

## Algebra Qualifying Exam

Fall 2006

**Test Instructions:** Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest is ignored). For multiple part problems, do as many parts as you can; however, not all parts count equally.

### Groups

- G1. List all finite groups  $G$  whose automorphism group has prime order. Justify your answer.
- G2. Let  $G$  be a finite group and  $H$  be a non-normal subgroup of  $G$  of index  $n > 1$ .
- (a) Show that if  $|H|$  is divisible by a prime  $p \geq n$ , then  $H$  cannot be a simple group;
  - (b) Show that there is no simple group of order  $504 = 2^3 \cdot 3^2 \cdot 7$ . (Hint: Choose a good prime  $\ell$ , and let  $G$  act on the set of Sylow  $\ell$ -subgroups getting an embedding of  $G$  into a permutation group, and apply (a). You may use the fact that the alternating group  $A_n$  is simple if  $n \geq 5$ .)
- G3. Let

$$SL_2(\mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, ad - bc = 1 \right\},$$

where  $p$  is an odd prime.

- (a) Prove that any subgroup of a cyclic group is cyclic.
- (b) Compute the order of  $G$ .
- (c) Prove that for any odd prime  $\ell$ , the Sylow  $\ell$ -subgroup of  $G$  is cyclic. (Hint: You may use the fact that the multiplicative group  $F^\times$  of a finite field  $F$  is cyclic).

### Rings

- R1. Determine all prime ideals in the polynomial ring  $\mathbb{Z}[x]$ . Justify your work.
- R2. Let  $R$  be a noetherian domain. A non-zero element  $x$  in  $R$  is called a *prime element* if  $(x)$  is a prime ideal. Prove all of the following:
- (a) Every nonzero non-unit in  $R$  is a product of irreducible elements.
  - (b) Every nonzero ideal  $I \neq R$  in  $R$  contains a (finite) product of non-zero prime ideals.

- (c) If every nonzero prime ideal in  $R$  contains a prime element then every irreducible element in  $R$  is a prime element.

[You may not use theorems about UFD's]

- R3. Let  $R$  be a commutative ring and  $M$  a finitely generated  $R$ -module. Suppose there exists a positive integer  $n$  and a surjective  $R$ -module homomorphism  $\varphi : M \rightarrow R^n$ . Show that  $\ker \varphi$  is also a finitely generated  $R$ -module.

### Fields

- F1. Let  $F$  be a finite field of positive characteristic  $p$ . Show that the unit group  $F \setminus \{0\}$  of  $F$  is a cyclic group and that  $F$  is a Galois extension of  $\mathbb{Z}/p\mathbb{Z}$ .
- F2. Let  $f(x)$  be the polynomial  $x^6 + 3$  over  $\mathbb{Q}$  (the field of rational numbers). Determine the Galois group of  $f(x)$ , i.e., the Galois group of  $K/\mathbb{Q}$  where  $K$  is a splitting field of  $f(x)$ .
- F3. Let  $f(x)$  be an irreducible polynomial over  $F$  and  $K/F$  a normal extension. Show that  $f(x)$  factors into irreducible polynomials over  $K$  all of the same degree.

### Linear Algebra

- L1. Let  $V$  be an  $n$ -dimensional vector space over a field  $F$  (for finite  $n$ ) and  $\sigma : F \rightarrow F$  be a field homomorphism (sending 1 to 1). We regard  $V \otimes_{F,\sigma} F$  as an  $F$ -vector space via the  $F$ -multiplication given by  $a(v \otimes \alpha) = (av) \otimes \alpha$  for  $a, \alpha \in F$  and  $v \in V$ .
- (a) Compute the formula of  $\dim_F(V \otimes_{F,\sigma} F)$  if  $F$  is a finite extension of a field  $k$  fixed by  $\sigma$ , where  $V \otimes_{F,\sigma} F$  is the tensor product over  $F$  regarding  $F$  as  $F$ -module via  $\sigma$ .
- (b) Let  $p$  be a prime. If  $F = \mathbb{F}_p(x_1, \dots, x_m)$  ( $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ) and  $\sigma(\phi) = \phi^p$  for all  $\phi \in F$ , compute the formula of  $\dim_F(V \otimes_{F,\sigma} F)$ . Here  $\mathbb{F}_p(x_1, \dots, x_m)$  is the field of fractions of the polynomial ring  $\mathbb{F}_p[x_1, \dots, x_m]$  of  $m$  variables.
- (c) Give an example of a field  $F$  of characteristic 3 and a homomorphism  $\sigma : F \rightarrow F$  such that  $\dim_F V \otimes_{F,\sigma} F = 2 \dim_F V$ , and justify your example.
- L2. Let  $V$  be a finite dimensional vector space over the rational number field  $\mathbb{Q}$  and  $I : V \times V \rightarrow \mathbb{Q}$  be an alternating bilinear map (that is,  $I(x, y) = -I(y, x)$  for all  $x, y \in V$ ). We call  $I$  *degenerate* if there exists nonzero  $x \in V$  such that  $I(x, V) = 0$ , and  $I$  is *non-degenerate* if  $I$  is not degenerate.
- (a) Show that if  $V$  is two dimensional and  $I$  is non-degenerate then there exists a basis  $x, y$  such that  $I(x, y) = 1$ .

- (b) Show that  $I$  is degenerate if  $\dim_{\mathbb{Q}} V$  is odd.
  - (c) If  $\dim_{\mathbb{Q}} V = 2m$  is even and  $I$  and  $I'$  are two nondegenerate alternating forms on  $V$ , show that there exists an invertible linear transformation  $T : V \rightarrow V$  such that  $I'(x, y) = I(Tx, Ty)$  for all  $x, y \in V$ .
- L3. Let  $T : V \rightarrow V$  be a linear transformation on an  $n$ -dimensional vector space  $V$  over  $\mathbb{C}$  with  $n \geq 1$ . Suppose that  $T^n = 0$  but  $T^{n-1} \neq 0$ .
- (a) Compute  $\dim \text{Ker}(T^{101})$ .
  - (b) If  $S : V \rightarrow V$  is a linear transformation with  $ST = TS$  and  $\dim S(V) = n - 2$ , compute  $\dim S^{101}(V)$ .