1. Let S be a subset of $\mathbb{R}^n$ with the distance function $d(x,y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ so that $(S, d|_{S\times S})$ is a metric space.

a) Given $y \in S$, is $E = \{x \in S : d(x,y) \geq r\}$ a closed set in S?
b) Is the set E in part a) contained in the closure of $\{x \in S : d(x,y) > r\}$ in S?

Prove your answers.

2. Let $f : (a,b) \to \mathbb{R}$ be continuous on (a,b) and differentiable in $(a,b)\backslash\{c\}$. If $\lim_{x\to c} f'(x) = d \in \mathbb{R}$, show that f is differentiable at c, and $f'(c) = d$.

3. Let T be a linear transformation of the vector space V into itself. If Tv and v are linearly dependent for each $v \in V$, show that T must be a scalar multiple of the identity.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and its second derivative, f'', satisfies $|f''(x)| \leq B$.

a) Prove that

$$|2Af(0) - \int_{-A}^{A} f(x)dx| \leq \frac{A^3}{3}B$$

b) Use the result of part a) to justify the following estimate:

$$\Big|\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(a + \frac{2k-1}{2n}(b-a))\Big| \leq Cn^{-2},$$

where C is a constant that does not depend on n.

5. a) Show that, given a continuous function, $f : [0,1] \to \mathbb{R}$, which vanishes at $x = 1$, there is a sequence of polynomials vanishing at $x = 1$ which converges uniformly to f on $[0,1]$.
b) If f is continuous on $[0,1]$, and

$$\int_0^1 f(x)(x-1)^k dx = 0 \text{ for each } k = 1, 2, ...,$$

show that $f(x) \equiv 0$.

6. Let T be a linear transformation from a finite dimensional vector space V into a finite dimensional vector space W. Compute (with proof)

$$\dim\ (\text{Null } T) + \dim\ (\text{Range } T)$$

and

$$\dim\ (\text{Null } T^*) + \dim\ (\text{Range } T)$$

in terms of the dimensions of V and W. Here T^* denotes the adjoint of T.

7. Let $A(x)$ be a function on $\mathbb{R}$ whose values are $n \times n$ matrices. Starting from the definition that the derivative $A'(x)$ is the matrix you get by differentiating the entries in $A(x)$, show that when $A(x)$ is invertible and differentiable for all x, $A^{-1}(x)$ is differentiable, and

$$(A^{-1})'(x) = -A^{-1}(x)A'(x)A^{-1}(x).$$

8. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Does it follow that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = \infty?$$

Prove your answer.

9. Suppose $u_n : \mathbb{R} \to \mathbb{R}$ is differentiable and solves

$$u_n'(x) = F(u_n(x), x),$$

where F is continuous and bounded.

a) Suppose $u_n \to u$ uniformly. Show that u is differentiable and solves

$$u'(x) = F(u(x), x).$$

b) Suppose

$$u'(x) = F(u(x), x), u(x_0) = y_0$$

has a unique solution $u : \mathbb{R} \to \mathbb{R}$ and $u_n(x_0)$ converges to y_0 as $n \to \infty$. Show that u_n converges uniformly to u on $-N \leq x \leq N$ for any N.

10. Suppose that $\{\vec{v}_j\}_{j=1}^n$ is a basis for the complex vector space $\mathbb{C}^n$.

a) Show that there is a basis $\{\vec{w}_j\}_{j=1}^n$ such that $(\vec{w}_j, \vec{v}_k) = \delta_{jk}$. Here $(\cdot,\cdot)$ is the standard inner product, $(\vec{w}, \vec{v}) = \overline{w}_1 v_1 + \overline{w}_2 v_2 + \cdots + \overline{w}_n v_n$, and $\delta_{jk} = 1$ when $j = k$ and 0 otherwise.

b) If the $\vec{v}_j$'s are eigenvectors for a linear transformation T of $\mathbb{C}^n$, show that the $\vec{w}_j$'s are eigenvectors for T^*, the adjoint of T with respect to (,).

11. Let f be bounded real function on [0,1]. Show that f is Riemann integrable if and only if f^3 is Riemann integrable.

12. a) Suppose that $x_0 < x_1 < \cdots < x_n$ are points in [a,b]. Define linear functions on $\mathbb{P}^n$, the vector space of polynomials of degree less than or equal n, by setting

$$l_j(p) = p(x_j) \quad j = 0, \ldots, n$$

Show that the set $\{l_j\}_{j=0}^n$ is linearly independent.

b) Show that there are unique coefficients c_j such that

$$\int_a^b p(x)dx = \sum_{j=0}^n c_j l_j(p)$$

for all $p \in \mathbb{P}^n$.