

### Basic Exam, Spring 2007

1. Let  $A$  be a real  $m \times n$  matrix,  $m > n$ , whose columns are linearly independent and  $\mathbf{b} \in \mathbb{R}^m$ . Show that the vector  $\mathbf{x}^* \in \mathbb{R}^n$  that minimizes the functional

$$g(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2$$

is given by the solution of the normal equations

$$A^t A \mathbf{x} = A^t \mathbf{b}.$$

Here  $\|\mathbf{z}\|_2^2 = \langle \mathbf{z}, \mathbf{z} \rangle = \sum_i z_i^2$ .

2. Let  $V, W, Z$  be  $n$ -dimensional vector spaces and  $T : V \rightarrow W$  and  $U : W \rightarrow Z$  be linear transformations. Prove that if the composite transformation  $UT : V \rightarrow Z$  is invertible, then both  $T$  and  $U$  are invertible. (Do not use determinants in your proof!)

3. Consider the space of infinite sequences of real numbers

$$\mathcal{S} = \{(a_0, a_1, a_2, \dots) : a_n \in \mathbb{R}, n = 0, 1, 2, \dots\}$$

endowed with the standard operations of addition and scalar multiplication:

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots); \quad c(a_0, a_1, \dots) = (ca_0, ca_1, \dots), \quad c \in \mathbb{R}.$$

For each pair of real numbers  $A$  and  $B$ , prove that the set of solutions  $(x_0, x_1, x_2, \dots)$  of the linear recursion

$$x_{n+2} = Ax_{n+1} + Bx_n, \quad n = 0, 1, 2, \dots$$

is a linear subspace of  $\mathcal{S}$  of dimension 2.

4. Suppose that  $A$  is a symmetric  $n \times n$  real matrix with distinct eigenvalues  $\lambda_1, \dots, \lambda_l$ , ( $l \leq n$ ). Find the sets

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : \lim_{k \rightarrow \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} \text{ exists} \right\}$$

and

$$L = \left\{ \lim_{k \rightarrow \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} : \mathbf{x} \in X \right\},$$

where  $\mathbb{R}^n$  is identified with the set of real column vectors, and  $\mathbf{x}^t$  denotes the transpose of  $\mathbf{x}$ .

5. Let  $T$  be a normal linear operator on a finite dimensional complex inner product linear space  $V$ . Prove that if  $\mathbf{v}$  is an eigenvector of  $T$ , then  $\mathbf{v}$  is also an eigenvector of its adjoint  $T^*$ .

6. Consider the integral equation

$$(*) \quad y(t) = y_0 + \int_0^t f(s, y(s)) ds$$

where  $f(t, y)$  is continuous on  $[0, T] \times \mathbb{R}$  and is Lipschitz in  $y$  with Lipschitz constant  $K$ . Assume that you have shown that the iterates defined by

$$y^n(t) = y_0 + \int_0^t f(s, y^{n-1}(s)) ds, \quad y^0(t) \equiv y_0$$

converge uniformly to a solution  $y(t)$  of (\*). Show that if  $Y(t)$  is a solution of (\*) and satisfies  $|Y(t) - y_0| \leq C$  for some constant  $C$  and all  $t \in [0, T]$ , then  $Y(t) \equiv y(t)$  on  $[0, T]$ .

7. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a twice continuously differentiable function with  $f''$  uniformly bounded, and with a simple root at  $x^*$  (i.e.,  $f(x^*) = 0, f'(x^*) \neq 0$ ). Consider the fixed point iteration

$$x_n = F(x_{n-1}) \quad \text{where} \quad F(x) = x - \frac{f(x)}{f'(x)}.$$

Show that if  $x_0$  is sufficiently close to  $x^*$ , then there exists a constant  $C$  so that for all  $n$ ,

$$|x_n - x^*| \leq C|x_{n-1} - x^*|^2.$$

8. Suppose the functions  $f_n$  are twice continuously differentiable on  $[0, 1]$  and satisfy

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for all } x \in [0, 1], \text{ and}$$
$$|f'_n(x)| \leq 1, \quad |f''_n(x)| \leq 1 \quad \text{for all } x \in [0, 1], n \geq 1.$$

Prove that  $f(x)$  is continuously differentiable on  $[0, 1]$ .

9. (a) Define “ $f$  is Riemann integrable on  $[0, 1]$ ”.

(b) Prove that every continuous function on  $[0, 1]$  is Riemann integrable.

10. Suppose the functions  $f_n(x)$  on  $\mathbb{R}$  satisfy:

(i)  $0 \leq f_n(x) \leq 1$  for all  $x \in \mathbb{R}$  and  $n \geq 1$ .

(ii)  $f_n(x)$  is increasing in  $x$  for every  $n \geq 1$ .

(iii)  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for each  $x \in \mathbb{R}$ , where  $f$  is continuous on  $\mathbb{R}$ .

(iv)  $\lim_{x \rightarrow -\infty} f(x) = 0$  and  $\lim_{x \rightarrow \infty} f(x) = 1$ .

Show that  $f_n(x) \rightarrow f(x)$  uniformly on  $\mathbb{R}$ .

11. (a) Consider the equations

$$u^3 + xv - y = 0, \quad v^3 + yu - x = 0.$$

Can these equations be solved uniquely for  $u, v$  in terms of  $x, y$  in a neighborhood of  $x = 0, y = 1, u = 1, v = -1$ ? Explain your answer.

(b) Give an example in which the conclusion of the implicit function theorem is true but the hypothesis is not.

12. Let  $c_0$  be the normed space of real sequences  $x = (x_1, x_2, \dots)$  such that  $\lim_{k \rightarrow \infty} x_k = 0$  with the supremum norm  $\|x\| = \sup_k |x_k|$ .

(a) Show that  $c_0$  is complete.

(b) Is the unit ball  $\{x \in c_0 : \|x\| \leq 1\}$  compact? Prove your answer.

(c) Is the set  $\{x \in c_0 : \sum_k k|x_k| \leq 1\}$  compact? Prove your answer.