

Algebra Qualifying Exam  
Fall 2007

**Test Instructions:** Each problem is worth 20 points. You must attempt to do at least two problems in each of the four sections. Your total score will be computed using only the two best scoring problems in each section. Whether you pass or fail depends on your performance in each section, not only on the total score. In each problem you may lose points if you do not explain clearly your reasoning or any theorems which you quote.

**Groups**

- G1. Let  $F$  be a finite field of characteristic  $p$  and let  $G$  be a subgroup of order  $p^a$  of the group  $GL(N, F)$  of invertible  $N$  by  $N$  matrices with entries in  $F$ . Show that there is a non-zero vector  $v$  in  $F^N$  such that  $gv = v$  for every  $g \in G$ .
- G2. Let  $M$  be the submodule of  $\mathbf{Z}^3$  generated by elements  $(0, 3, 2)$ ,  $(6, 48, 24)$  and  $(6, 24, 12)$ . Describe the quotient group  $\mathbf{Z}^3/M$  by giving a product of cyclic groups to which  $\mathbf{Z}^3/M$  is isomorphic.
- G3. A group  $G$  acts doubly transitively on a set  $X$  if for each pair  $(x_1, x_2)$  and  $(y_1, y_2)$ , with  $x_1 \neq x_2$  and  $y_1 \neq y_2$  of pairs of points of  $X$ , there exists a  $g \in G$  such that  $gx_1 = y_1$  and  $gx_2 = y_2$ . Show that if a finite  $G$  acts non-trivially and doubly transitively on  $X$ , then the stabilizer  $S_x$  of any point  $x$  in  $X$  is a maximal proper subgroup of  $G$ . (Here a subgroup  $M$  of  $G$  is *maximal proper* if  $M$  is not equal to  $G$  and if, for any subgroup  $H$  of  $G$  which contains  $M$ , either  $H = M$  or  $H = G$  holds. )

**Rings**

- R1. Let  $F$  be a field and  $A$  be a commutative  $F$ -algebra. Suppose  $A$  is of finite dimension as a vector space of  $F$ .
  - (a) Prove that if  $A$  is a domain,  $A$  is a field.
  - (b) Prove that even if  $A$  is not a domain, there are only finitely many prime ideals of  $A$ .
- R2. Let  $A$  be a commutative ring with identity, and write  $V$  for the set of all prime ideals of  $A$ . Put  $D(x) = \{P \in V \mid x \notin P\}$  for  $x \in A$ . Prove
  - (a)  $D(a) = D(a^n)$  for integers  $n > 0$ ;
  - (b)  $V = D(a) \cup D(b) \cup D(c)$  if  $a^3 + b^3 + c^3$  is invertible in  $A$ .
- R3. Determine all isomorphism classes of modules over the polynomial ring  $\mathbb{F}_2[X]$  which are of dimension 2 over  $\mathbb{F}_2$ , and justify your answer. Here  $\mathbb{F}_2$  is a field of two elements.

### Fields

- F1. Let  $F$  be a field. Show that the unit group  $F \setminus \{0\}$  of  $F$  is finitely generated if and only if  $F$  is finite.
- F2. Let  $f(x)$  be the polynomial  $x^4 - 2x^2 - 2$  over  $\mathbb{Q}$  and  $K$  be a splitting field of  $f(x)$ . Determine the Galois group  $\text{Gal}(K/\mathbb{Q})$ , and find the number of Galois extensions of  $\mathbb{Q}$  inside  $K$ . Prove your answer.
- F3. Let  $f(x)$  be an irreducible polynomial over the field  $F$  and let  $K/F$  be a finite extension.
- (a) Define what it means for the extension  $K/F$  to be *normal*.
  - (b) Show that if  $K$  is normal over  $F$ , then, in  $K[X]$ ,  $f(x)$  factors into a product of irreducible polynomials of the same degree.
  - (c) Show by example that this result does not hold for  $K$  not normal.

### Linear Algebra

- LA1. Let  $A$  be an  $N$  by  $N$  matrix with entries in  $\mathbb{C}$ .
- (i) Let  $g$  be an invertible  $N$  by  $N$  matrix. Show that  $\lim_{n \rightarrow \infty} A^n = 0$  if and only if  $\lim_{n \rightarrow \infty} (gAg^{-1})^n = 0$ .
  - (ii) Give necessary and sufficient conditions in terms of the conjugacy class of  $A$  only for  $\lim_{n \rightarrow \infty} A^n = 0$  to hold.
- Here, if  $A_n = (a_{n,(i,j)})$  is a sequence of  $N$  by  $N$  matrices with entries  $a_{n,(i,j)}$ ,  $\lim_{n \rightarrow \infty} A_n = 0$  if and only if  $\lim_{n \rightarrow \infty} a_{n,(i,j)} = 0$  for all  $i$  and  $j$ .
- LA2. Let  $A$  be a 3 by 3 matrix with complex entries. Suppose that  $A$  satisfies the relation  $A^2 + A + I_3 = 0$ , where  $I_3$  denotes the 3 by 3 identity matrix.
- (i) List the possible Jordan normal forms of  $A$ .
  - (ii) Suppose  $A$  has entries in  $\mathbb{R}$ . List the possible Jordan normal forms of  $A$ .
- LA3. Let  $F$  be a field of characteristic 0 and let  $Q$  be an invertible  $n$  by  $n$  symmetric matrix with entries in  $F$ .
- (i) Show that there exists an invertible matrix  $A$  such that  $AQA^t$  is diagonal.
  - (ii) Does the same remain true if  $Q$  is not invertible? Explain.