Math 167 Final Study Guide
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Disclaimer: This is just a guide to help you study for the final. The material in this study guide will not necessarily be on the final and vice versa. This is just what I would know/study if I were a student in Math 167. There may be some typos/mistakes so do not *just* use this to study (i.e. be sure to look at old homeworks, notes, book, etc.). Some of the exercises have been taken from the text.

Feel free to ask me questions during office hours, discussion or via email (`akashiwada@ucla.edu`) if anything here is unclear.

1 Topics to Know

The following is a list of topics that I feel are the most important that we've covered since the midterm. (Basically Chapters 3-6.) As I haven't seen your final, it is possible that are will be other topics covered on your exam that are not listed.

- Preferences and their properties/definitions
- **Von Neumann and Morgenstern utility functions**
  - Know how to construct a VNM utility function
  - Know how to calculate $\mathcal{E}u(\mathbf{L})$ for any lottery $\mathbf{L}$. Remember that a player who is following VNM rationality will prefer the lottery with the larger $\mathcal{E}u(\mathbf{L})$.
  - A player will pay up to \$ x to participate in lottery $\mathbf{L}$ if $u(x) = \mathcal{E}u(\mathbf{L})$
  - Risk
    - A player who is risk-averse will have a concave utility function, i.e $u(\mathcal{E}\mathbf{L}) > \mathcal{E}u(\mathbf{L})$, and is always willing to sell a lottery $\mathbf{L}$ for $\mathcal{E}\mathbf{L}$.
    - A player who is risk-loving will have a convex utility function, i.e. $u(\mathcal{E}\mathbf{L}) < \mathcal{E}u(\mathbf{L})$, and is always willing to buy a lottery $\mathbf{L}$ for $\mathcal{E}\mathbf{L}$.
    - A player who is risk-neutral has a linear utility function and is always indifferent between buying/selling a lottery for its expected value.
- Know how to calculate a payoff function
- Know how to construct a bimatrix game from the extensive form of a game with chance moves.
- Know how to use **domination** to solve a game (note that the order of deletions matters when deleting weakly dominated strategies).
- Cooperative Game Theory
  - Cooperative payoff region: convex hull of the vectors $(\pi_1(s_i, t_j), \pi_2(s_i, t_j))$ for all i, j.

  - Free disposal of utils: players can agree to get rid of utils before making a decision (e.g. players agree that PII has to "burn" \$2 before play which we have to subtract 2 for each of PII's payoffs). Get this region by extending $\max_{i,j} \pi_1(s_i, t_j)$ line down and $\max_{i,j} \pi_2(s_i, t_j)$ line to the left.
  - Transferable utils: after an agreement is made, players can trade utils (e.g. money). Get the region with *both* free disposal and transferable utils, one needs to find $\max_{i,j} \pi_1(s_i, t_j) + \pi_2(s_i, t_j) = M$. Then the region below the line $x_1 + x_2 = M$ is the cooperative payoff region we want.

- Bargaining set: set of points in the cooperative payoff region that are *Parteo optimal* and *individually rational*.
  - A point $\vec{x}$ is Pareto optimal if $\vec{x} < \vec{y}$ implies $\vec{y}$ is not in our cooperative payoff region. Recall that $\vec{x} < \vec{y}$ if $x_i \leq y_i$ for all i but $\vec{x} \neq \vec{y}$.
  - Given a disagreement point d, a payoff x satisfies individual rationality if $x_i \geq d_i$ for all i.

  The bargaining set of a cooperative payoff region is usually the north-east boundary of the region satisfying individual rationality.

- Nash Bargaining Solution to (X, d)
  - For the generalized Nash bargaining solution, we need to be given player I's bargaining power α and player II's bargaining power β. In the regular Nash bargaining solution, $\alpha = \beta = \frac{1}{2}$.
  - The Nash bargaining solution can only be found under certain conditions (the cooperative payoff region X is convex, closed and bounded from above, and free disposal is allowed) and satisfies 4 axioms (see page 184).
  - Finding the Nash bargaining solution *geometrically*:
    1. "Recalibrate" axes so the disagreement point d is the origin.
    2. Find points r and t on the horizontal and vertical axes, respectively, such that the line through r and t is a supporting line to X and $s = \alpha r + \beta t \in X$.
       Warning: α must go with r and β with t.
    3. s is the Nash bargaining solution.
  - Finding the Nash bargaining solution *algebraically*:
    1. The Nash bargaining solution s is the max of the Nash product $(x_1 - d_1)^\alpha (x_2 - d_2)^\beta$.
    2. When finding this max, you do not need to use α and β as given, just the ratio (e.g. if $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$, maximizing $(x_1 - d_1)^{1/3}(x_2 - d_2)^{2/3}$ is the same as maximizing $(x_1 - d_1)^1 (x_2 - d_2)^2$).

- Minimax and Maximin

  Here we are only considering pure strategies in strictly competitive games. However, these methods can be applied to bimatrix games *if* we are only looking at one of the players' payoff matrices (not the entire game).

  - The minimax is the minimum of the maximum payoffs in each column, i.e.

    $$\overline{m} = \min_t \max_s \pi(s, t),$$

    and the maximin is the maximum of the minimum payoffs in each row

    $$\underline{m} = \max_s \min_t \pi(s, t).$$

    Theorem 1. *(σ, τ) is a saddle point (in pure strategies) if and only if $\overline{m} = \underline{m} = \pi(\sigma, \tau)$.*

- Security Strategies and Security Levels
  - Player I's security level is the best he can do under the assumption that player II is trying to minimize player I's payoff. As similar definition holds for player II's security level.
  - A strategy is a security strategy for player i if it guarantees a payoff of at least that player's security level.

Theorem 2. *If player I's payoff matrix has saddle point (σ, τ) then his security level is $\pi_1(\sigma, \tau)$ and σ is one of his security strategies.*

– If we are in a zero-sum game then saddle points of $-A^T$ correspond to security strategy/level for player II. Otherwise, we need to let A be player II's payoff matrix and find the opposite "saddle points" of A (the strategy pair that corresponds to entry of A that is largest in row and smallest in column).

- Mixed Strategies
  – In $m \times n$ game, player I's mixed strategy p is a $m \times 1$ vector where each entry p_i is the probability that he used pure strategy s_i. Similarly, player II's mixed strategy q is a $n \times 1$ vector where q_i is the probability she plays pure strategy t_i.
  – Finding mixed security strategies:
    1. Look for saddle points in pure strategies. If one exists, you do not need to look for a solution in mixed strategies.
    2. If the game is larger than $m \times 2$ or $2 \times n$, try reducing the game by either using symmetry or domination.
    Note: If you are asked to find *all* mixed security strategies or Nash equilibria, then you cannot delete weakly dominated strategies. But if you just need to find one, then deleting weakly dominated strategies will work.
    3. Once you have a manageable size matrix, you know 3 methods for finding mixed strategies: equalizing expected payoffs, separating hyperplanes, and oddments.
    Note: I would not recommend using separating hyperplanes if you are asked to analyze a $m \times 2$ game as I think the procedure turns out to be different (but I'm not sure...).

  Warning: Do not forget that mixed strategies need to be the size of the original matrix so if you use domination or rule out pure strategies, you need to say that they have zero probability of being chosen.

  Theorem 3. *1. If $\underline{v} = \max_p \min_q \Pi(p, q)$ and $\overline{v} = \min_q \max_p \Pi(p, q)$ then $\underline{v} \leq \overline{v}$.*
  2. $(\tilde{p}, \tilde{q})$ is a saddle point iff $\underline{v} = \overline{v} = \Pi(\tilde{p}, \tilde{q})$.
  3. If $(\tilde{p}, \tilde{q})$ is a saddle point of player I's payoff matrix then his security level is $\Pi(\tilde{p}, \tilde{q})$ and $\tilde{p}$ is a security strategy.
  4. Any finite two player zero-sum game has a value $v = \underline{v} = \overline{v}$.
  5. If A is player I's payoff matrix in a zero-sum game then saddle points $(\tilde{p}, \tilde{q})$ are Nash equilibria.

2 Sample Problems

Again, I also suggest looking at the problems in the text that are similar to the assigned homework problems. Problems marked with (^{s}) are modified problems from Straffin.

1. Exercise 5.9.15 on page 213.
2. ^{s} Take a 2×2 game with a saddle point in pure strategies. What happens when you try to find a mixed strategy equilibrium? *This problem illustrates why it is important to look for saddle point in pure strategies before doing a mixed strategy analysis.*
3. Suppose player I's VNM utility function is $u_1(x) = x^2$ and player II's VNM utility function is $u_2(x) = \log_2(x + 1)$.
   (a) Are the players risk-averse? Give two different explanations for your answer.
   (b) Suppose they are given the lottery **L** below. How much would each be willing to pay to participate in **L**?

L =

0	3	4	5
0.3	0.3	0.2	0.2

4. ^{s} Find the minimax and maximin of each zero-sum game. Find a security strategy for both players and the value of the following games (this is a good opportunity to practice the 'separating hyperplanes' method):

   (a) $\begin{bmatrix} -4 & 2 & 0 & 3 & -2 \\ 4 & -1 & 0 & -3 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 5 & 2 & 1 \\ 4 & 1 & 3 \\ 3 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

5. What is the cooperative payoff region of the game

	t_1	t_2
s_1	(3,2)	(2,1)
s_2	(4,3)	(1,4)

   How does it change if we allow free disposal of utils? Free disposal and tranferable utils? Find a Nash equilibrium of this game. Is it Parto optimal?

6. ^{s} Consider the game

	t_1	t_2	t_3
s_1	(0,-1)	(0,2)	(2,3)
s_2	(0,0)	(2,1)	(1,-1)
s_3	(2,2)	(1,4)	(1,-1)

   (a) Show that $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)^T$ is a security strategy for player I. What is his security level?
   (b) Find player II security strategy and security level.
   (c) What is the expected payoff of each player if they both use their security strategies?

7. ^{s} Find both players security level in

	t_1	t_2
s_1	(2,6)	(10,5)
s_2	(4,8)	(0,0)

   Using their security levels as the disagreement point, i.e. $d = (\text{PI security level}, \text{PII security level})$, find the regular Nash bargaining solution.

8. Consider the following game with disagreement point $d = (2, 1)$

	t_1	t_2	t_3
s_1	(6,3)	(4,7)	(2,8)
s_2	(3,6)	(-2,3)	(2,-1)

(a) Find the regular Nash bargaining solution.

(b) Find the Nash bargaining solution when $\alpha = \frac{1}{10}$ and $\beta = \frac{9}{10}$.

(c) Find the Nash bargaining solution when $\alpha = \frac{13}{14}$ and $\beta = \frac{1}{14}$.

9. Toby figures there are 4 ways to get from his apartment to UCLA: walk, bike, take the bus, or drive. He likes driving the most and walking the least. He likes taking the bus $\frac{2}{3}$ as much as driving, but riding his bike takes much longer so he only likes it about $\frac{1}{4}$ as much as driving. Use these to construct the lotteries that Toby thinks are equivalent to taking the bus and bicycling to campus. Write his Von Neumann and Morgenstern utility function for traveling.

   Now suppose Toby is looking to rent a new apartment and he narrows his choices to Apartment A or Apartment B. In trying to figure out which to choose, he figures out the proportions of each mode of transportation he would use from both apartments which is given in the following tables. Based on the VNM utility function you just found, which apartment will Toby like better?

Apartment A =

car	bus	bike	walk
0.1	0.7	0	0.2

Apartment B =

car	bus	bike	walk
0.2	0.5	0.2	0.1