PYTHAGOREAN THEOREM AND DISTANCE FORMULA

Participants review carefully the Pythagorean theorem, prove it geometrically and algebraically, and use it to solve real-world problems. Participants focus on the connection between the Pythagorean theorem and the distance formula, and use the distance formula to calculate distances between points on a square grid and in a coordinate plane.

Lesson Goals
❑ Understand several proofs of the Pythagorean theorem, both geometric and algebraic ❑ Use the Pythagorean theorem to solve real-world problems ❑ Understand the connection between the Pythagorean theorem and the distance formula ❑ Calculate the distance between two points in a coordinate plane

Word Bank
❑ right triangle ❑ hypotenuse ❑ leg ❑ Pythagorean theorem ❑ distance formula ❑ circle

Focus Questions
❑ What is the Pythagorean theorem? ❑ How is the Pythagorean theorem proved? ❑ What are some real-world applications of the Pythagorean theorem? ❑ How are the Pythagorean theorem and the distance formula related?

SELECTED CA MATH STANDARDS

Grade 6

AF 1.1 Write and solve one-step linear equations in one variable.
AF 1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.
MR 2.3 Estimate unknown quantities graphically and solve for them by using logical reasoning and arithmetic and algebraic techniques.
MR 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.
MR 3.3 Develop generalizations of the results obtained and the strategies used and apply them in new problem situations.

Grade 7

NS 1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.
NS 2.4 Use the inverse relationship between raising to a power and extracting the root of a perfect square integer; for an integer that is not square, determine without a calculator the two integers between which its square root lies and explain why.
NS 2.5 Understand the meaning of the absolute value of a number; interpret the absolute value as the distance of the number from zero on a number line; and determine the absolute value of real numbers.
AF 4.1 Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.
MG 2.2 Estimate and compute the area of more complex or irregular two-and three-dimensional figures by breaking the figures down into more basic geometric objects.
MG 3.2 Understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, and determine their image under translations and reflections.
MG 3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.
MG 3.4 Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.

Algebra

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as 3(2x-5) + 4(x-2) = 12.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

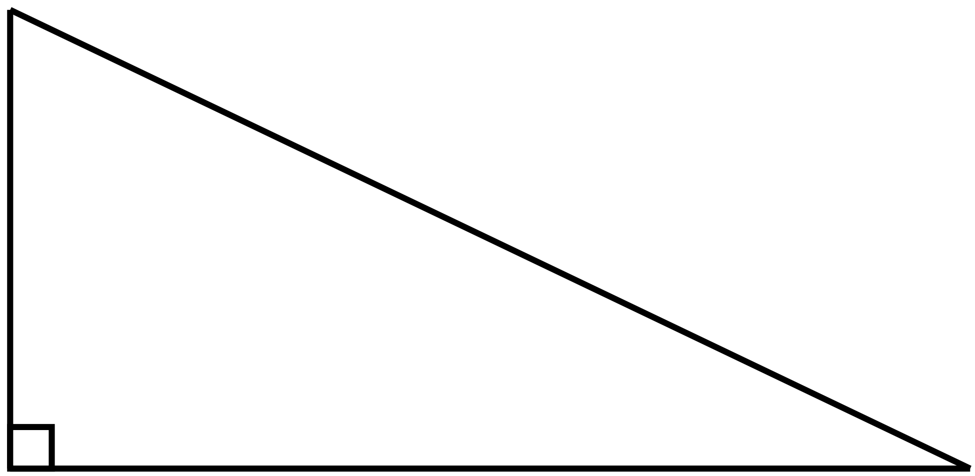
Geometry

14.0 Students prove the Pythagorean theorem.
15.0 Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

Activity	Grade 6	Grade 7	Algebra	Geometry
Pythagorean Theorem	AF **1.1**, 1.2	NS **1.2**, 2.4 MG **3.3**	2.0, 4.0, 5.0	15.0
Geometric Proof		MG 2.2, **3.4**		14.0
Algebraic Proof		AF **4.1**	2.0, 4.0, 5.0	14.0
Applications	AF **1.1**, 1.2 MR 2.3, 3.2, 3.3	NS **1.2** AF **4.1** MG **3.3**	4.0, 5.0	15.0
Derivation of Distance Formula		NS **2.5** AF **4.1** MG 3.2	2.0, 4.0, 5.0	15.0
Equation of a circle	MR 3.2, 3.3	MG 3.2	2.0, 4.0, 5.0	15.0

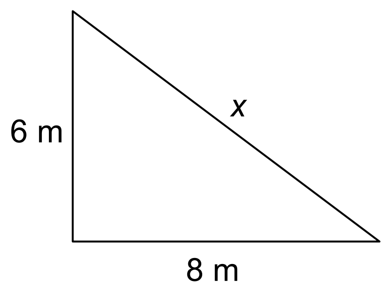
NS: Number Sense
MG: Measurement and Geometry
AF: Algebra and Functions
MR: Mathematical Reasoning

PYTHAGOREAN THEOREM



FIND THE MISSING PART

1)



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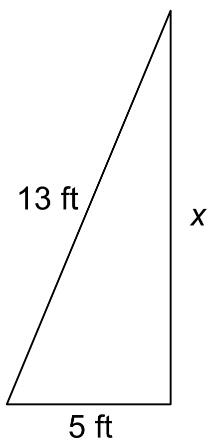

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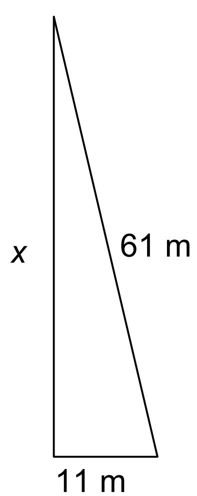

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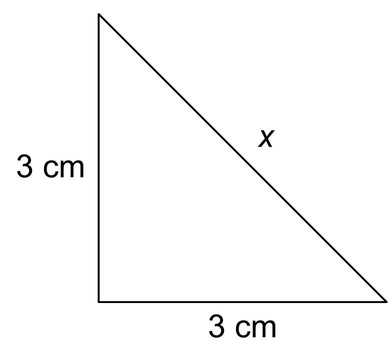

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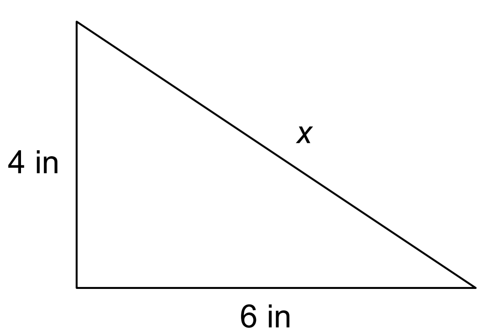
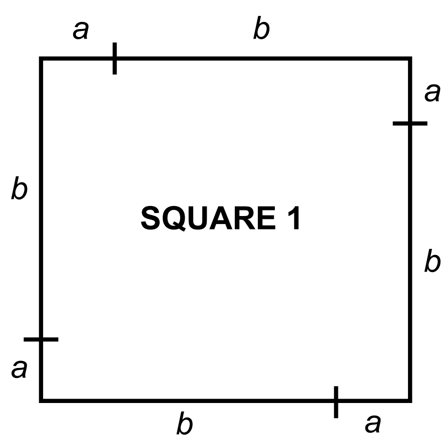


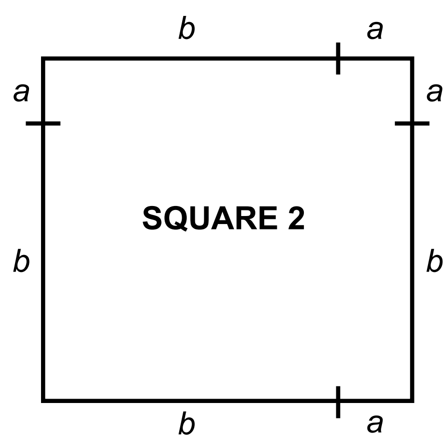
GEOMETRIC PROOF

The following steps lead to a geometric proof of the Pythagorean theorem.

1. For this exploration, you will need two 8.5" x 11" sheets of paper of different colors. Make a square from each sheet by folding one corner of the sheet over so that one of the shorter edges lies along one of the longer edges. Cut off the excess rectangular portion. Save the excess portion to measure lengths *a* and *b* in the next step. Label your squares SQUARE 1 and SQUARE 2. (The two squares are congruent.)

2. Make a mark along the edge of SQUARE 1 (other than the midpoint). Label the first length *a* and the second length *b*. Use the excess portion to record these two lengths. Rotate the square 90° counterclockwise. Mark the same lengths, first *a*, then *b*, along this edge of SQUARE 1. Rotate the square again and repeat this process for the remaining two edges of SQUARE 1. The finished square should look like the one below on the left.



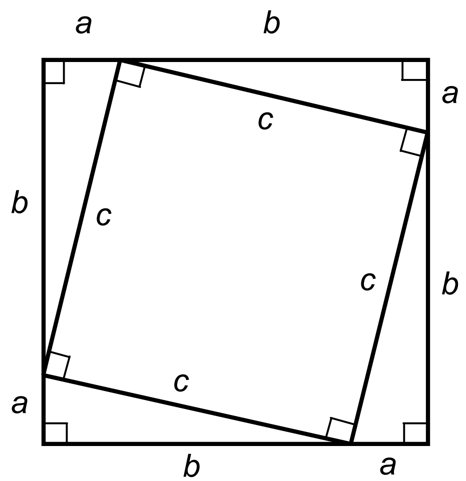


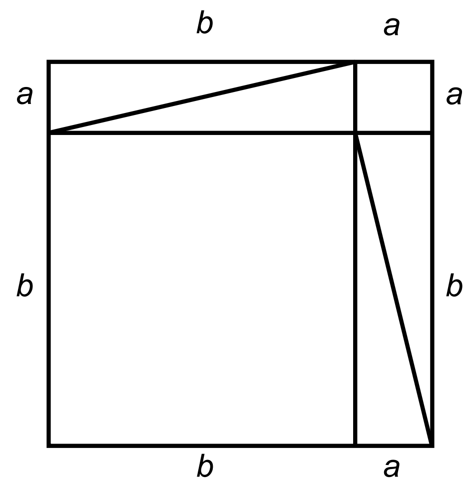
3. With a straight edge connect the consecutive marks to form four right triangles in the corners of SQUARE 1. Label the hypotenuse of each triangle c. The finished square should look like the one below (next page) on the left.

4. Mark one edge of SQUARE 2 first with length *a* and then with length *b* (the same lengths you used for SQUARE 1). Use the excess portion you made in Step 1 as your guide to mark the lengths accurately. Rotate the square 90° counterclockwise and mark the edge first with length *a* and then with length *b*. Rotate the square 90° counterclockwise again, and mark this edge first with length *b*, then with length *a*, so that the *b* lengths are adjacent to each other. Finally, rotate the square 90° counterclockwise again, and mark this edge first with length *b*, then with length *a*. The finished square should look like the one above on the right.

5. With a straight edge connect the marks on opposite sides of SQUARE 2 (connect *a* to *a*, *b* to *b*) to form two squares and two congruent rectangles.

6. With a straight edge draw the diagonals of the two rectangles. The finished square should look like the one below on the right.





7. Using the lines you drew as guides, cut SQUARE 1 and SQUARE 2 into pieces.

8. Compare one triangular piece from SQUARE 1 to one of the triangle pieces from SQUARE 2. What do you notice? What are the areas of the triangles in SQUARE 1 and SQUARE 2?

9. If you remove the four triangles from SQUARE 1, what shape remains? What is the area of this shape?

10. If you remove the four triangles from SQUARE 2, what shapes remain? What are the areas of these shape?

11. Write an expression for the area of SQUARE 1 that is the sum of the areas of the four triangles and the middle square. Then write an expression for the area of SQUARE 2 that is the sum of the areas of the four triangles and the two smaller squares. Are these two expressions equal? How do you know?

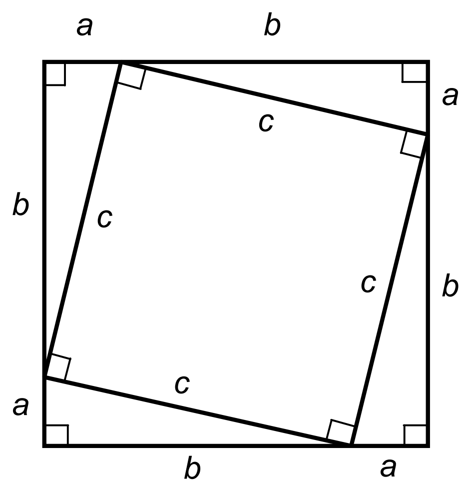
12. Finally, take one of the right triangles and place the two smaller squares along each of its legs, matching the edge lengths. Line the large square along the hypotenuse of the triangle. What does this demonstrate about the relationship of the legs and hypotenuse of a right triangle?

(This exploration is adapted from *College Preparatory Mathematics, Mathematics 1 (Algebra 1, Units 7-2) v. 3.1)*

ALGEBRAIC PROOF

DIRECTIONS: Use the diagram below and the area formulas for squares and triangles to give an algebraic proof of the Pythagorean theorem.

Pythagorean Theorem: For a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

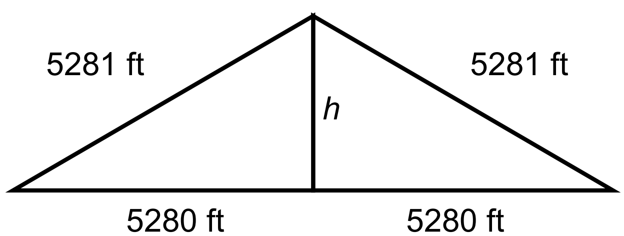


APPLICATIONS

Solve the following problems. In solving these problems, keep in mind that it is always helpful to draw pictures to guide your intuition and reasoning.

RAILROAD LINE PROBLEM: A high-speed railroad line was built through the desert. In order to reduce derailments along a two-mile stretch, the track was made using straight rails one mile long. The rails were laid in the winter, and in the summer, due to the heat, each rail expanded one foot in length. Ordinarily, the rails would push each other to the side, but in this case they jutted upwards where the ends meet. How high above the ground are the rails at the joints? (Historical note: when the RT Metro in Sacramento, California was first running, the trains experienced a problem similar to this situation.)

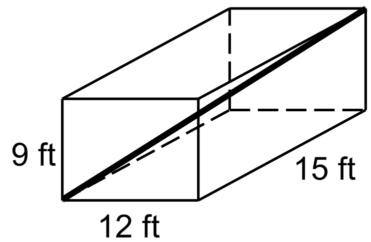
(This problem is adapted from *College Preparatory Mathematics, Mathematics 1 (Algebra 1, Units 7-2) v. 3.2*)



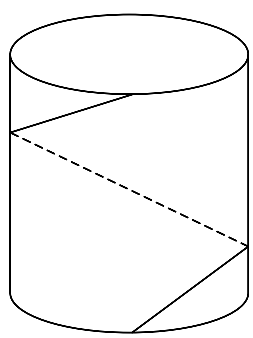
NOT DRAWN TO SCALE

EUCLID STREET PROBLEM: In order to go to school each day, Michelle walks about 3 miles east from her home on Pythagoras Avenue, then walks north along Euclid Street to the front of her school. She knows that the shortest distance from her home to the front of the school is about 9 miles (as the crow flies). How far does she walk along Euclid Street each school day?

RECTANGULAR BOX PROBLEM: You have a huge rectangular box with dimensions 9 ft. x 12 ft. x 15 ft. What is the length of the longest rod you could fit in your box?



CYLINDRICAL CAN PROBLEM: The circumference of the base of a right cylindrical can is 24 inches; the height is 7 inches. A shortest possible spiral that winds once from the top to the bottom (see below) is painted on the can, so that the bottom of the spiral is directly below the top of the spiral. What is the length of the spiral?

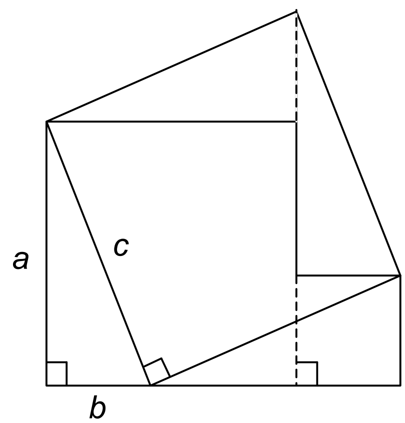


JOURNAL

Four proofs of the Pythagorean theorem are sketched in the following pages. Select one of the four proofs, study it, and figure out why it works. Use the fourfold way to explain the proof. Be prepared to share your findings with others.

FOUR PROOFS
First Proof

The Pythagorean theorem is well known to most high school geometry students. What is not so well known is that there are numerous ways to prove this famous theorem. On this page you have a diagram that suggests a particular version of the proof. Your task is to complete the proof along the lines suggested below. Be ready to present your proof to the class.



Proof sketch: Let A be the area of the entire figure. Let T be the area of the right triangle with legs a and b and hypotenuse c. The figure contains four congruent copies of the right triangle. (Why are they congruent?) Decompose A in two different ways to see that

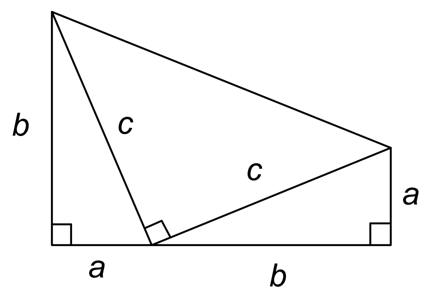
$$A = a^2 + b^2 + 2T,$$

$$A = c^2 + 2T.$$

Conclude that $a^2 + b^2 = c^2$.

FOUR PROOFS
Second Proof

James Abram Garfield (1831-1881) was elected the country's twentieth president in 1880. Earlier he had taught mathematics, and he discovered a proof of the Pythagorean theorem in 1876, while he was a member of the House of Representatives. Find Garfield's proof of the Pythagorean theorem using the diagram below and expressing the area of the trapezoid in two different ways. Be ready to present your proof to the class.



Proof sketch: Let A be the area of the entire figure, which is a trapezoid. The area formula for the trapezoid is

$$A = \text{(width) x (average height)} = \left(\frac{(a+b)(a+b)}{2}\right) = \left(\frac{a^2+2ab+b^2}{2}\right).$$

(Why is this formula true? You can find this formula by joining two copies of the trapezoid together along the slanted edge to form a rectangle.)

Decompose the trapezoid into three triangles and express A as the sum of the areas of three triangles,

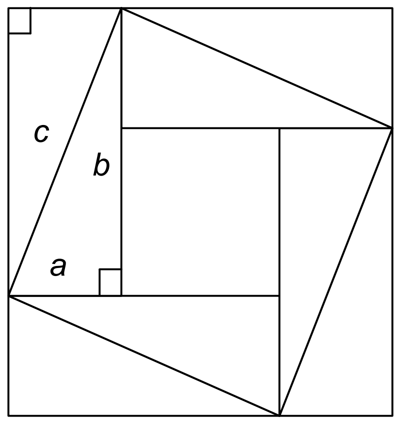
$$A = \frac{1}{2}(ab) + \frac{1}{2}(c^2) + \frac{1}{2}(ab) = \frac{(c^2+2ab)}{2}.$$

Equate these two expressions for A and conclude that

$$c^2 = a^2 + b^2.$$

FOUR PROOFS
Third Proof

On this page you have a diagram that suggests yet another version of the proof. Your task is to prove the theorem using the decomposition of the larger square suggested by the diagram below. Be ready to present your proof to the class.



Proof sketch: Assume $a \leq b$. Let A be the area of the largest square. Then

$$A = (a+b)^2.$$

Note that the side-length of the inside square is b-a. (Why?) Decompose A into four rectangles and the inside square, to see that

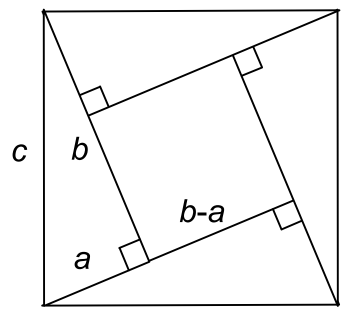
$$A = 4ab + (b-a)^2 .$$

Equate these two expressions for A and conclude that

$$c^2 = a^2 + b^2.$$

FOUR PROOFS
Fourth Proof

On this page you have a diagram that suggests a version of the proof that is closely related to the third proof. (How is it related?) Your task is to prove the theorem using the decomposition of the square suggested by the diagram. Be ready to present your proof to the class.



Proof sketch: Assume $a \leq b$. Let A denote the area of the outside square. Then

$$A = c^2.$$

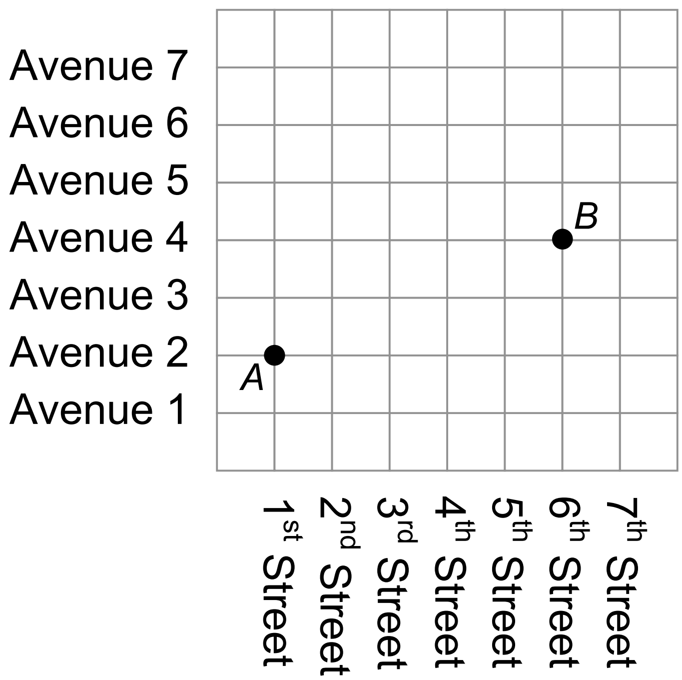
The inside figure is a square (why?) with sides of length a - b (why?). Decompose A into four triangles and the inside square, to see that

$$A = \frac{4ab}{2} + (b - a)^2.$$

Equate these two expressions for A and conclude that

$$c^2 = a^2 + b^2.$$

POINT A TO POINT B



PRACTICE WITH DISTANCE ON A GRID

Find the distance from point A to point B.

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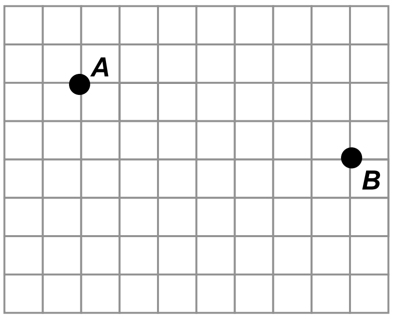

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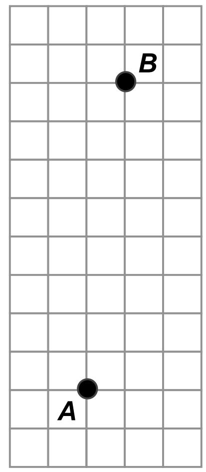

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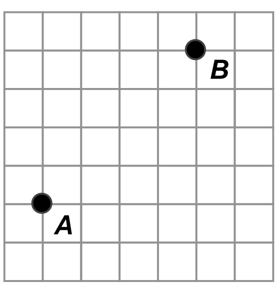

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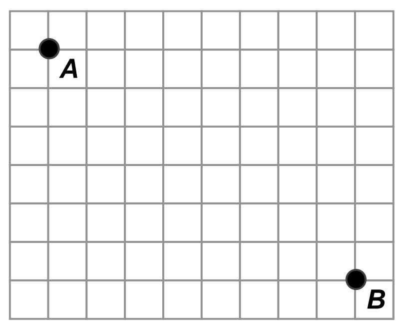
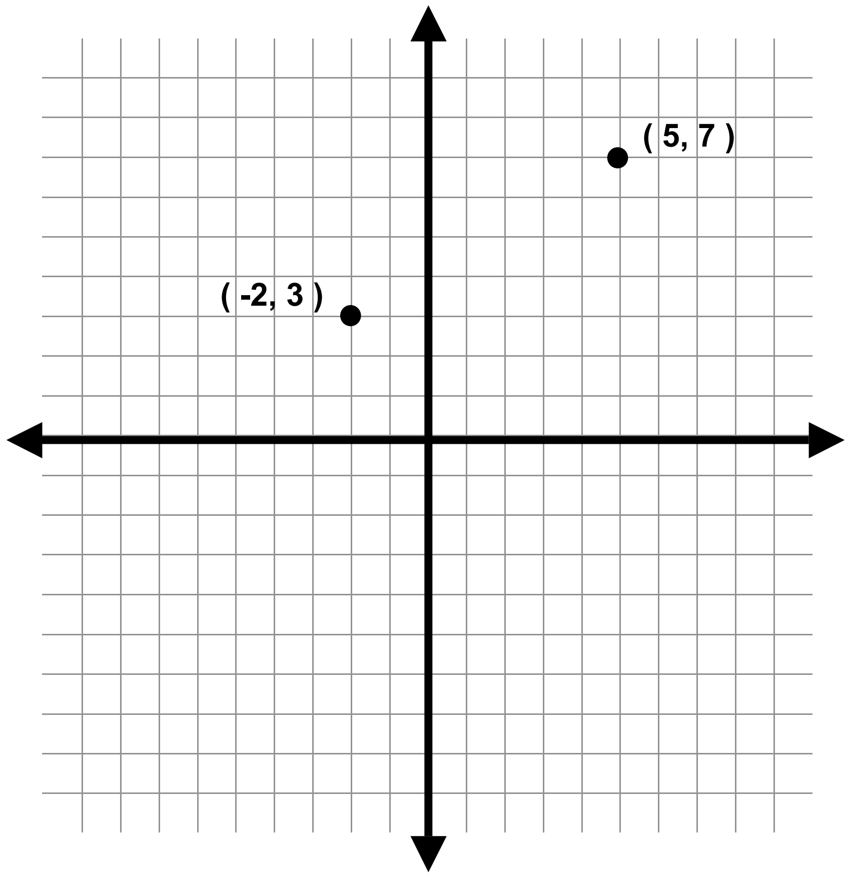


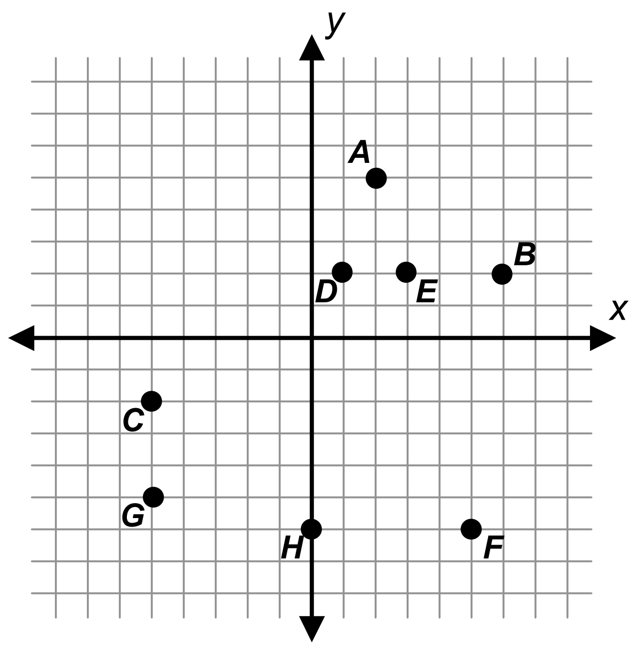
DISTANCE IN A COORDINATE PLANE



PRACTICE WITH DISTANCES IN A COORDINATE PLANE

Use the distance formula to find the distance between the points given.

1. *A* to *B*



2. *C* to *D*

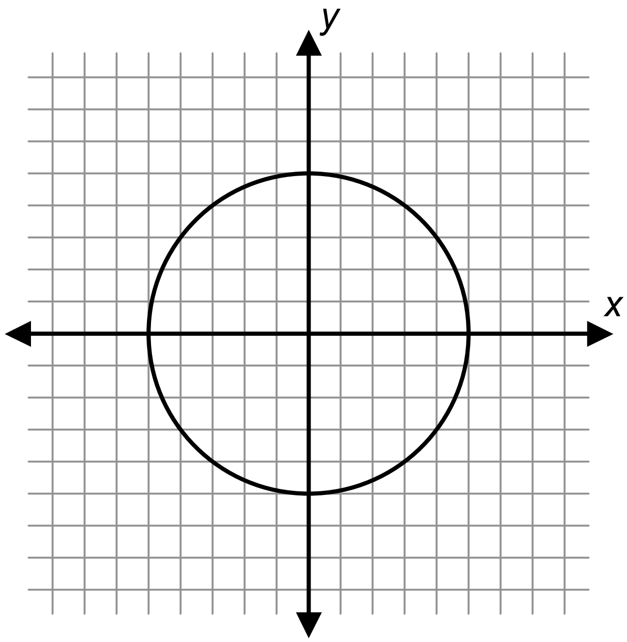
3. *E* to *F*

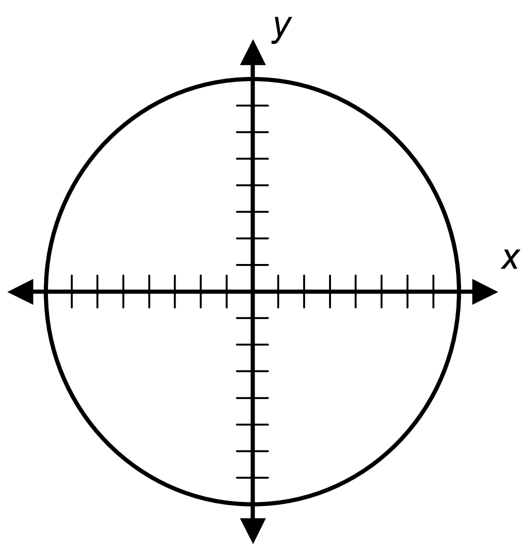
4. *G* to *H*

5. (0, 3) to (-7, 4)

6. (-12, 4) to (3, -10)

CIRCLES IN THE PLANE





PRACTICE WITH CIRCLES

Find the equation of the circle.

1. Center at the origin, radius = 7 units

2. Center at the origin, radius = 1 unit

3. Center at the origin, radius = 15 units

4. (BONUS!) Center at (1,2), radius = 3 units

5. (BONUS!) Center at (-2,7), radius = 5 units

CLASSROOM CONNECTION

Identify three Standards at your grade level that focus on prior knowledge necessary to understand the equation of a circle.

Identify a lesson in your textbook that addresses one of these Standards. Discuss strengths of the lesson and modifications you might make (if any) to increase understanding for all learners.

THE RECTANGLE PARADOX

Cut up a 5 x 21 rectangle as shown in the picture. Put the pieces back together as shown in the second picture to make an 8 x 13 rectangle. Some area has been lost. Explain how.