Math 164 - Fall'02 - Final Exam

1. (8 points) Let f be concave function on $\mathbb{R}^2$ and $a \in \mathbb{R}$. Show that

$$U = \{x \in \mathbb{R}^2 \ : \ f(x) \geq a\}$$

is convex set.

2. (12 points) Consider the problem

$$\left.\begin{array}{ll} \text{minimize} & z = c_1x_1 + c_2x_2 \\ \text{subject to} & a_1x_1 + a_2x_2 \leq b \\ & x_1, x_2 \geq 0 \end{array}\right\} \qquad (*)$$

(a) (4 points) Under what conditions is the feasible set not empty?
(b) (8 points) If an optimal solution to $(*)$ exists, derive a simple rule to find one.

3. (10 points) Given the constraints

$$\begin{array}{ll} \text{I} & x_1 + 4x_2 \leq 6 \\ \text{II} & x_2 \leq 2 \\ \text{III} & 5x_1 - x_2 \leq 4 \\ \text{IV} & x_1 - x_2 \leq 2 \end{array}$$

the point $\bar{x} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and the at $\bar{x}$ feasible direction $p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, perform a ratio test to determine the maximal step length $\bar{\alpha}$ such that $\bar{x} + \bar{\alpha}p$ remains feasible. Show work and justify your steps.

4. (8 points) Consider the following (primal) linear programming problem P.

$$\left.\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0\ . \end{array}\right\} (P)$$

(a) (4 points) Find the dual D of P.
(b) (4 points) With respect to infeasibility and unboundedness, what can you say about one of the two problems P and D, and using duality theory, what can you conclude about the other? Explain your answers by mentioning the relevant theorems or corollaries you are using.

5. (10 points) Consider

$$f(x_1, x_2) = x_1^4 + x_2^4 - 4x_1x_2 + 1$$

Find all stationary points and classify them according to local minimizer, maximizer or saddle showing that the respective FONC and SOSC hold.

6. (12 points) Consider

$$\left.\begin{array}{ll} \text{maximize} & f(x_1,x_2) = x_1^2 x_2^2 \\ \text{subject to} & x^T x = 1 \end{array}\right\} \qquad (*)$$

(a) (8 points) Show that on the feasible set given by the above constraint f has a local maximum of $\frac{1}{4}$ at $x^* = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}^T$ with Lagrange multiplier $\lambda^* = \frac{1}{2}$.

(b) (4 points) Given two positive numbers a_1 and a_2, define

$$\bar{x}_i = \frac{\sqrt{a_i}}{\sqrt{a_1 + a_2}}\ , i = 1, 2\ .$$

Show that $\bar{x} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^T$ is feasible for the problem $(*)$ and deduce from your result from part (a)[1] that the geometric mean $\sqrt{a_1 a_2}$ of the two positive numbers a_1 and a_2 is no greater than their arithmetic mean $\frac{1}{2}(a_1 + a_2)$. Therefore assume[2] that the local maximum from part (a) is a global maximum on the feasible set.[3]

7. (14 points)

(a) (10 points) Using the method of Lagrange multipliers solve[4]

$$\begin{array}{ll} \text{minimize} & f(x_1,x_2,x_3) = -x_1x_2 - x_1x_3 - x_2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 3\ . \end{array}$$

(b) (4 points) Consider now the perturbed problem

$$\begin{array}{ll} \text{minimize} & f(x_1,x_2,x_3) = -x_1x_2 - x_1x_3 - x_2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 3 + \delta\ . \end{array}$$

Using the Lagrange multiplier from part (a)[5] find a first-order estimate of the minimum value of the perturbed problem.

8. (8 points) Consider the constraint

$$g(x_1,x_2) = (x_1 - x_2^2)(x_1 + x_2) = 0$$

Find the tangent cone and the null space of the Jacobian matrix[6] of g at $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

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[1] You can do this even if you have not succeeded in 'solving' part (a)!

[2] This can be shown, but you do not need to do this.

[3] This problem can be generalized to n variables to show that the geometric mean of n positive numbers is no greater than their arithmetic mean.

[4] This means find a global minimizer.

[5] If you have not solved part (a), use λ as your Lagrange multiplier.

[6] Note that here the Jacobian is simply the gradient, because $g : \mathbb{R}^2 \to \mathbb{R}$.