Exercise 1.

Solution.

Proof.
(a) It is compact. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 + y^2$. Then $f$ is continuous since for every $\epsilon > 0$ we have that

$$|f(x_0, y_0) - f(x, y)| = |x^2 - x_0^2 + y^2 - y_0^2|$$

$$\leq |x - x_0||x + x_0| + |y - y_0||y + y_0|$$

$$\leq ||(x - x_0, y - y_0)||_2 \left(||x + x_0|| + |y + y_0|\right)$$

$$\leq d_{\mathbb{R}^2}((x, y), (x_0, y_0))(1 + 2|x_0| + 1 + 2|y_0|)$$

$$< \epsilon$$

provided

$$d_{\mathbb{R}^2}((x, y), (x_0, y_0)) < \delta < \min \left\{1, \frac{\epsilon}{1 + 2|x_0| + 1 + 2|y_0|}\right\}.$$

Since $f$ is continuous and the set $\{3\}$ is a closed subset of $\mathbb{R}$ we know that

$$f^{-1}(\{3\}) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 3\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3\}$$

is a closed subset of $\mathbb{R}^2$. If $(x, y) \in f^{-1}(\{3\})$, then

$$d((0, 0), (x, y)) = \sqrt{x^2 + y^2} = \sqrt{3}$$

and so $f^{-1}(\{3\}) \subseteq B((0, 0), \sqrt{3} + 1)$. So $f^{-1}(\{3\})$ is bounded. Since it is a closed and bounded subset of $\mathbb{R}^2$, we know it is compact.

(b) It is not compact since it is not bounded. (You’d still need to prove it’s unbounded).

(c) It is not compact since it is not closed. (The sequence $(1, 1/n)$ converges to $(1, 0) \notin \{(1, 1/n) \in \mathbb{R}^2 : n \in \mathbb{N}\}$).

(d) It is not compact since it is not closed. The sequence $\{(0, 3 - 1/n)\}_{n=1}^{\infty}$ is a sequence in $\{(x, y) : x^2 + y^2 < 3\}$ that converges to $(0, 3) \notin \{(x, y) : x^2 + y^2 < 3\}$. So $\{(x, y) : x^2 + y^2 < 3\}$ is not closed, and cannot be compact.

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x$ and $g(x, y) = x^2 - y$. Since $f$ and $g$ are continuous,

$$f^{-1}([0, 1]) \text{ and } g^{-1}([1, \infty))$$

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are closed sets, being the pre-image of a closed set under a continuous function. But then
\[
\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\} = \{ (x, y) : 0 \leq x \leq 1 \} \cap \{ (x, y) : 0 \leq y \leq x^2\}
\]
\[= f^{-1}([0, 1]) \cap g^{-1}([0, \infty))
\]
is closed, being an intersection of two closed sets. It suffices to show this set is bounded. If \((x, y) \in \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}\), then
\[
d((0, 0), (x, y)) = \sqrt{x^2 + y^2} \leq \sqrt{1 + x^4} \leq \sqrt{1 + 1} = \sqrt{2}
\]
and so it is bounded as well.

\[\square\]

**Exercise 2.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Let \(f : X \to Y\) be a continuous function. Let \(E\) be a connected subset of \(X\). Show that \(f(E)\) is connected.

**Solution.**

**Proof.** We proceed by contradiction. Suppose \(E\) is connected but \(f(E)\) is disconnected. Then there exists \(A, B\) relatively open subsets of \(E\) with \(A \cap B = \emptyset\) and \(A \cup B = E\). Since \(f\) is continuous, we see that \(f^{-1}(A)\) and \(f^{-1}(B)\) are relatively open subsets of \(E\) with
\[
E = f^{-1}(f(E)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)
\]
and
\[
\emptyset = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).
\]
So \(E\) is disconnected, a contradiction. \(\square\)

**Exercise 3.**

**Solution.**

**Proof.** Let \(f : X \to \mathbb{R}\) be a continuous function and \(E\) a connected subset of \(E\). Say \(a, b \in E\) and without loss of generality \(y \in [f(a), f(b)]\). We know from the previous exercise that \(f(E)\) is connected. By Theorem 8.4 this implies since \(f(a) \in f(E)\) and \(f(b) \in f(E)\) that \([f(a), f(b)] \subseteq f(E)\). In particular, \(y \in f(E)\). So there exists at least one \(x \in E\) with \(f(x) = y\), as desired. \(\square\)

**Exercise 4.**

**Solution.**

**Proof.**

1. Assume (i). Say \((x^{(j)})_{j=1}^{\infty}\) is a sequence in \(E\) which converges to \(x_0\) with respect to the metric \(d_X\). Let \(\epsilon > 0\). Since (i) holds there exists \(\delta > 0\) so that
\[
d_Y(f(x), L) < \epsilon
\]
for every \(x \in E\) with \(d_X(x, x_0) < \delta\). Now since \((x^{(j)})_{j=1}^{\infty}\) converges to \(x_0\) there exists \(N\) sufficiently large so that \(d(x^{(j)}, x_0) < \delta\) for all \(j \geq N\). But then
\[
d_Y(f(x^{(j)}), L) < \epsilon
\]
for all \(j \geq N\), as desired.
(2) Assume \((ii)\). Say that \((i)\) fails, so that
\[
\lim_{x \to x_0, x \in E} f(x) \neq L.
\]
Then there exists \(\epsilon > 0\) so that for every \(\delta > 0\) there exists \(x \in E\) with \(d_X(x, x_0) < \delta\) yet \(d_Y(f(x), L) > \epsilon\). Taking \(\delta = 1/n\) for each \(n \in \mathbb{N}\) this produces, for every \(n \in \mathbb{N}\), an \(x_n \in E\) with \(d_X(x_n, x_0) < 1/n\) with \(d_Y(f(x_n), L) > \epsilon\). But then \(x_n \to x_0\) with respect to \(x_n\). So by \((ii)\), there exists \(N \in \mathbb{N}\) so that for all \(n \geq N\) we have \(d_Y(f(x_n), L) < \epsilon\), a contradiction. \[\square\]

**Exercise 5.**

**Solution.**

**Proof.** Let \(x_0 \in X\). Let \(\epsilon > 0\). Since \(f_j\) converges uniformly to \(f(x)\), there exists \(N \in \mathbb{N}\) so that
\[
d(f_j(x), f(x)) < \epsilon/3
\]
for all \(j \geq N\) and \(x \in X\). Since \(f_N\) is continuous, there exists \(\delta > 0\) so that
\[
d_Y(f_N(x_0), f_N(x)) < \epsilon/3
\]
for all \(x \in X\) with \(d_X(x, x_0) < \delta\). Now, by the triangle inequality it follows that for all \(x \in X\) with \(d_X(x, x_0) < \delta\) we have
\[
d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x_0)) + d_Y(f_N(x_0), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]
So \(f\) is continuous at \(x_0\), as desired. \[\square\]