PYTHAGOREAN THEOREM AND DISTANCE FORMULA

Participants carefully review the Pythagorean theorem, prove it geometrically and algebraically, and use it to solve real-world problems. Participants focus on the connection between the Pythagorean theorem and the distance formula, and use the distance formula to calculate distances between points on a square grid and in a coordinate plane.

Lesson Goals
❑ Understand several proofs of the Pythagorean theorem, both geometric and algebraic
❑ Use the Pythagorean theorem to solve real-world problems
❑ Understand the connection between the Pythagorean theorem and the distance formula
❑ **Calculate the distance between two points in a coordinate plane**

Word Bank
❑ right triangle
❑ hypotenuse
❑ leg
❑ Pythagorean theorem
❑ distance formula
❑ circle

Focus Questions
❑ What is the Pythagorean theorem?
❑ How is the Pythagorean theorem proved?
❑ What are some real-world applications of the Pythagorean theorem?
❑ How are the Pythagorean theorem and the distance formula related?

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Estimated Time: 3 hours

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<td>Lesson Summary

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Universal Access Strategies</td>
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<tr>
<td>Materials

paper (2 colors)
scissors
cardboard cylinder (from paper towel roll or toilet paper roll)</td>
<td>Reproducibles</td>
<td>Prepare Ahead</td>
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<tr>
<td>Participant Pages

PP1: Summary Page
PP2: Selected CA Math Standards
PP3: Pythagorean Theorem
PP4: Find the Missing Part
PP5-6: Geometric Proof
PP7: Algebraic Proof
PP8-9: Applications
PP10: Journal
PP11-14: Four Proofs
PP15: Point A to Point B
PP16: Practice with Distance on a Grid
PP17: Distance in a Coordinate Plane
PP18: Practice with Distance in a Coordinate Plane
PP19: Circles in the Plane
PP20: Practice with Circles
PP21: Classroom Connection
PP22: The Rectangle Paradox</td>
<td>Overhead Transparencies

OH1: Focus Questions
OH2: CA Math Standards – In Brief
OH3: Pythagorean Theorem
OH4: Proof of the Theorem
OH5: Railroad Line Problem
OH6: Euclid Street Problem
OH7: Box Problem
OH8: Cylindrical Can Problem
OH9: Point A to Point B
OH10: Distance in a Coordinate Plane
OH11: Circles in the Plane
OH12: Post-it Planning</td>
<td>Focus Questions

• What is the Pythagorean theorem?
• How is the Pythagorean theorem proved?
• What are some real-world applications of the Pythagorean theorem?
• How are the Pythagorean theorem and the distance formula related?</td>
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<td>Problem of the Day Idea

Revisit the “Rectangle Paradox” Problem. Explain the paradox using the Pythagorean theorem or the distance formula.</td>
<td>Journal Idea

Select one of the four proofs of the Pythagorean theorem and figure out why it works. Use the fourfold way to explain the proof.</td>
<td>Assessment Idea

Find the length of the diagonal of a cube with an edge of length 6 cm.</td>
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PYTHAGOREAN THEOREM AND DISTANCE FORMULA

This lesson is divided into two parts. The first part of the lesson (2 hours) explores the Pythagorean theorem. Knowledge of the Pythagorean theorem (and also of its converse) appears in the California Mathematics Standards as a key Standard for Grade 7. The proof of the theorem appears only later, in the high school geometry standards. However, for teachers of Grade 7, it is essential to have some understanding of how the theorem is proved in order to teach the theorem with confidence. (Recall PUFM.) There are more than 400 proofs of the Pythagorean theorem, many of them simple variations of each other. A sampling of proof strategies is presented in this lesson.

The second part of the lesson (1 hour) explores the distance formula in the plane, which is developed from the Pythagorean theorem. The distance formula leads quite naturally to the equation of a circle.

Preview

- Use OH1 (Focus Questions) and PP1 (Summary Page) to preview the goals of the lesson.

- Use OH2, PP2 (California Math Standards) to introduce standards addressed in the lesson. Participants may want to make a note of them on appropriate participant pages.

Introduce 1

- Use OH3, PP3 (Pythagorean Theorem). ***What is the Pythagorean theorem?*** Write down the Pythagorean theorem as some of the participants say it (for example, $a^2 + b^2 = c^2$). Ask participants to identify *a, b,* and *c*. [*a* and *b* are the lengths of the legs, *c* is the length of the hypotenuse of a right triangle.]

> A common response to "What is the Pythagorean theorem?" is "*a* squared plus *b* squared equals *c* squared." This is the "slogan form" of the Pythagorean theorem, which is a good mnemonic device for remembering the theorem. However, be sure to give a complete statement of the theorem, citing explicitly the lengths of the legs and hypotenuse of a right triangle. Distinguish carefully between the "squares on the leg" and the "length of the leg."

- Review parts of the right triangle (hypotenuse and legs). Be sure to discuss the location of the hypotenuse (opposite the right angle) and to identify it as the longest side. Have participants label their diagrams.

- Write out the Pythagorean theorem completely and carefully: ***For a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.***

An alternative statement of the theorem, more in the spirit of Euclid: **For a right triangle, the sum of the squares on the legs is equal to the square on the hypotenuse.** Note the use of "on" rather than "of." In this version, the "square on the hypotenuse" is the square with the hypotenuse as one side, and the "squares on the legs" are two squares with the legs as sides. The statement of the Pythagorean theorem then refers to the areas of these squares.

- Ask if anyone knows what the converse to the Pythagorean theorem is. [If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the triangle is a right triangle.] If participants are up to it, write down also the converse to the Pythagorean theorem, and discuss why it is a converse.

Summarize

- Use PP4 (Find the Missing Part). Demonstrate how to do the first problem. Have participants solve the remaining problems. Discuss solutions.

FIND THE MISSING PART – SOLUTIONS

leg_1 = 6 m,	leg_2 = 8 m,	hypotenuse = ?	Answer: x = 10 m
leg_1 = 5 ft,	leg_2 = ?,	hypotenuse = 13 ft	Answer: x = 12 ft
leg_1 = ?,	leg_2 = 11 m,	hypotenuse = 61 m	Answer: x = 60 m
leg_1 = 3 cm,	leg_2 = 3cm,	hypotenuse = ?	Answer: $x = \sqrt{18}\text{cm} = 3\sqrt{2}\text{cm}$
leg_1 = 4 in,	leg_2 = 6 in,	hypotenuse = ?	Answer: $x = \sqrt{52}\text{in} = 2\sqrt{13}\text{in}$

Participants who use a calculator will get a decimal approximation for their solutions to problems (d) and (e). This is a good time to discuss the correctness of the radical form (an exact answer), the simplified radical form (a simplified exact answer), and the decimal approximation. All three are acceptable, though the decimal approximation should be labeled as such.

This is a good time to discuss "simplified" radical form. $3\sqrt{2}$ is "simpler" than $\sqrt{18}$ only when a computation with it must be done by hand. Calculators make it unnecessary to simplify. Also, when comparing the sizes of radicals, "simplified" radicals are <u>harder</u> to compare.

Which is bigger? $\sqrt{20}$ ***or*** $\sqrt{50}$ ***?*** $2\sqrt{5}$ ***or*** $5\sqrt{2}$ ***?***

Explore

- (Pairs) Use OH4 (Proof of the Theorem) and PP5-6 (Geometric Proof). Distribute scissors and sheets of paper of two different colors. Participants follow the directions for the geometric proof of the Pythagorean theorem.

- (Whole group) Discuss the proof. ***Why is the proof a geometric proof? What geometric facts are used in the proof?***

- (Whole group) Use OH4 (Proof of the Theorem) and PP7 (Algebraic Proof). Work out the algebraic proof as a group. Discuss the proof, and compare it to the geometric proof already given. ***How does this proof differ from our first proof?*** [The algebraic proof depends on the formula for the area of a triangle and on some algebraic manipulations. The geometric proof depends on the congruence of certain geometric figures.]

Practice

- (Whole group) Use OH5 (Railroad Line Problem) and PP8-9 (Applications). Introduce the problem. (1 mile = 5280 ft)

> **RAILROAD LINE PROBLEM**
>
> A high-speed railroad line was built through the desert. In order to reduce derailments along a two-mile stretch, the track was made using straight rails one mile long. The rails were laid in the winter, and in the summer, due to the heat, each rail expanded one foot in length. Ordinarily, the rails would push each other to the side, but in this case they jutted upwards where the ends meet. How high above the ground are the rails at the joints? (Historical note: when the RT Metro in Sacramento, California was first running, the trains experienced a problem similar to this situation.)
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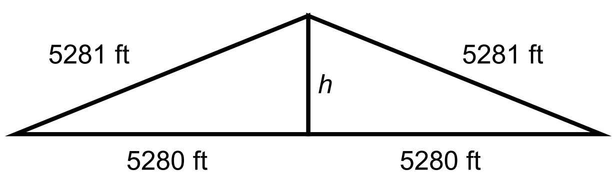
> 
>

- Encourage participants to predict the answer before doing any computations.

  - ***How high will the railroad track buckle?***
  - ***Just high enough to slide your arm between the ground and the tracks?***
  - ***Just high enough for an average height person to walk through?***
  - ***Just high enough for this room to fit under?***
  - ***More than the height of this room?***

- Complete the Railroad Line Problem as a group. [$\sqrt{10561}$ feet or approximately 102.77 feet.]

- (Pairs) Use PP8-9 (Applications). Participants solve Problems 2-4.

> Bring a cardboard cylinder from a paper towel roll, or from an empty toilet paper roll, to serve as a model for Problem 2.

Summarize

- (Whole group) Use OH6 (Euclid Street Problem), OH7 (Box Problem), OH8 (Cylindrical Can Problem), and PP8-9 (Applications). Share solutions. [Euclid Street Problem: $\sqrt{72} = 6\sqrt{2}$ or approximately 8.5 miles; Rectangular Box Problem: $\sqrt{450} = 15\sqrt{2}$ or approximately 21.2 feet; Cylindrical Can Problem: 25 inches.]

> For the Box Problem, label the top left end of the diagonal *A* and the three visible vertices on the bottom *B*, *C*, and *D*, so that the diagonal is *AB*. Apply the Pythagorean theorem twice, to the triangles *ABC* and *ACD*, and do some algebra. For the Cylindrical Can Problem, we must use the fact that the length of the spiral is not changed if we cut the can along the straight line segment from the top to the bottom of the spiral and flatten out the can. The flattened can is a rectangle, and the spiral becomes a diagonal of the rectangle. We use here the fact that the shortest spiral corresponds to the shortest curve between opposite vertices of the rectangle, which is a straight line segment. (The shortest path between two points is along a straight line.)

Journal

- Use PP10 (Journal) with PP11-14 (Four Proofs). Each participant should select one of the four proofs of the Pythagorean theorem, study it to figure out why it works, and use the fourfold way to explain the proof. Participants may be asked to share their analyses of the proofs.

Extend (Optional)

- Show the CalTech video on the Pythagorean theorem.

Introduce 2 (Pythagorean Theorem on a Grid)

- Use OH9, PP15 (Point *A* to Point *B*).

  - ***How can you go from point A to point B using the streets and avenues?*** [Many solutions. Getting from point *A* to point *B* using the streets and avenues is called "taxicab geometry."]

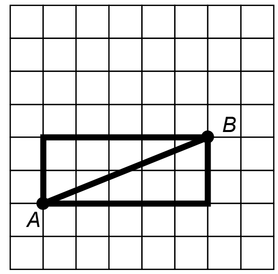
  - ***What is the shortest distance from point A to point B using the streets and avenues? How many different ways can you get from point A to point B in the shortest distance?*** [2 streets north and 5 streets east; 5 streets east and 2 streets north; 1 street north, 5 streets east, and 1 more street north; etc.]

  - ***Is there an even shorter way to get from point A to point B?*** [If allowed to cut through blocks, there are many shorter ways to get from *A* to *B*. The shortest way is the straight line from *A* to *B*.]

  - ***How can we calculate the distance along the straight-line path from A to B?*** [Use the Pythagorean theorem to find the distance "as the crow flies." The straight-line distance is the length of the line segment joining *A* and *B*. This segment is the hypotenuse of a right triangle whose legs lie on a street and an avenue.]

- Invite participants to show (or demonstrate) how to use the Pythagorean theorem to find the straight-line distance.

$$\begin{aligned} leg^2 + leg^2 &= hypotenuse^2 \\ (2)^2 + (5)^2 &= c^2 \\ 29 &= c^2 \\ \sqrt{29} &= c \end{aligned}$$



Practice

- (Pairs) Use PP16 (Practice with Distance on a Grid). Demonstrate by solving the first problem if needed. Participants solve the remaining problems.

- Discuss solutions. [1) $\sqrt{53}$; 2) $\sqrt{65}$; 3) $\sqrt{32} = 4\sqrt{2}$; 4) 10]

Introduce 3 (Derive Distance Formula)

- Use OH10, PP17 (Distance in a Coordinate Plane). Have participants find the distance from Point *A* to Point *B*. [$\sqrt{(4^2 + 7^2)} = \sqrt{65}$ units, which is approximately 8.06 units]. Discuss the similarities of this graph and the Point *A* to Point *B* street/avenue grid.

  - ***Besides counting, how might we find the leg lengths of 7 units and 4 units***? [Compute vertical distance using the difference of the *y*-values of the two coordinates: 7 – 3 = 4 or |3 – 7| = 4. Compute horizontal distance using difference of the x-values of the two coordinates: 5 – (-2) = 7 or | -2 – 5| = 7.]

  - ***Why is it important to take the absolute value of 3 – 7?*** [We are looking for a distance, and distances cannot be negative.]

- Relate calculations using coordinates to the computation of the distance from Point *A* to Point *B*.

$$|AB| = \sqrt{(4^2 + 7^2)}$$

$$|AB| = \sqrt{(7-3)^2 + (5-(-2))^2} = \sqrt{65} \text{ units}$$

- Generalize for any two points, (x_1, y_1) and (x_2, y_2). The distance between them, denoted by *d*, is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \, ,$$

or more simply,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

In solving problems, it is usually easiest to work with d^2, the distance squared. If you do this, you must remember at the end to take the square root to get the distance d.

- Compare the distance formula to the formula for the slope of a line,

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}.$$

  - ***When calculating slope, the order we subtract the coordinates is important. Does the order matter for the distance formula? Why or why not?*** [If (say) we interchange y_1 and y_2 in the slope formula, the slope m switches sign. If we interchange y_1 and y_2 in the distance formula, the distance d stays the same.]

Practice

- (Pairs) Use PP18 (Practice with Distances in a Coordinate Plane). Demonstrate how to solve the first problem if needed. Participants solve the remaining problems.

- (Whole group) Discuss solutions. [1) 5; 2) $\sqrt{52} = 2\sqrt{13}$; 3) $\sqrt{68}$; 4) $\sqrt{26}$; 5) $\sqrt{50}$; 6) $\sqrt{421}$]

Explore (optional)

- Use PP22 (The Rectangle Paradox). Revisit the Rectangle Paradox POD from the Slopes Module in the Perspectives on Geometry course. Explain the paradox using the Pythagorean theorem or the distance formula.

Summarize

- ***How does the distance formula relate to the Pythagorean theorem?***

Extend (Equation of a circle centered at the origin)

Note: Participants will likely recognize formulas for circumference (linear units) and area (square units) of circle. This extension derives the formula for the set of ordered pairs that make up a circle in a coordinate plane.

- Write the following formulas on the overhead:

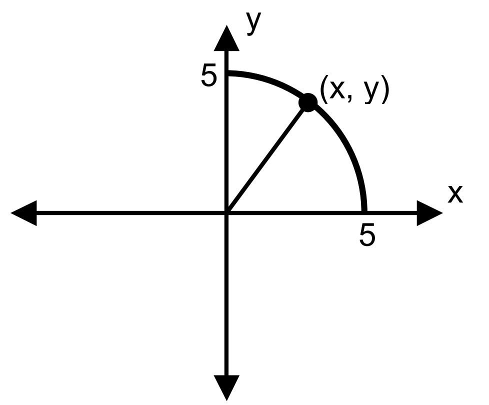
$$C = 2\pi r, \qquad A = \pi r^2, \qquad x^2 + y^2 = r^2.$$

- Participants identify the meaning of the variables and use of each formula.

- Use OH11, PP19 (Circles in the Plane). Refer to the top circle. ***What is the radius of the circle?*** [5 units] ***Name four points on the circle.*** [Many possibilities to choose from: (5, 0), (0, 5), (-5, 0), (0, -5), (3,4), (-3,4), $(1, 2\sqrt{2})$, etc.]

If (x,y) lies on a circle centered at the origin, then so do its reflection $(x,-y)$ in the x-axis, its reflection $(-x,y)$ in the y-axis, its rotation $(y,-x)$ by 90°, and the various other reflections and rotations of these points.

- Pick a random point (x,y) on the circle in the first quadrant. Draw a segment from the origin to this point (x,y).



- Write an equation for the distance of this line segment. Share solutions.
  [$(x-0)^2 + (y-0)^2 = 5^2$, or $x^2 + y^2 = 5^2$.]

- Refer to the bottom circle and repeat the process. Share solutions.
  [$(x-0)^2 + (y-0)^2 = 8^2$, or $x^2 + y^2 = 8^2$.]

- Generalize the equation of a circle centered at (0, 0) with radius r.

The general equation for a circle centered at the origin (0, 0) with radius r is $(x-0)^2 + (y-0)^2 = r^2$, or $x^2 + y^2 = r^2$

- (Pairs) Use PP20 (Practice with Circles). Participants complete Problems 1-3. Challenge more advanced participants to find equations for circles not centered at the origin (Problems 4-5).

- Share solutions. [1) $x^2 + y^2 = 7^2$; 2) $x^2 + y^2 = 1^2$; 3) $x^2 + y^2 = 15^2$; 4) $(x-1)^2 + (y-2)^2 = 3^2$; 5) $(x+2)^2 + (y-7)^2 = 5^2$.]

The equation of a circle of radius r centered at the point (h, k) is $$(x-h)^2 + (y-k)^2 = r^2$$ This equation can also be obtained directly from the definition of a circle and the distance formula.

Summarize

- (Whole group) Ask participants to describe the equation of the circle using the fourfold way.

Classroom Connection

- Use PP21 (Classroom Connection) to explore skills studied at various grade levels that lead to understanding of the equation of a circle.

Textbook Connection: Post-It Planning (10-20 minutes)

- Use OH12 (Post-It Planning). Participants create 3-5 post-it reminders of "aha's," important ideas, activities, and possible misconceptions that they would like to remember when they return to the classroom. They write a textbook reference page on a post-it, or stick a post-it on the appropriate page in their textbook.

Closure

- Use OH2, PP2 (California Math Standards) to revisit standards. Connect module activities to student outcome goals.

Activity	Grade 6	Grade 7	Algebra	Geometry
Pythagorean Theorem	AF **1.1**, 1.2	NS **1.2**, 2.4 MG **3.3**	2.0, 4.0, 5.0	15.0
Geometric Proof		MG 2.2, **3.4**		14.0
Algebraic Proof		AF **4.1**	2.0, 4.0, 5.0	14.0
Applications	AF **1.1**, 1.2 MR 2.3, 3.2, 3.3	NS **1.2** AF **4.1** MG **3.3**	4.0, 5.0	15.0
Derivation of Distance Formula		NS **2.5** AF **4.1** MG 3.2	2.0, 4.0, 5.0	15.0
Equation of a circle	MR 3.2, 3.3	MG 3.2	2.0, 4.0, 5.0	15.0

- Use OH1 (Focus Questions) and PP1 (Summary Page) to revisit the goals of the lesson. Tie up loose ends.