THE PYTHAGOREAN THEOREM

In this lesson, students use their knowledge of exponents, roots, and area to prove the Pythagorean theorem. Students work with numbers, pictures and algebraic symbols to understand this result. They use the Pythagorean theorem and its converse to solve problems.

This lesson falls near the end of a cluster of lessons that apply algebra readiness concepts of length, area, and volume. In earlier lessons, students found perimeters and areas of circles and rectangles, and they found surface areas and volumes of prisms and cylinders. They also established formulas for triangles and parallelograms using dissection proofs. In the last geometry lesson they will continue to use the Pythagorean theorem and its converse to solve problems.

Math Goals

- Explore the Pythagorean theorem numerically, algebraically, and geometrically
- Understand a proof of the Pythagorean theorem
- Use the Pythagorean theorem and its converse to solve problems.

Abridged California Standards	Algebra Readiness • AF 1.1 Use variables and appropriate operations to write an expression or an equation that represents a verbal description • MG 3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement. General Mathematics • MG2.1 Use formulas routinely for finding the perimeter and area of basic two-dimensional figures, including rectangles, squares, and triangles.
Summative Assessment	• Week 26 The Pythagorean Theorem (MG 3.3)

PLANNING INFORMATION

Estimated Time: 75 - 90 Minutes		
Student Pages SP1: Ready, Set, Go SP2: Two Right Triangles SP3-4: Pythagorean Theorem (Part 1) SP5: Pythagorean Theorem (Part 2) SP6-7: Pythagorean Theorem Practice	**Overhead Transparencies** OH1: Ready, Set, Go OH2: Two Right Triangles OH3: Right Triangle ABC	**Reproducibles** R1 Pythagorean Theorem Cut Ups this will be the two squares from SP3. I think two sets of squares will fit on one page.
Materials Rulers Scissors Calculators (optional) Envelopes (optional)	**Prepare Ahead** Make two squares from R1 into overheads and cut them for demonstration.	**Management Reminders** To save some class time, pre-cut R1 into large squares.
Homework Pythagorean Theorem Practice (SP6)	**Assessment** SPx: Knowledge Check A1: Formative Quiz	**Strategies for Special Needs** Manipulating polygons creates a visual explanation for the simplification of the expressions $\frac{1}{2}ab+\frac{1}{2}ab=ab$ and $4\left(\frac{1}{2}ab\right)=2ab$

THE WORD BANK

A right triangle has exactly one right angle. The longest side of a right triangle is called its hypotenuse. The hypotenuse is opposite the right angle. The other two sides are called legs. The legs are both shorter than the hypotenuse, and they form the sides of the right angle.

The Pythagorean theorem states that, for a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

24.1 The Pythagorean Theorem

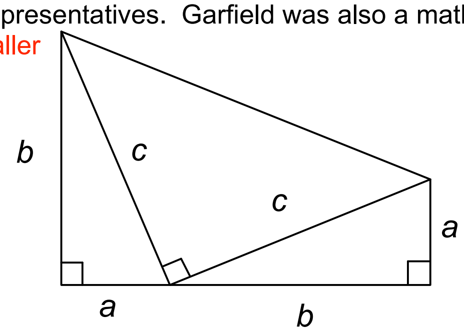
MATH BACKGROUND	
DOES $a^2 + b^2 = c^2$?	
Math Background 1 Summarize 1, Summarize 2	Caution: The slogan "*a* squared plus *b* squared equals *c* squared" is an incomplete statement of the Pythagorean theorem because there is no reference to a right triangle nor identification of the meaning of the variables. Here are preferred statements: 1. For a right triangle, the sum of the squares on the legs is equal to the square on the hypotenuse. (A geometric focus: illustrated in the "Two Triangles" exercise) 2. For a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. (A numerical focus: derived in the right triangle cut-up proof) 3. For a right triangle with legs of length *a* and *b* and hypotenuse of length *c*, *a* squared plus *b* squared equals *c* squared. (More precise restatement of the slogan)

ANOTHER PROOF OF THE PYTHAGOREAN THEOREM

Math Background 2
Extend

The proof in this lesson is only one of more than 400 proofs that have been recorded on this famous theorem. Many are simple variations of each other.

Here is another proof, discovered in 1876 by President James A. Garfield while a member of the House of Representatives. Garfield was also a mathematics teacher. cary, please make diagram smaller



Let *A* be the area of the entire figure, which is a trapezoid. The area of the trapezoid is:

$$A = \left(\frac{(a+b)(a+b)}{2}\right) = \left(\frac{a^2 + 2ab + b^2}{2}\right).$$

Area A can also be dissected into three triangles:

$$A = \frac{1}{2}(ab) + \frac{1}{2}(c^2) + \frac{1}{2}(ab) = \frac{(c^2 + 2ab)}{2}.$$

By substitution:

$$\frac{a^2 + 2ab + b^2}{2} = \frac{c^2 + 2ab}{2}$$

So: $a^2 + b^2 = c^2$

24.1 The Pythagorean Theorem

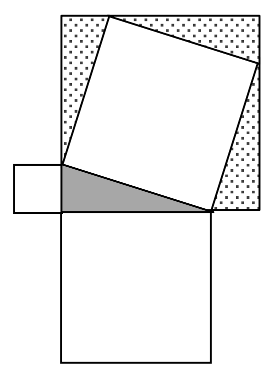
TEACHING TIPS

COMBINING LIKE TERMS – MISCONCEPTION ALERT	
Teaching Tip 1 Preview/Warmup	Since the proof of the Pythagorean theorem in this lesson requires some symbol manipulation, practice is provided in the warmup. Be prepared for student mistakes when combining like terms. If students falter by stating that $ab + ab = a^2b^2$, remind them of previous experiences with variables. For example, the cost of two slices of pepperoni pizza can be represented as *p* + *p* = 2*p*. Therefore, *a* + *a* =2*a* might represent the cost of two apples. It follows that $\frac{1}{2}a + \frac{1}{2}a = 1a = a$ might represent the cost of two halves of apples, which is the cost of one apple, or simply a. Later in the lesson, polygon cutouts link a visual model to this symbol manipulation. Emphasize that two triangles combined from the second square are congruent to one rectangle in the first square. Since the area of each triangle is $\frac{1}{2}ab$, and the area of each rectangle is *ab,* it follows that $\frac{1}{2}ab + \frac{1}{2}ab = ab$ and $4(\frac{1}{2}ab) = 2ab$

EXPLORING THE PYTHAGOREAN THEOREM AND ITS CONVERSE	
Teaching Tip 2 Practice	In general, if we have a theorem that says, "If A is true, then B is true", then the converse of that statement is, "If B is true, then A is true." Therefore, Pythagorean Theorem: If a triangle is a right triangle, then the sum of the squares of the two shorter legs is equal to the square of the hypotenuse. Converse of the Pythagorean theorem: If the sum of the squares of the two shorter legs is equal to the square of the hypotenuse, then the triangle is a right triangle. Replace "legs" with "sides" and "hypotenuse" with "longest side" – the converse is starting with a mystery triangle so avoid right triangle names. Students can use the Pythagorean theorem together with its converse to verify numerically whether a triangle is right. Draw one triangle with sides of length 5, 11, and 12 units, and another triangle with sides of length 5, 12, and 13 units. Do not label right angles on either triangle. Ask students to determine which of the triangles are right (if any) by using only their side lengths. Since $5^2 + 11^2$ is not equal to 12^2, they can deduce that the 5-11-12 triangle is NOT right by the Pythagorean theorem. On the other hand, since $5^2 + 12^2 = 13^2$, the converse to the Pythagorean theorem says that the 5-12-13 triangle is in fact a right triangle.

24.1 The Pythagorean Theorem

PREVIEW / WARMUP	
Whole Class ➢ SP1, OH1 Ready, Set, Go Teaching Tip 1	• Introduce the goals and standards for the lesson. Underline important vocabulary. • Students find the areas of the figures given and simplify the given expressions. Discuss answers and possible misconceptions as needed.

	INTRODUCE 1
Whole Class ➢ SP2, OH2 Two Right Triangles	• Discuss basic properties of a right triangle (three sides, two acute angles, one right angle) and vocabulary (legs, hypotenuse) associated with the naming of the sides of a right triangle. ***Which side of a right triangle must be the longest? Why?*** The hypotenuse must be the longest side because it is opposite the largest (right) angle. • Focus attention on the small triangle on the left. Demonstrate how to draw squares on the legs. ***What is the area of the square on the shorter leg?*** 9 square units ***on the longer leg?*** 16 square units ***How do the side lengths relate to the areas of these squares?*** Side length squared is equal to area, and square root of area is equal to side length. • Demonstrate how to draw a square on the hypotenuse using the given vertices as guides. Then encase the square on the hypotenuse with a larger square as shown. ***What steps can be taken to find the area of the square on the hypotenuse?*** Find the area of the outside square and subtract the area of four triangles. ***Once you know the area of the square on the hypotenuse, how can you find the length of the hypotenuse?*** Take square root of the area of the square. This will give the length of the side of the square. 

	EXPLORE 1
Pairs/ Individual ➢ SP2 Two Right Triangles	• Students find the area of the square on the hypotenuse for the small triangle and then find the length of the hypotenuse by taking its square root. Ask questions to guide computation as needed. ***What is the area of the large square?*** 49 sq units ***The area of each triangle?*** 6 sq units ***The four triangles?*** 24 sq units ***The square on the hypotenuse?*** 25 sq units ***How are the areas of the squares on the legs related to the area of the square on the hypotenuse?*** 9 + 16 = 25. • Students find lengths of sides and areas of squares on the sides for the larger triangle. Note that the hypotenuse of the larger triangle is a non-integral square root. This length may be left in square root form, or it may be estimated.

	SUMMARIZE 1
Whole Class ➢ SP2, OH2 Two Right Triangles	• Invite students to explain their work and calculations on the overhead or board, leading them to conjecture the Pythagorean theorem based on two numerical illustrations. ***What appears to be a relationship between the area of the square on the hypotenuse and the areas of the squares on the two legs?*** In a right triangle, the area of the square on the hypotenuse appears to be equal to the sum of the squares on the two legs. • Explain to students that this conjecture is among the most well known mathematical theorems. It was discovered thousands of years ago, and it is called the Pythagorean theorem. The Pythagorean Theorem was understood to varying degrees in many ancient civilizations. Ancient Babylonians recorded many Pythagorean triples on stone tablets between 1900 and 1600 BCE. Although the theorem as we know it today is commonly attributed to the Greek philosopher and mathematician Pythagoras (who lived during the 6th century BCE), the earliest known formal proofs of the Pythagorean theorem and its converse are recorded in Euclid's *Elements*. (References: *1*. Boyer and Merzbach, A History of Mathematics, Wiley 1991, *2*. Wikipedia)

INTRODUCE 2

Whole Class

➢ SP3-4 Pythagorean Theorem (Part 1)

R1 Pythagorean Theorem Cut Ups

- Lead students through a cut-up proof of the Pythagorean theorem. Show how each of the squares was constructed using side lengths from the right triangle. Label some right angles and lengths.

  What does it mean to say that two shapes are congruent? They have the exact same shape and size.

  Are the two large shapes congruent? Yes. ***How do you know?*** They are both squares with a side length of $a + b$.

  How do their areas compare? They are same.

- Write the area inside each triangle, rectangle, and square.

- Cut out both squares, and cut them into the smaller polygons.

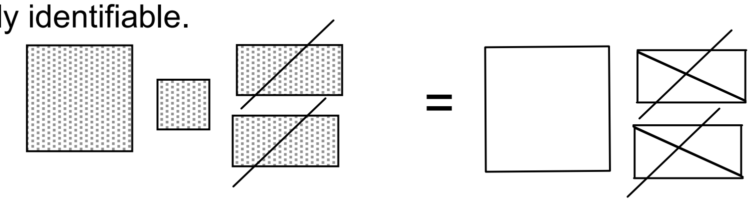
- Arrange two triangles to form a rectangle.

  What does this mean geometrically? The area of two triangles is the same as the area of one rectangle. ***algebraically?*** $\frac{1}{2}ab + \frac{1}{2}ab = ab$

- Separate the cut up pieces into two piles, keeping dissected pieces from each square together. Ask students to state an equation that shows that the sum of the area of the shaded pieces is equal to that of the unshaded pieces. Record the equation on the board.

  What equation is illustrated? $a^2 + ab + ab + b^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$

- Have students rearrange pieces so that the pieces with equal area are clearly identifiable.



  What simplified equation does the picture suggest? $a^2 + b^2 = c^2$

EXPLORE 2	
Individual/Pairs ➢ SP3-4 Pythagorean Theorem (Part 2)	• Students answer questions that lead them to record for themselves this common proof of the Pythagorean theorem. Circulate as students work, giving reminders and hints only if needed.

SUMMARIZE 2	
Whole Class ➢ SP5, OH Pythagorean Theorem (Part 2) Math Background 1	• Ask individuals or pairs to come to the overhead to explain the different parts of the problem. • Congratulate students for proving the Pythagorean theorem. ***What is the Pythagorean theorem?*** For a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. • Using one triangle and the three squares, arrange the pieces to show that the sum of the squares on the legs of the right triangle is equal to the square on the hypotenuse. Remind students that this was their earlier conjecture and it is the Pythagorean theorem.

PRACTICE	
Individuals ➢ SP6-7 Pythagorean Theorem Practice Teaching Tip 2	• This group of problems uses both the Pythagorean theorem and its converse. Use for additional practice or homework.

EXTEND	
Whole Class Math Background 2	• Share the Garfield proof of the Pythagorean theorem with students if desired.

CLOSURE	
Whole Class ➢ SP1, OH1 Ready, Set	• Review the goals and standards for the lesson.

24.1 The Pythagorean Theorem

SOLUTIONS

SP1-Ready Set Go

1. 48 sq. units 2. xy sq. units 3. 24 sq. units 4. 1/2xy sq. units.

5. a+a=2a 6. ab+ab=2ab
7. (1/2)a+(1/2)a = a 8. (1/2)ab+(1/2)ab=ab

SP2—Two Right Triangles

1. 3, 4
2. 4, 7
3. 9, 16
4. 16, 49
5. 25, 65
6. 5, $\sqrt{65} \approx 8$
7. In a right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two legs.

SP5—Right Triangle ABC

1. $a^2 + ab + ab + b^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$

$$a^2 + 2ab + b^2 = 4(\frac{1}{2}ab) + c^2$$

2. $a^2 + 2ab + b^2 = 2ab + c^2$

$$a^2 + b^2 = c^2$$

3. In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

4. Pythagorean theorem

24.1 The Pythagorean Theorem

SP6-7—Pythagorean Theorem Practice

1. 9 + 4 = 13; $3^2 + 2^2 = (\sqrt{13})^2$
2. 4 squared + 5 squared does not equal 9 squared. 16+25 does not equal 81.
3. 13
4. no
5. 3-4-5, 6-8-10

6. only the 6-8-10 triangle is a right triangle, by the converse to the Pythagorean Theorem. The 4-6-8 triangle is not a right triangle, by the Pythagorean theorem.

7. This cannot be right because of the converse to the Pythagorean Theorem. The given triangle is not a right triangle. Note: The Pythagorean Theorem itself does not justify Tommy's answer. The converse to the Pythagorean Theorem goes beyond that, saying that Tommy's answer also cannot be a lucky guess.