is stable, so $\text{tp}(ea/B_0)$ is also stable. By Theorem 3.2 this implies that $e \downarrow_{B_0a} B$. Now, if $e \not\downarrow_B a$ then by transitivity we get $e \not\downarrow_{B_0} a$ which proves (ii). \square

4. DOMINATION, WEIGHT AND STABILITY

In this section we will introduce the basic concepts and some of the main technical results which will allow us to state and prove the main results of the paper, which are included in Sections 5 and 6. The section is divided into two parts. We will first define domination and

weight and some related concepts, and prove some basic results about these. Then we will need to work out some results about generically stable types, which will play a key role in order to establish stationarity of types which are dominated by a stable type in a dependent theory.

ns domination

4.1. Orthogonality and domination: definitions. First, let us recall the most general concepts. We will give the definitions without assuming anything on the theory; of course, they do not always give rise to well-behaved notions. Note that for unstable types we only define domination over a fixed base set.

fn:domination

Definition 4.1.

- We say that a tuple a dominates a tuple b over a set A (and denote this by $b \triangleleft_A a$) if for every tuple c we have

$$a \downarrow_A c \implies b \downarrow_A c$$

- Let $p, q \in S(A)$. We say that p dominates q over A or $q \triangleleft_A p$ if for every $a \models p$ there exists $b \models q$ such that $b \triangleleft_A a$.
- If $p \triangleleft_A q$ and $q \triangleleft_A p$, we say that they are domination equivalent over A and write $p \bowtie_A q$.

Note that domination equivalence over a fixed base A is clearly an equivalence relation on $S(A)$.

rmk:acl_dom

Remark 4.1. $a \triangleleft_A b$ if and only if $a \triangleleft_{\text{acl}(A)} b$.

Definition 4.2. Two types $p(x)$ and $q(x)$ are weakly orthogonal if they are defined over a common domain B and for every tuple $a \models p'$ and $b \models q'$ we have $a \downarrow_B b$.

In this paper we will be particularly interested in domination by stable types, so it is natural to assume that the dominating type is stable and explore some consequences.

end_base_dom

Observation 4.2. Assume $p \triangleleft_A q$, q is stable, $A \subseteq B$, p', q' nonforking extensions of p, q (respectively) to B . Then $p' \triangleleft_B q'$.

Proof. By Remark 4.1, we may assume that A and B are algebraically closed.

Let a realize p' , so in particular it realizes p , hence there is b realizing q such that $a \triangleleft_A b$. By applying an automorphism over A , we may assume $b \models q'$ (so a realized an A -conjugate of p'). So we have $a \downarrow_A B$ and $b \downarrow_A B$.

Now assume $b \downarrow_B c$. Since $q := tp(b/A)$ is stable by transitivity we have $b \downarrow_A Bc$ so that $a \downarrow_A Bc$ and hence $a \downarrow_B c$ by monotonicity.

We can conclude that p' is dominated over B by some A -conjugate of q' ; but by stationarity of stable types and Remark 4.1, this proves the observation. \square

on descent

Remark 4.3. The converse of the above theorem has not been proved (or disproved) in the most general case. The proof for the setting that we were able to find (see the end of section 6) uses the ‘‘Theorem of Descent’’, which is a result analogous to that in [Z] for stably dominated types, and has a very technical proof.

Let us now define the following technical notion of domination with respect to stable types. As we will see in Section 5 (Corollary 5.2), it is equivalent to domination when realizations of stable types are involved.

st-domination

Definition 4.3. We say that b dominates a over A with respect to stable types, $a \triangleleft_A^{st} b$, if for every c with $\text{tp}(c/A)$ stable, we have

$$b \downarrow_A c \implies a \downarrow_A c$$

The notions of regularity, orthogonality and domination between types with different domains do not make much sense without stationarity (or, like in the simple case, some version of amalgamation of non forking extensions –the “Independence Theorem”– that permits to analyze extensions of both types at the same time). Since in the general context we lack this, we restrict the definitions to the case where we have some stability.

a:stable-orth

Definition 4.4.

- Two stable types p and q are orthogonal if every non forking extensions p' and q' of p and q respectively to a common domain are weakly orthogonal.
- A stable type $r(x)$ over A is regular if given any $B \supset A$, any forking extension $q(x)$ of $r(x)$ to B , and any non forking extension $p(x)$ of $r(x)$ to B , $q(x)$ is weakly orthogonal to $r(x)$.
- Let $p(x)$ and $q(x)$ be stationary stable types over A and B respectively. We will say that $p(x)$ dominates $q(x)$ if there are a, b realizations of p and q respectively such that $a \downarrow_A B$ and $b \downarrow_B A$ and for every c we have $b \not\downarrow_{A \cup B} c$ implies $a \not\downarrow_{A \cup B} c$. In this case we write $q \triangleleft p$.
- We say that stable types p and q are domination equivalent if they dominate each other. In this case we write $p \bowtie q$.

Note that whereas domination over a fixed base set is clearly transitive, for domination this is not straightforward from the definitions. In general this is not the case, even in simple theories. For stable types (like in stable theories), the situation is nice because of stationarity:

Observation 4.4. Let p, q, r be stable types, $p \triangleleft q \triangleleft r$. Then $p \triangleleft r$.

Proof. Let $p \in S(A)$, $q \in S(B)$, $r \in S(C)$. Without loss of generality (see Remark 4.1), A , B and C are algebraically closed. So we have nonforking extensions p', q' of p, q respectively to $D = \text{acl}(A \cup B)$ such that $p' \triangleleft_D q'$. Similarly, there are nonforking extensions q'', r'' of q, r to $E = \text{acl}(B \cup C)$ with $q'' \triangleleft_E r''$.

Now taking nonforking extensions to $D \cup E$, applying Observation 4.2 and transitivity of domination over a fixed base, we get the desired conclusion. Note that we are only using that q and r be stable. \square

As a conclusion, domination equivalence is an equivalence relation on stable types.

4.2. **Weight: definitions.** We now recall the general notion of weight.

Definition 4.5.

- Let $p(x)$ be any type over some set A . We will say that $a, \langle b_i \rangle_{i < \lambda}$ witnesses $\text{pwt}(p(x)) \geq \lambda$ (pre-weight of p is at least λ) if $a \models p(x)$, $\{b_i\}_{i < \lambda}$ is A -independent and $a \not\downarrow_A b_i$ for all i . If λ is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i < \lambda}$ witnesses $\text{pwt}(p(x)) = \lambda$ and that p has pre-weight λ .

dfn:weight

- Let $p(x)$ be any type over some set A . We will say that $a, B, \langle b_i \rangle_{i < \lambda}$ witnesses $\text{wt}(p(x)) \geq \lambda$ (weight of p is at least λ) if $a \models p(x)$, $a \perp_A B$, $\{b_i\}_{i < \lambda}$ is B -independent sequence and $a \not\perp b_i$ for all i . If λ is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i < \lambda}$ witnesses $\text{wt}(p(x)) = \lambda$ and that p has weight λ .
- We say that $p(x)$ has rudimentarily finite weight if there is no $a, B, \langle b_i \rangle_{i < \omega}$ witnessing $\text{wt}(p(x)) \geq \omega$.

We now define stable weight (which as we will see later, precisely captures the weight of the “stable part” of a type). Let us for simplicity restrict ourselves to finite weight and pre-weight (since we are not going to be concerned with infinite weight in this paper), although the general definition can be given similarly.

stable weight

Definition 4.6.

- Let $p(x)$ be any type over some set A . We will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses $\text{stp}(p(x)) \geq n$ (stable pre weight of p is at least n) if $a \models p(x)$, $\{b_i\}_{i=1}^n$ is A -independent, $\text{tp}(b_i/A)$ is stable and $a \not\perp_A b_i$ for all i . If n is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses $\text{stp}(p(x)) = n$ and that p has stable pre-weight n .
- Let $p(x)$ be any type over some set A . We will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses $\text{stw}(p(x)) \geq n$ (stable weight of p is at least n) if $a \models p(x)$, $a \perp_A B$, $\{b_i\}_{i=1}^n$ is B -independent sequence, $\text{tp}(b_i/B)$ is stable and $a \not\perp b_i$ for all i . If n is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i=1}^n$ witnesses $\text{stw}(p(x)) = n$ and that p has stable weight n .
- As before, a type has rudimentarily finite stable weight if there is no witness of infinite stable weight. Specifically, $p \in S(A)$ has rudimentarily finite stable weight if there are no $B, a \models p|B$ (a nonforking extension of p to B) and $\{b_i\}_{i < \omega}$ an B -independent set, such that $\text{tp}(b_i/B)$ is stable and $a \not\perp_B b_i$ for all i .

The following observation is very easy using the properties of forking for stable types. Note that a similar result for the general notion of weight is far from being clear.

weight monotone

Observation 4.5. Suppose $a \perp_A B$. Then $\text{stw}(\text{tp}(a/B)) = \text{stw}(\text{tp}(a/A))$.

Proof. Clearly, it is enough to show that $\text{stp}(a/B) \geq \text{stp}(a/A)$. So suppose $\bar{b} = \langle b_i : i < \alpha \rangle$ witness $\text{stp}(a/A) \geq \alpha$. Without loss of generality $\bar{b} \perp_A B$ (recall that $\text{tp}(\bar{b}/A)$ is stable, hence does not fork over A). By symmetry, $B \perp_A \bar{b}$. It is easy to see that \bar{b} is an independent set over B .

Now suppose $a \perp_B b_i$; then by transitivity $Ba \perp_A b_i$, so $a \perp_A b_i$, a contradiction. So \bar{b} witness $\text{stp}(a/B) \geq \alpha$, as required. \square

Finally, we define stable domination. Note that our definition is slightly more general than the original one given by Haskell, Hrushovski and Macpherson, since we allow $\pi_i(x)$ to be an arbitrary (partial) stable type, not just a stable set. This definition is borrowed from [5] and [12].

stabledomination

Definition 4.7. A type $p \in S(A)$ is called stably dominated if there exists a collection of stable types $\bar{\pi} = \langle \pi_i : i < \alpha \rangle$ and definable functions $f_i : p^{\mathcal{C}} \rightarrow \pi_i$ such that for every set $B \supseteq A$ and $a \models p$, if $f_i(a) \perp_A^{st} B$ for all i (which in this context just means that $\text{tp}(f_i(a)/B)$ is definable over A), then $\text{tp}(B/A\bar{f}(a)) \vdash \text{tp}(B/Aa)$ where \bar{f} denotes $\langle f_i : i < \alpha \rangle$.

In this case we also say that p is stably dominated by $\bar{\pi}$ via \bar{f} .

4.3. Strongness and weight. It is already well-known (see [Ad], [Us], [OnUs], [OnUs1], [H], [U2], [S], [U]) that strong dependence, and, more generally, Adler's notion of strongness is related to finiteness of weight. It will follow from our analysis in Section 5 (just like in stable [H], simple [U6], and rosy [U] theories) that in a strong theory any stable type has finite weight. In this subsection we only observe the fact that in a strong theory the stable weight of any type has to be *rudimentarily finite*.

The following is easy and probably well known:

Lemma 4.6. *Let $\{b_\alpha : \alpha < \lambda\}$ be a collection of elements realizing stable types over a set A . Then there are sequences I_α starting with a_α respectively such that I_α is a Morley sequence (in particular, indiscernible) over $AI_{\neq\alpha}$.*

Proof. The proof is an easy adaptation of the proof of Lemma 1.3 in [OnUs]. \square

Lemma 4.7. *Let $A, a, \{b_i : i < \omega\}$ be such that*

- $\text{tp}(b_i/A)$ is stable for all $i < \omega$
- The set $\{b_i : i < \omega\}$ is independent over A
- $a \not\perp_A b_i$ for all i

Then $\text{tp}(a/A)$ is not strong.

Proof. By the previous Lemma, we can construct sequences I_i mutually Morley over A starting with b_i respectively. By Fact 3.3 $a \not\perp_A b_i$ implies that $\text{tp}(a/Ab_i)$ divides over A for all i witnessed by I_i .

Now clearly the sequences I_i give a dividing system for $\text{tp}(a/A)$ of infinite depth, as required. \square

4.4. Generic stability and domination. Here we investigate some properties of generically stable types in a dependent theory, extending results proven in [U2]. This will be central to the main result of Section 5.

As we have already mentioned, it was shown in [U2] that in a dependent theory generic stability leads to “stable” behavior of nonforking. In particular, any nonforking sequence in a generically stable type is an independent set. The next lemma will allow us to get a converse. Recall that a type is generically stable if and only if some Morley sequence in it is an indiscernible set.

Lemma 4.8. *(T dependent)*

Let A be a set, $\langle a_i : i \in I \rangle$ an indiscernible sequence over A which is also an A -independent set (that is, $a_i \perp_A a_{I \setminus \{i\}}$ for all i). Then $\langle a_i : i \in I \rangle$ is an indiscernible set over A , and it is therefore hence $\text{Av}(I, A)$ is generically stable.

Proof. Assume not. Without loss of generality $I = \mathbb{Q}$ and there is a formula $\varphi(x, y) = \varphi(\bar{a}_{<0}, x, y, \bar{a}_{>1})$ such that $\varphi(a_0, a_1) \wedge \neg\varphi(a_1, a_0)$.

Clearly $\varphi(a_{\frac{1}{2}}, a_1) \wedge \neg\varphi(a_{\frac{1}{2}}, a_0)$, and since $\bar{a}_{<0}a_0\bar{a}_{>1} \equiv_A \bar{a}_{<0}a_1\bar{a}_{>1}$, we get $\text{tp}(a_{\frac{1}{2}}/A\bar{a}_{\leq 0}\bar{a}_{\geq 1})$ splits strongly and therefore forks (by Fact 2.4) over A , a contradiction. \square

As a consequence, we obtain the following nice theorem. It strengthens Lemma 7.8 in [U2].

Theorem 4.9. *(T dependent)*

Let $p, q \in S(A)$, $q \triangleleft_A p$, p is generically stable. Then q is also generically stable.

Proof. We can find sequences $I = \langle a_i : i \in \mathbb{Q} \rangle$, $J = \langle b_i : i \in \mathbb{Q} \rangle$ such that:

- $a_i \models p$, $a_i \downarrow_A \bar{a}_{<i} \bar{b}_{<i}$ and
- $b_i \models q$, $b_i \triangleleft_A a_i$.

Clearly I is a nonforking sequence in p . By the properties of nonforking independence for generically stable types (Fact 2.5, see also section 7 of [U2]), I is an independent set over A . By the choice of b_i and since $a_i \downarrow_A \bar{b}_{<i}$, it is also clear that J is a nonforking sequence. Notice that we are only interested in $\text{tp}(a_i b_i / A \bar{a}_{<i} \bar{b}_{<i})$ so we may assume without loss of generality that J is an A -indiscernible sequence and that the sequence of pairs $\langle a_i b_i : i \in \mathbb{Q} \rangle$ is A -indiscernible.

The following technical claim is the key for the rest of the proof.

Claim 4.10. Let $l < k \in \mathbb{Q}$, let $i_l \leq l$, $i_k \geq k$ and $i_j \in (l, k)$, and let $\bar{b}_{\leq l} = \langle b_{i_l} \rangle$, $\bar{b}_{> k} = \langle b_{i_k} \rangle$ and $\bar{b}_{(l,k)} = \langle b_{i_j} \rangle$ be finite tuples of elements in J .

Then, for every formula $\varphi(\bar{z}, x, \bar{y})$ over A , we have

$$\varphi(\bar{b}_{\leq l}, a_k, \bar{b}_{> k}) \iff \varphi(\bar{b}_{\leq l}, a_k, \bar{b}_{(l,k)}).$$

Proof. Assume $\varphi(\bar{b}_{\leq l}, a_k, \bar{b}_{(l,k)})$. Note that by indiscernibility for most $i > k$ we have $\varphi(\bar{b}_{\leq l}, a_i, \bar{b}_{> k})$. So $\neg \varphi(\bar{b}_{\leq l}, a_k, \bar{b}_{> k})$ would imply that $\neg \varphi(\bar{b}_{\leq l}, a_i, \bar{b}_{> k})$ for every $i \in (l, k)$, contradicting definition of dependence (Definition 2.5). \square

It follows that $a_i \downarrow_A \bar{b}_{<i} \bar{b}_{>i}$ if and only if $a_i \downarrow_A \bar{b}_{<i}$ (which we have by construction). But we know that $b_i \triangleleft_A a_i$, so by definition J is an A -independent set; by Lemma 4.8 J is an indiscernible set, which implies by Fact 2.5 that q is generically stable. \square

Remark 4.11. In general one would like to prove that stable weight and stable pre-weight behave in much the same manner as weight and pre weight.

The problem is that a stable type $r(x)$ over A might not have any stable restrictions to some subset $A_0 \subset A$ and it is not clear that we have a way to find stable types which play the role of $r(x)$. In this paper we will discuss a good example of this.

Let $p(x) \in S(A)$ which is dominated by a stable type $r(x)$ over A and such that $p(x)$ does not fork over \emptyset . It should be the case that $p(x)$ is stably dominated over \emptyset . However, we believe this is still unknown except when $\text{tp}(A/\emptyset)$ is an automorphism invariant type. And even in this case, finding the element satisfying a stable type over \emptyset , which will eventually dominate $p(x)$ is quite hard (it is pretty much a construction given in [Z] which we managed to adapt to our context, involving $\omega \cdot 2$ -“Morley” sequences of the type of A over \emptyset —see Fact 6.15 and Appendix A).

It might be that tools for this sort of constructions will eventually be found, but with the evidence we have right now it is also quite likely that stable types are, in some way, essentially sensitive to their base.

5. DOMINATION, STABLE DOMINATION AND STABLE WEIGHT

section4

This section presents what is perhaps the main result of this article. We first prove that domination by a stable type can always be witnessed within the algebraic closure of a realization of the (dominated) type. We will then prove that in a dependent theory, (forking) domination by a stable type is enough to achieve what is defined as stable domination in [U1] and [U2] (which is a small variation from the definition in [Z]). This implies that domination by a stable type

is an equivalent (and probably more manageable - certainly, easier to verify) definition for the notion which has been so productive in the study of algebraically closed valued fields (see [Z]).

We refer the reader to Subsection 4.1 for the definitions.

Lemma 5.1. *Let e/A be a stable type, and suppose $e \not\triangleleft_A c$. Then the lack of domination is witnessed by an element satisfying a stable type over A . That is, there exists d such that d/A is stable, $e \not\downarrow_A d$, $c \downarrow_A d$.*

Proof. Let e/A be a stable type and $e \not\triangleleft_A c$. By definition there is some b such that $e \not\downarrow_A b$ and $c \downarrow_A b$.

Let $d = Cb(e/Ab)$. In particular, $d \in \text{acl}(Ab)$ and d/A is stable. Since $e \not\downarrow_A b$, $c \downarrow_A b$ and we know that by definition $e \downarrow_{Ab} d$. So by transitivity $e \not\downarrow_A d$ and $c \downarrow_A d$. \square

Corollary 5.2. *The following are equivalent for a, b and A such that $\text{tp}(a/A)$ is stable:*

- (i) $a \triangleleft_A b$
- (ii) For every c such that $\text{tp}(c/A)$ is stable, we have

$$b \downarrow_A c \implies a \downarrow_A c$$

Theorem 5.3. *Any type which is dominated by a stable type is domination equivalent to a stable type.*

Moreover, let e/A be a stable type such that e dominates a over A . Then there is some $a_0 \in \text{dcl}(Aa)$ such that $\text{tp}(a_0/A)$ is stable and a is domination equivalent to a_0 over A .

Proof. Let e, a and A be as in the statement of the lemma, let $a'_0 := Cb(e/Aa)$, and let a_0 be the weak canonical base defined in Lemma 3.7(i) (this is, a_0 is the tuple consisting of sets of Aa -conjugates of the elements in the tuple a'_0). So $\text{tp}(a_0/A)$ is stable by Remark 3.6, $e \downarrow_{Aa'_0} a$ by definition, and $e \downarrow_{Aa_0} a$ by construction and transitivity. Suppose that $a \not\triangleleft_A a_0$ witnessed by d , so that $a_0 \downarrow_A d$ and $a \not\downarrow_A d$.

We know that $\text{tp}(e/Aaa_0)$ is stable, so we can take the non forking extension to Aaa_0d and after using a permutation find some d' such that $e \downarrow_{Aaa_0} d'$ and $\text{tp}(d'/Aaa_0) = \text{tp}(d/Aaa_0)$. So without loss of generality $e \downarrow_{Aaa_0} d$. By transitivity $e \downarrow_{Aa_0} ad$ and $e \downarrow_{Aa_0} d$. But by hypothesis $a_0 \downarrow_A d$, so by Theorem 3.2 (both $\text{tp}(e/A)$ and $\text{tp}(a_0/A)$ are stable) we get that $\text{tp}(ea_0/A)$ is stable and $ea_0 \downarrow_A d$, so $a_0 \downarrow_{Ae} d$ and $e \downarrow_A d$ again by Theorem 3.2. But by hypothesis e dominates a over A so $a \downarrow_A d$, a contradiction.

So $a \triangleleft_A a_0$ and, since $a_0 \in \text{dcl}(Aa)$, it is clear that $a_0 \triangleleft_A a$ which completes the proof. \square

Now we can prove the main result of this section. The proof uses strongly that types which are dominated by a stable type are generically stable (Theorem 4.9), and therefore have automorphism invariant non forking extensions (Fact 2.5). If one analyzes the content of the definition of stable domination, one immediately notices that this invariance is essential to the proof which gives a strong hint that the result cannot hold outside the dependent context (see the following Remark 5.5).

Theorem 5.4. *(T dependent) Let a be a tuple and $C := \text{acl}(C)$ be a set such that $\text{tp}(a/C)$ is dominated by a stable type. Then $\text{tp}(a/C)$ is stably dominated.*

Proof. By Theorem 5.3, let f be a C -definable function such that a is domination equivalent to $f(a)$ over C and such that $tp(f(a)/C)$ is stable. By Theorem 4.9 we know that $tp(a/C)$ is generically stable so it has unique non forking extensions.

Let D and D' be such that $f(a) \downarrow_C D$ and $D' \equiv_{Cf(a)} D$. We would like to show $D' \equiv_{Ca} D$.

Let a' be such that $tp(a'D'/Cf(a)) = tp(aD/Cf(a))$ so that in particular $f(a') = f(a)$ and $tp(a'/C)$ is dominated by $tp(f(a)/C)$. By definition of domination we have that both $a \downarrow_C D$ and $a' \downarrow_C D$. But this implies that $tp(a/CD)$ and $tp(a'/CD)$ are non forking extensions of $tp(a/C) = tp(a'/C)$, and by uniqueness of non forking extensions for generically stable types (Fact 2.5 (ii)) we can conclude the theorem. \square

remark

Remark 5.5. Notice that we are proving that in order to prove “domination” based on logic implication for generic realizations (stable domination) it is enough to verify that one has a forking based domination. This implicitly assumes that one has automorphism invariance of non forking extensions. It is therefore very unlikely that any version of the previous theorem would hold without assuming that the theory is dependent.

Let us conclude by observing that the other direction holds as well (no assumptions on T are needed here; in fact, we are not even using stability - the notion of domination defined in Definition 4.7 is, in the general context, simply stronger than forking domination).

Observation 5.6. *A stably dominated type $p \in S(A)$ is dominated over A by a stable type.*

Proof. Let p be stably dominated by $\bar{\pi}$ via \bar{f} (see Definition 4.7). Note that $\bar{\pi}$ is a stable type and \bar{f} is an A -definable function from p to $\bar{\pi}$. Let $a \models p$, and $\bar{b} = \bar{f}(a)$; so $\bar{b} \models \bar{\pi}$. Clearly $tp(\bar{b}/A)$ is stable, and we show $a \triangleleft_A \bar{b}$.

So assume $\bar{b} \downarrow_A c$ for some c ; in particular, $f_i(a) \downarrow_A c$ for all i . By stable domination we know that $tp(c/A\bar{b}) \vdash tp(c/Aa)$.

Let $c' \equiv_{A\bar{b}} c$ such that $a \downarrow_{A\bar{b}} c'$. By transitivity $a\bar{b} \downarrow_A c'$. Since $c \equiv_{A\bar{b}} c'$, we have $c \equiv_{Aa} c'$, so $a \downarrow_A c'$, as required. \square

section5

6. WEIGHT AND STABLE WEIGHT

Theorem 5.3 is a central result for this paper and, before proving Theorem 5.3, maybe the most surprising part was what followed the “moreover” in the statement, in the sense that whenever a type $p(x)$ was dominated by a stable type one could witness this domination by a tuple which was in the definable closure of the realization of the type. In a sense, we were able to find a stable part of the type which witnessed the domination.

This of course cannot possibly be repeated without assuming domination by a stable type. However, one can try to repeat the process without hoping for domination, but for “as much domination as one can hope for by a stable type”. This is the purpose of this section; of course, the first step is to start making sense of what “as much domination as one can hope for by a stable type” can possibly mean.

Domination has traditionally be linked to the notion of weight. Notice that if we assume finite weight and that weight is linked to domination as in simple theories (see Chapter 5 of [16]), then Theorem 4.9 should imply that stable domination is equivalent to being able to witness the weight using only stable types. In this spirit, the first part of this section will be devoted to

studying the properties of stable weight, which under some circumstances will capture precisely the notion of the size (or weight) of the largest stable part of the realization of a given type. Again, we refer the reader to Subsection 4.1 (specifically Definition 4.6) for the definitions.

6.1. Stable weight. We will start with some easy observations.

stable weight

Lemma 6.1. *The following hold.*

- (i) If $a \perp_A b$ then $\text{stw}(a/A) = \text{stw}(a/Ab)$.
- (ii) $\text{stw}(ab/A) \leq \text{stw}(a/A) + \text{stw}(b/A)$. Equality holds whenever $a \perp_A b$.

Proof. The proofs are the same as proofs of Lemmas 5.2.2 and 5.2.4 in [16]. The only thing to notice is that since all of the proofs involve having realizations of stable types at one side or the other of the forking independence, all of the properties of forking independence (symmetry, transitivity, etc.) will hold in this context. \square

We will now show that if we limit ourselves to stable types, then stable weight works just as in stable theories. The proofs of the following Lemma is very similar to Lemma 2.16 of [9] (which is in turn based on Hyttinen's work [4]), although certain arguments need to be added.

t1-domination

Lemma 6.2. *Let $p \in S(A)$ stable, and assume that*

- (i) $a, A', \{b_1, \dots, b_n\}$ witness $\text{stw}(p) \geq n$ so that $\{b_1, \dots, b_n\}$ is a A -independent set of elements each of which satisfies a stable type over A , $a \perp_A A'$, independent over A' and $a \not\perp_{A'} b_i$ for all i .
- (ii) *There is no c such that the following three conditions hold:*
 - (a) $\text{tp}(c/A')$ is stable
 - (b) $a \perp_{A'} c$
 - (c) $b_1 b_2 \dots b_{n-1} \perp_{A'} c b_n$
 - (d) $b_n \not\perp_{A'} c$.

Then

- (1) *For any c , if $a \perp_{A'} c$ and $a \perp_{A' b_n} c$ then $b_n \perp_{A'} c$.*
- (2) *If, furthermore, $\text{stw}(\text{tp}(b_n/A')) > 1$, then there are B and b'_n, b'_{n+1} such that $a, B, \{b_1, \dots, b_{n-1}, b'_n, b'_{n+1}\}$ witness $\text{stw}(p) \geq n + 1$.*

Proof. (1) Suppose towards a contradiction that we can find c such that $a \perp_{A'} c$, $a \perp_{A' b_n} c$ and $b_n \not\perp_{A'} c$.

Claim 6.3. We may assume $\text{tp}(c/A')$ is stable.

Proof. As in Lemma 5.1, we define $c' = \text{Cb}(\text{tp}(b_n/A')c)$ so that $c' \in \text{acl}(A'c)$, $\text{tp}(c'/A')$ is stable and $c' \perp_{A'c} b_n$. It follows that $a \perp_A c'$, $a \perp_{A' b_n} c'$ (since trivially $c' \in \text{acl}(A' b_n c)$) and by transitivity $c' \not\perp_A b_n$. Replacing c by c' we have the conclusion of the claim. \square

Let $\langle b'_i : i < n \rangle \equiv_{A' b_n a} \langle b_i : i < n \rangle$ such that $\langle b'_i : i < n \rangle \perp_{A' b_n a} c$. Then $\langle b'_i : i < n \rangle a \perp_{A' b_n} c$ by left transitivity, so $\langle b'_i : i < n \rangle \perp_{A' b_n} c$, hence by symmetry $c \perp_{A' b_n} \langle b'_i : i < n \rangle$.

So we have shown that without loss of generality $c \downarrow_{A'b_n} b_1 \dots b_{n-1}$. Let $C = A'c$. It is easy to see that (a),(b),(c) above hold for c (e.g. (b) holds by symmetry and transitivity), contradicting assumption (ii) of the Lemma.

(2) Assume $\text{stw}(\text{tp}(b_n/A')) > 1$. This means that there is $B \supseteq A'$ and c, d which witness this, so that

- $b_n \downarrow_{A'} B$
- $c \downarrow_B d$
- $b_n \not\downarrow_B c$ and $b_n \not\downarrow_B d$.

Without loss of generality $ab_1 \dots b_{n-1} \downarrow_{A'b_n} Bcd$ (recall that the type $\text{tp}(ab_1 \dots b_{n-1}/A'b_n)$ is stable). Clearly, all of the assumptions of the Lemma hold for $a, B, \{b_1, \dots, b_n\}$. Since the conclusion of part (1) fails for the tuples b_n, d, B and b_n, c, B we can conclude that $a \not\downarrow_B c$ and $a \not\downarrow_B d$. Choosing $b'_n = c, b'_{n+1} = d$, we are done. □

The proofs of the following two results go through now word by word as in Section 2 of [\[9\]](#), so we state them without proofs.

weight1-st

Lemma 6.4. *Let $p \in S(A)$ be a stable type of finite stable weight. Then p is non-orthogonal to a stable type of stable weight 1.*

Moreover, suppose that $a \models p, B = \{b_i : i < m\}, d$ are such that $a, A, \{b_i : i < m\} \cup \{d\}$ witness $\text{stw}(p) \geq m + 1$. Then there exist $D \supseteq A$ and d' such that

- $\text{stw}(d'/A') = 1$
- $a, D, \{b_i : i < m\} \cup \{d'\}$ witness $\text{stw}(p) \geq m + 1$

Proof. Exactly the same as the proof of Lemma 2.17 in [\[9\]](#), using Lemma 6.2 instead of what Lemma 2.16 in [\[9\]](#). □

weight-st

Theorem 6.5. *Let $p \in S(A)$ be a stable non-algebraic type of rudimentarily finite stable weight. Then p is domination equivalent to a finite free product of stable weight-1 types. That is, there exist $a, A', \{b_i : i < n\}$ such that*

- $a, A', \{b_i : i < n\}$ witness that $\text{stw}(p) \geq n$
- $\text{stw}(b_i/A') = 1$ for all i
- $a \boxtimes_{A'} b_0 \dots b_{n-1}$

So in particular $\text{wt}(p) < \omega$.

Proof. Given the previous results, the proof of Theorem 2.21 in [\[9\]](#) goes through word by word. □

Corollary 6.6. *Let p be a stable type in a strong theory (in particular strongly dependent). Then p is domination equivalent to a free product of finitely many weight-1 stable types.*

Proof. By the previous theorem and Lemma 4.7. □

weight-eg

Corollary 6.7. *A stable regular type of finite stable weight has (stable) weight 1.*

Proof. Suppose not, and let $p \in S(A)$ be a stable regular type of stable weight at least 2. Without loss of generality (since a non forking extension of a stable regular type is stable and regular), there exists a witness $a, \{b_1, b_2\}$ for $\text{stp}_w(p) \geq 2$. Moreover, by Lemma 6.4 we may assume that $\text{stw}(\text{tp}(b_1/A)) = \text{stw}(\text{tp}(b_2/A)) = 1$.

Let a' be such that $\text{tp}(a'/Ab_1) = \text{tp}(a/Ab_1)$, $a' \perp_{Ab_1} b_2$. Then clearly $a' \perp_A b_2$ (as b_1, b_2 are independent over A).

Now notice:

- $a \perp_{Ab_2} a'$: The type p is regular, so $\text{tp}(a/Ab_2)$ and $\text{tp}(a'/Ab_2)$ are weakly orthogonal.
- $b_1 \not\perp_{Ab_2} a$: We know $b_1 \perp_A b_2$ and $b_1 \not\perp_A a$.
- $b_1 \not\perp_{Ab_2} a'$: This follows from $a' \equiv_{Ab_1} a$ (so $a' \not\perp_A b_1$), and $b_1 \perp_A b_2$.

So we have a witness for $\text{stw}(\text{tp}(b_1/Ab_2)) \geq 2$, but this type is a non forking extension of $\text{tp}(b_1/A)$, a contradiction. \square

The following is a straightforward adaptation of the proof of Theorem 5.2.5 in [\[Wag16\]](#).

5.2.5

Fact 6.8. *Any superstable type has finite stable weight. In particular, the hypothesis in Theorem 6.5 hold for any superstable type, so it must be domination equivalent to a finite tuple of weight one types.*

Another proof that translates from the stable theory context into stable types is the following.

Corollary 6.9. *Non orthogonality is an equivalence relation among regular stable types of finite weight. In fact, it is an equivalence relation among types of stable weight 1.*

Proof. By Corollary 6.7 every regular stable type has stable weight 1 so it is enough to prove the second part of the statement.

Let p, q, r be stationary stable regular types and suppose that $p \not\perp q$ and $q \not\perp r$. Let A and B be sets and a, b_1, b_2, c be elements such that $a \models p|A$, $b_1 \models q|A$, $b_2 \models q|B$, $c \models p|B$, $a \not\perp_A b_1$ and $c \not\perp_A b_2$. As in the stable theory context, we can extend all the types to $A \cup B$ and using stationarity and automorphisms we may assume that $b_1 = b_2 = b$ and $A = B = A \cup B$.

Now, if $a \perp c$ then $b, A, \langle a, c \rangle$ would witness that q has stable weight at least 2, a contradiction. \square

6.2. The stable part of a type. The following easy observation follows straight from the definitions.

observation

Observation 6.10. *Let p be any type. Then $\text{stp}_w(p) \leq \text{pwt}(p)$ and $\text{stw}(p) \leq \text{wt}(p)$.*

If p is stable, we have equality:

nt for stable

Observation 6.11. *Let $p(x)$ be a stable type. Then the stable weight of p is equal to the weight of p .*

Proof. Suppose that $a, \langle b_i : i < \alpha \rangle, A'$ witnesses that $\text{wt}(p) \geq \alpha$; we may assume without loss of generality that $A = A'$.

Let $b'_i := Cb(a/Ab_i)$. We know by construction that $\text{tp}(b'_i/A)$ is stable and $a \perp_{Ab'_i} b_i$ for $i < \alpha$. By transitivity we know that $b'_i \not\perp_A a$, and by definition $b'_i \perp_A b'_{<i}$ (recall that $b'_i \in \text{acl}(Ab_i)$). So $a, \langle b'_i : i < \alpha \rangle, A$ witnesses $\text{stw}(p) \geq \alpha$. \square

We will soon extend the above equality to types dominated by stable types. Let us first identify the “stable part” of a given type p of finite stable weight.

stable part

Theorem 6.12. *Let $\text{tp}(a/A)$ be any type with stable pre-weight α . Then there is $d \in \text{dcl}(aA)$ such that $\text{tp}(d/A)$ is stable and has stable pre-weight α .*

Proof. It is clearly enough to show that $\text{stw}(\text{tp}(a/A)) \leq \text{stw}(\text{tp}(d/A))$ for some $d \in \text{acl}(aA)$. Let $a, \bar{b} := \langle b_i \rangle_{i \in \alpha}$ witness that $\text{tp}(a/A)$ has stable pre-weight α ; let $d := \text{Cb}(\bar{b}/aA)$.

Then $\bar{b} \downarrow_{Ad} a$, so in particular $b_i \downarrow_{Ad} a$ for all $i \in \alpha$. Since by hypothesis $b_i \not\downarrow_A a$ by transitivity (both $\text{tp}(b_i/A)$ and $\text{tp}(d/A)$ are stable) we can conclude that $d \not\downarrow_A b_i$. By definition, d, \bar{b} witness that $\text{tp}(d/A)$ has stable pre-weight greater than or equal to α , as needed. \square

Corollary 6.13. *Let $p(x)$ be any type of finite stable weight. Then there is some non forking extension $\text{tp}(a/B)$ of $p(x)$ and some $d \in \text{acl}(aB)$ such that $\text{tp}(d/B)$ is stable and $\text{wt}(\text{tp}(d/B)) = \text{stw}(\text{tp}(d/B)) = \text{stw}(p(x))$.*

stable domination

Theorem 6.14. *Let $p(x) = \text{tp}(a/A)$ be a type. Then*

- *If p is dominated by a stable type over A , then $\text{wt}(p) = \text{stw}(p)$.*
- *Assume that $\text{wt}(p) < \omega$. If $\text{wt}(p) = \text{stw}(p)$, then p has a nonforking extension which is dominated by a stable type.*

In particular, if T is dependent, then p has a nonforking stably dominated extension, so p itself is generically stable.

Proof. The first clause is easy at this point: if $\text{tp}(a/A)$ is dominated by a stable type, then by Theorem 4.9 there is some $d \in \text{dcl}(aA)$ such that $\text{tp}(d/A)$ dominates $\text{tp}(a/A)$ over A . It follows that $\text{wt}(\text{tp}(a/A)) = \text{wt}(\text{tp}(d/A)) = \text{stw}(\text{tp}(d/A)) = \text{stw}(\text{tp}(a/A))$ by the definitions and Observation 6.11.

For the second part notice that if $\text{wt}(\text{tp}(a/A))$ is finite then trivially so is the stable weight of $\text{tp}(a/A)$. Let $\text{tp}(a/A)$ be a type of stable weight n ; by taking a nonforking extension, we may assume without loss of generality that the stable pre weight is n . By Theorem 6.12, let $d \in \text{dcl}(aA)$ be a tuple such that $\text{tp}(d/A)$ is stable and such that the stable weight of $\text{tp}(d/A)$ is n . By Corollary 6.11 we know that the weight of $\text{tp}(d/A)$ must be n and by Theorem 6.5 d is dominated by a sequence $\bar{b} := \{b_i : i < n\}$ of stable weight 1 types.

If $\bar{b} := \{b_i : i < n\}$ does not dominate a over A there is, by definition, some b' such that $b' \downarrow_A \bar{b}$ but $b' \not\downarrow_A a$. But then \bar{b}, b' would contradict that $n := \text{stw}(p) = \text{stpw}(p)$.

To prove the last statement of the theorem, notice that if T is dependent then dominated by stable implies stably dominated by Theorem 5.4. Stable domination implies generic stability and a nonforking restriction of a generically stable type is itself generically stable by [12]. \square

HHM

Fact 6.15. ([2], Theorem 4.9, “Descent”) *Let p, q be global types invariant over a set A . Assume that for all $b \models q \upharpoonright A$, the type $p \upharpoonright Ab$ is stably dominated in the Haskell-Hrushovski-Macpherson sense. Then $p \upharpoonright A$ is stably dominated.*

The equivalent theorem of descent does not follow immediately from the statement of Fact 6.15 because the definition of stable domination used in [2] is not precisely the same one we use. However, the *proof* of the analogue fact using our definition of stably dominated types is precisely the same (with a very minor modification of the definition of $St_b(X)$). We will not

repeat the proof but we will give in Appendix A a step by step account of the changes needed to adapt the proof of Theorem 4.9 in [Z] to our context.

From now on we will assume Fact 6.15 with our definition of stably dominated types.

The following is a more natural reformulation of Fact 6.15 with our definitions. We will state it over a model, which assures that nonforking is equivalent to being automorphism-invariant. It is possible that one could replace M with an arbitrary extensible set A by working with Lascar-invariance, but this would involve a deeper manipulation of the proof in [Z], and we will not go into the details here.

5.16 **Observation 6.16.** *Let p, q be global types invariant over a set A . Assume that for some $b \models q \upharpoonright A$, the type $p \upharpoonright Ab$ is stably dominated. Then $p \upharpoonright A$ is stably dominated.*

Proof. It is enough to show that if $b' \equiv_A b$, then $p \upharpoonright Ab'$ is stably dominated. Let $a \models p \upharpoonright Ab$, and let σ be an automorphism over A taking b to b' . Denote $a' = \sigma(a)$. Since $a'b' \equiv_A ab$, clearly $\text{tp}(a'/Ab')$ is stably dominated. But $p = \sigma(p)$ by invariance, hence $a' \models \sigma(p) \upharpoonright Ab' = p \upharpoonright Ab'$, which completes the proof. \square

cor:descent

Corollary 6.17. *(T dependent) Let $p \in S(M)$ be a type over a model M , and assume that it has a nonforking extension $p' \in S(B)$ which is stably dominated. Then p is stably dominated.*

Proof. Recall that since T is dependent and M is a model, a global type is automorphism-invariant over M if and only if it does not fork over M . Note also that every type over M has a global nonforking extension. So the corollary follows immediately. \square

in dependent

Corollary 6.18. *Let T be dependent, p a type of finite stable weight over a model M of T . Then $\text{wt}(p) = \text{stw}(p)$ if and only if p is stably dominated.*

Proof. The “if” direction follows from Theorems 6.14 and 5.4. Now assume that $\text{wt}(p) = \text{stw}(p)$. By Theorem 6.14 again, there is a nonforking extension of p which is dominated by a stable type, hence stably dominated by Theorem 5.4.

Since p is a nonforking restriction of a stably dominated type, it is, by “descent” (Corollary 6.17), also stably dominated. \square

the appendix

APPENDIX A. PROOF OF DESCENT

In this section we will give the key steps to adapt the proof of descent in [Z] to our context. Since we proved that stable domination and domination by a stable type are equivalent (Theorem 5.3) we can prove either descent for a type stably dominated or prove that whenever a type $p(x)$ has a non forking extension $p'(x)$ which is dominated by a stable type then $p(x)$ is dominated by a stable type.

In order to adapt the proof in [Z] we need some notation first.

Definition A.1. *let d be a (possibly infinite) tuple and let C be a set. Then we define $St_B(d)$ to be the set of $a \in \text{dcl}(dC)$ such that $\text{tp}(a/C)$ is stable.*

Observation A.1. *Since stable types are closed under concatenation, it is clear that*

- $St_B(d)$ is closed under definable closure and
- If e is an enumeration of $St_B(d)$ then $e \in \text{dcl}(Bd)$ and $\text{tp}(e/B)$ is stable.

We will also need the following.

Definition A.2. *let d be a (possibly infinite) tuple and let C, D be a sets. Then we define the canonical base of $\text{tp}(d/CD)$ with respect to C –which we will denote $\text{Cb}(\text{tp}(d/CD); C)$ or $\text{Cb}(d/CD; C)$ – to be*

$$\text{Cb}(\text{tp}(St_C(dDC)/DC)).$$

Observation A.2. *Notice that, since $St_C(dDC)$ is by definition stable over C , we know that it is stable over DC and the existence of $\text{Cb}(\text{tp}(d/CD); C)$ follows by Theorem 2.3(i).*

Propositions A.4 and A.5 are the analogues of Proposition 3.22 in [HHM]. In our context, it follows from the existence and stability of the canonical base of a stable type (Theorem 2.3 and Remark 3.6).

We will first need an easy observation.

Observation A.3. *Let c be a tuple and X, Z be sets such that $\text{tp}(c/Z)$ is stable. Then $c \downarrow_Z X$ if and only if $c \downarrow_Z St_Z(X)$.*

Proof. One direction is clear by monotonicity. For the other direction, notice that if $c \not\downarrow_Z X$ then $c \not\downarrow_Z \text{Cb}(c/ZX)$. Since $\text{Cb}(c/ZX)$ is definable over ZX and stable over Z , the observation follows by monotonicity. \square

3.22-1

Proposition A.4. *Let Z be any set, let $b \in Z$ be a tuple and let c be a tuple such that $\text{tp}(c/b)$ is stable.*

The following are equivalent.

- (i) $St_Z(X) \downarrow_Z c$.
- (ii) $St_b(XZ) \downarrow_{St_b(Z)} c$.

Proof. Since $\text{tp}(c/Z)$ is stable and $b \in St_b(Z)$ it is clear by symmetry and the above observation that it is enough to prove that $c \downarrow_Z X$ if and only if $c \downarrow_{St_b(Z)} St_b(XZ)$. Let $d' := \text{Cb}(c/ZX)$, so that $d' \in \text{dcl}(ZX)$, $\text{tp}(d'/b)$ is stable –so $d' \in St_b(ZX)$ – and $c \downarrow_{bd'} ZX$.

If $c \downarrow_Z X$ then by definition $d' \in St_b(Z)$ and $c \downarrow_{St_b(Z)} XZ$. The left to right implication follows by monotonicity.

For the converse, notice that by monotonicity $c \downarrow_{St_b(Z)d'} ZX$ so that if $c \not\downarrow_Z X$ then by transitivity (every extension of $\text{tp}(c/b)$ is stable) we have that $c \not\downarrow_{St_b(Z)} d'$. \square

3.22-2

Proposition A.5. *Let Z be any set, let $b \in Z$ be a tuple such that $\text{tp}(Z/b)$ is stable and let c be a tuple such that $\text{tp}(c/b)$ is stable.*

Then the following are equivalent.

- (i) $St_Z(X) \downarrow_Z c$.
- (ii) $St_b(X) \downarrow_Z c$.

Proof. As before, we need to show that $c \downarrow_Z X$ if and only if $c \downarrow_Z St_b(X)$. One direction is clear by monotonicity. For the other direction, assume that $c \not\downarrow_Z X$ and let $d' := \text{Cb}(cZ/X)$. Since by assumption $\text{tp}(Z/b)$ is stable, we know that $\text{tp}(cZ/b)$ is stable so that d' exists and $d' \in St_b(X)$. By definition $cZ \downarrow_{bd'} X$ and by left transitivity we know that $c \downarrow_{Zd'} X$. By transitivity $c \not\downarrow_Z d'$ and the result follows by monotonicity. \square

By Theorem 5.3 it makes sense to use the following definition of dominated via a function.

Definition A.3. *We will say that $\text{tp}(a/C)$ is stably dominated via $f(x)$ if $f(x)$ is a C -definable function such that $\text{tp}(f(a)/C)$ is stable and $f(a)$ dominates a over C .*

The theorem of descent can now be stated as follows.

Theorem A.6. *Let $A := \text{acl}(A)$, let q be a global A invariant type and let p be a type over A such that for some $b \models q$ there is a non forking extension $p(x, b)$ of p over Ab such that $p(x, b)$ is stably dominated via $f(x, b)$. Then $p(x)$ is stably dominated over A .*

Proof. By Theorem 4.9 we know that $p(x, b)$ is a generically stable type and therefore so is the non forking restriction $p(x)$; so in particular, $p(x)$ is stationary. Let p' be the global non forking extension of $p(x)$ so that p' is A -invariant and $p'|Ab$ is stably dominated over Cb via $f(x, b)$. By Observation 6.16 we have that for any $b' \models q$ the type $p'|Ab'$ is stably dominated over Ab' via $f(x, b')$.

The construction of a function g over A such that p is dominated by g over A is now exactly the same as the construction in Section 4 of [Z]. We will give a summary of the construction.

In [Z] they find a tuple e which dominates a over A and such that $\text{tp}(e/A)$ is stable, in the following manner:

- Construct an A -indiscernible sequences $\langle b_i \rangle$ with $i \leq (\omega + 1)^2$ and such that $b_\lambda \models q|Aa\{b_i\}_{i < \lambda}$.
- Define $d_i := f(a, b_i)$, $b_J := \{b_i\}_{i \in (\omega+1)^2}$ and $d_J := \{d_i\}_{i \in (\omega+1)^2}$. Let

$$d'_J := \text{Cb} \left(\text{tp} \left(b_{(\omega+1)^2} d_{(\omega+1)^2} / b_J d_J \right); b_J \right).$$

In particular, there is an infinite sequence of functions f_J such that $f_J(b_J, a) := d_J$.

- One can show that the relation $(d_J, b_J) \sim (d'_J, b'_J)$ defined by two such tuples being equivalent if and only if there is some a such that $a \models p|b_J$ and $a \models p|b'_J$ and $f_J(b_J, a) := d_J$ and $f_J(b'_J, a) := d'_J$ is an equivalence relation which is in fact an intersection of infinite definable, finitary relations e_λ .
- One can then show that $\text{tp}(e_\lambda/A)$ is stable for every λ and that $e := \langle e_\lambda \rangle$ dominates a over A . □

With the same construction, some of the proofs in [Z] can be made a little simpler, partly because we can use that $\text{tp}(a/A)$ is generically stable and partly because proving that $f(a, b)$ dominates a over Ab is in general easier than proving that $f(a, b)$ stably dominates a over Ab . However, we were unable to simplify the construction and the proofs in any significant manner. We believe however that the insight provided in this paper can help simplify the proof of descent in at least one of the following ways:

First, it follows that $\text{St}_A(d_J b_J)$ is non empty and that it dominates $p(x)$ over A . A posteriori, it appears that one should be able to prove this without going through the actual construction of the tuple e .

Second, since each g_λ is finitary, it is definable over a finite sequence of $b_i d_i$'s and not all of them can be orthogonal to a over A . It follows that one should be able to inductively witness that the weight of $p(x)$ over A with a sequence of elements which are stable over A , thus showing

that there must be some e in $\text{dcl}(\{b_i\}_{i \in \omega}, \{d_i\}_{i \in \omega})$ which is stable over A and which dominates a over A (so that one does not have to go all the way to $(\omega^2 + 1)$ -sequences). Even in the proof of Theorem 4.2 in [HHM] it seems that going to $(\omega^2 + 1)$ -sequences could be simplified.

One thing which is not clear to us and which would imply a very significant simplification of the theorem of descent is whether one can avoid infinite “Morley” sequences of the b_i ’s. The following question seems to be the key for this to be possible.

Question A.1. Let $p(x)$ and $q(x)$ be A -invariant global types such that for every $b \models q(x)$ the type $p|Ab$ is stably dominated over Ab via $f(x, b)$. If $a \models p|Ab$ and $d := f(a, b)$, is a dominated over A by $St_A(bd)$?

REFERENCES

- [1] Hans Adler, *Strong theories and weight*, In preparation.
- [2] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson, *Stable domination and independence in algebraically closed valued fields*, submitted.
- [3] Ehud Hrushovski and Anand Pillay, *NIP and invariant measures*, preprint.
- [4] Tapani Hyttinen, *Remarks on structure theorems for ω_1 -saturated models*, Notre Dame J. Formal Logic **36** (1995), no. 2, 269–278.
- [5] Byunghan Kim, *Simple First Order Theories*, Ph.D. Thesis, University of Notre Dame (1996).
- [6] Daniel and Poizat Lascar Bruno, *An introduction to forking*, J. Symbolic Logic **44** (1979), no. 3, 330–350.
- [7] Alf Onshuus, *Properties and consequences of thorn-independence*, J. Symbolic Logic **71** (2006), no. 1, 1–21.
- [8] Alf Onshuus and Alexander Usvyatsov, *On dp-minimality, strong stability and weight.*, Submitted.
- [9] ———, *Orthogonality and domination in unstable theories.*, Submitted.
- [10] Saharon Shelah, *Dependent first order theories, continued*, To appear in Israel Journal of Mathematics, <http://shelah.logic.at/files/783.ps>.
- [11] ———, *Strongly dependent theories*, Submitted (Sh863).
- [12] Alexander Usvyatsov, *Generically stable types in dependent theories*, Journal of Symbolic Logic **74** (2009), no. 1, 216–250.
- [13] ———, *Morley sequences in dependent theories*, submitted.
- [14] Anand Pillay, *Geometric stability theory*, Oxford Logic Guides, vol. 32, The Clarendon Press Oxford University Press, New York, 1996. , Oxford Science Publications.
- [15] S. Shelah, *Classification theory and the number of nonisomorphic models*, Second, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990.
- [16] Frank O. Wagner, *Simple theories*, Mathematics and its Applications, vol. 503, Kluwer Academic Publishers, Dordrecht, 2000.

ALEXANDER USVYATSOV, UNIVERSIDADE DE LISBOA, CENTRO DE MATEMÁTICA E APLICAÇÕES FUNDAMENTAIS, AV. PROF. GAMA PINTO, 2, 1649-003 LISBOA, PORTUGAL

URL: <http://www.math.ucla.edu/~alexus>