Topological States of Matter

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Classification of MTC

Let $N_0=I$, N_1, N_2, …, N_{n-1} be $n\times n$ symmetric matrices with non-negative integer entries and commute with each other pairwise.

Let $S=(s_{ij})_{0\leq i,j\leq n-1}$ be a real, symmetric orthogonal matrix such that $SN_iS=D_i$, where $D_i=(s_{ia}/s_{0a}\delta_{a,b})_{0\leq a,b\leq n-1}$ is a diagonal matrix whose entries are all algebraic integers.

Fix n, find all such data

More Constraints

Given n, there exists an m and a diagonal matrix $T=(\theta_i \delta_{ij})_{0\leq i,j\leq n-1}$ such that

1. **$T^m=Id$**
2. **S, T leads to a rep ρ of SL(2,Z) into O(n)**
3. **ρ factors through $SL(2,Z_m)$**
4. **The diagonal matrices in $SL(2, Z_m)$ are sent to signed permutation matrices.**

Question: fix n, find all such data

Known for n=1,2,3,4 (Rowell, Stong, W.)

Maxwell's Mistake

E. H. Hall, 1879

On a new action of the magnet on electric currents

Am. J. Math. Vol 2, No.3, 287--292

"It must be carefully remembered that the mechanical force which urges a conductor carrying across the lines of the magnetic force, acts, not on the electric current, but on the conductor which carries it"

Maxwell, Electricity and Magnetism

Quantum Hall Effect

- **1980 von Klitzing ---IQHE**

  (1985 Nobel)
- **1982 Stormer, Tsui, Gossard---FQHE**

  Laughlin (1998 Nobel)

  quasi-particle with 1/3 electron charge

  and braiding statistics (abelian anyons)

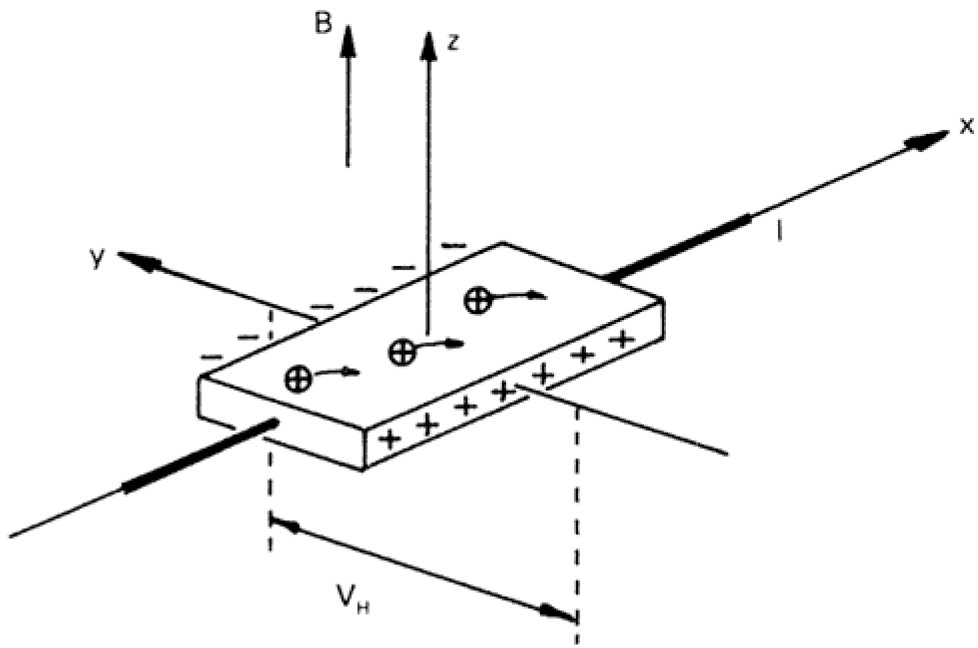
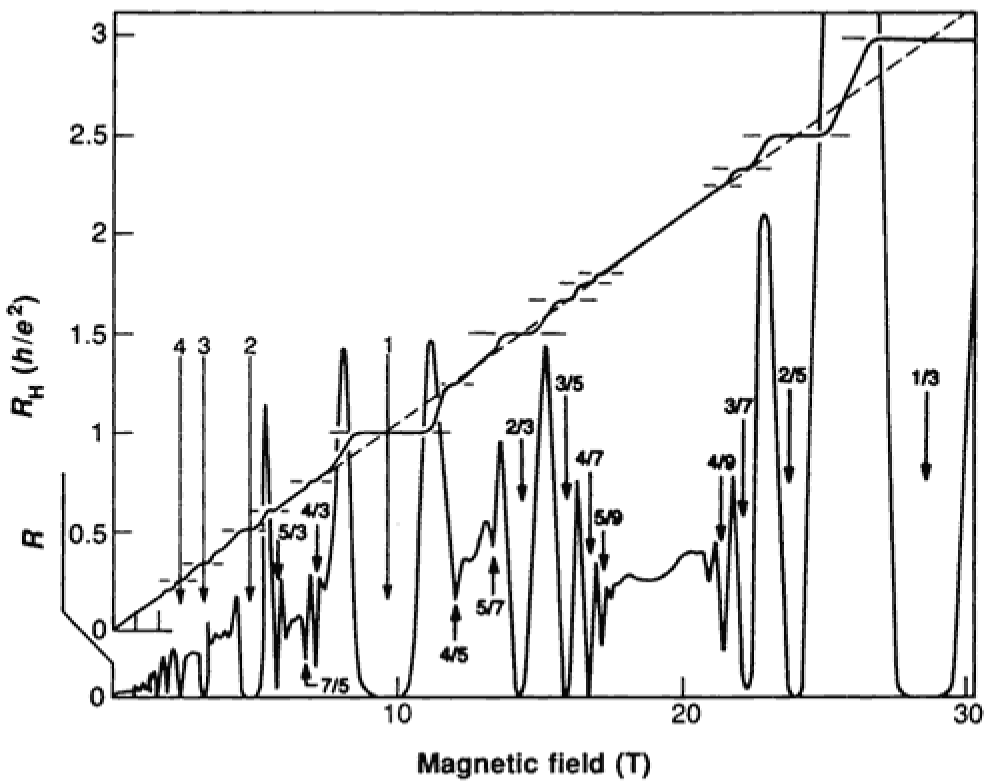


Fig. 1. A voltage V drives a current I in the positive x direction. Normal Ohmic resistance is V / I. A magnetic field in the positive z direction shifts positive charge carriers in the negative y direction. This generates a Hall potential (VH) and a Hall resistance (VH/ I) in the y direction. (Nobel Press Release 1998)



Hall resistance $R_{xy} = \nu^{-1}\, h/e^2$, ν an integer or a fraction with precision 10^{-10} (ν is the Landau filling fraction)

Motivating Questions

- How the bulk of electron liquids are modeled and how to classify them?
- What are the theories for boundaries and how to classify?
- What is the relation between bulk and boundary theories?

How to Model FQH liquids

Given an electron liquid on a closed surface Σ, then the ground states will be a Hilbert space $V(\Sigma)$.

An energy gap protects the ground states (if controlled below the gap). The Hamiltonian is 0, there will be no continuous evolutions.

This is the best place for quantum memory. To process the encoded information, we have to change the topologies.

Atiyah's Axioms of (2+1)-TQFT:

(i.e. a TQFT without excitations and central charge=0 or anomaly free)

Surface Σ^2 → $\mathbb{C}$-vector space $V(\Sigma)$

3-manifold M^3 → a vector $Z(M^3) \in V(\partial M^3)$

- $V(\emptyset) \cong \mathbb{C}$
- $V(\Sigma_1 \sqcup \Sigma_2) \cong V(\Sigma_1) \otimes V(\Sigma_2)$
- $V(\Sigma^*) \cong V^*(\Sigma)$
- $Z(\Sigma \times I) = Id_{V(\Sigma)}$
- $Z(M_1 \cup M_2) = Z(M_1) \cdot Z(M_2)$

Elementary Excitations

Elementary excitations are particle-like in electron liquids, and carry topological charges---particle types. They are modeled by a ribbon tensor category.

A ribbon tensor category with non-singular S-matrix is called a modular tensor category (MTC).

MTCs=TQFTs (Turaev)

TQFT can be extended to include surfaces with boundaries whose boundaries are labeled by particles types=simple object classes.

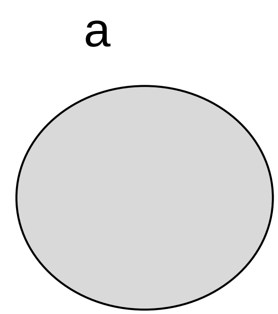
Walker, Segal, Moore-Seiberg,…

TQFT with corners

Let L={a,b,c,…} be the particle types (labels), $a \rightarrow a^*$, and $a^{**}=a$, 0=trivial type

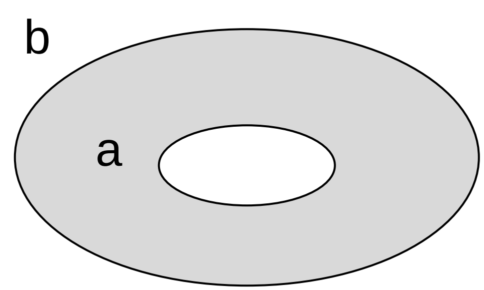
Disk Axiom:

$V(D^2, a)=0$ if $a \neq 0$, $\mathbb{C}$ if $a=0$



Annulus Axiom:

$V(A,a,b)=0$ if $a \neq b^*$, $\mathbb{C}$ if $a=b^*$

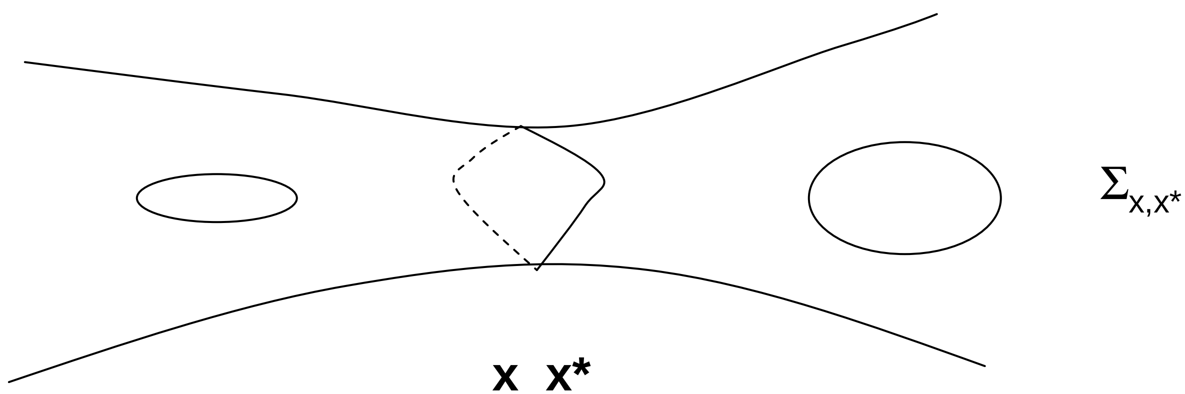


Gluing Axiom

Locality in the theory is encoded in the cutting and gluing of surfaces.

The Gluing Formula:

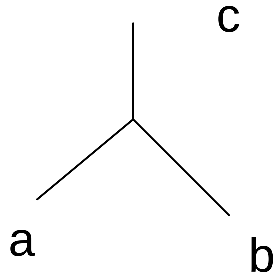
$$V(\Sigma) \cong \bigoplus_{x \in L} V(\Sigma_{x,x^*})$$

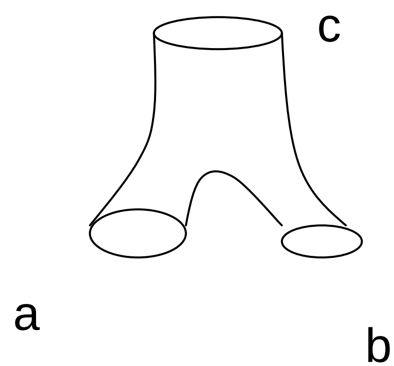


SU(2)-Chern-Simons Theory

Fix a level k=r-2, label set L={0,1,…r-2}

Set $V_{a,b,c}=V(P_{a,b,c})$. Then a basis of $V_{a,b,c}$ is





V(Σ) can be obtained from the disk axiom, annulus axiom, and this data by the gluing formula for any Σ

Simulation of TQFTs by QCM

Given a TQFT, set $X=\oplus_{a,b,c}V_{a,b,c}\cong \mathbb{C}^p$,

By the gluing axiom, we have:

$$\begin{array}{ccc} V(\Sigma) & \to & (\mathbb{C}^p)^{\otimes m} \\ \rho(f)\downarrow & & \downarrow U_L\ (?) \\ V(\Sigma) & \to & (\mathbb{C}^p)^{\otimes m} \end{array} \qquad f: \Sigma \to \Sigma$$

Studying pants decompositions by Hatcher-Thurston theory leads to an efficient simulation.

Model Hamiltonian

Given a TQFT, and a triangulation of a surface Σ, can $V(\Sigma)$ be constructed as the ground states of a local Hamiltonian on $(\mathbb{C}^2)^{\otimes n}$, where n is the number of edges (or vertices or faces)?

Yes if the TQFT is a Drinfeld double.

Should be Yes for all ???

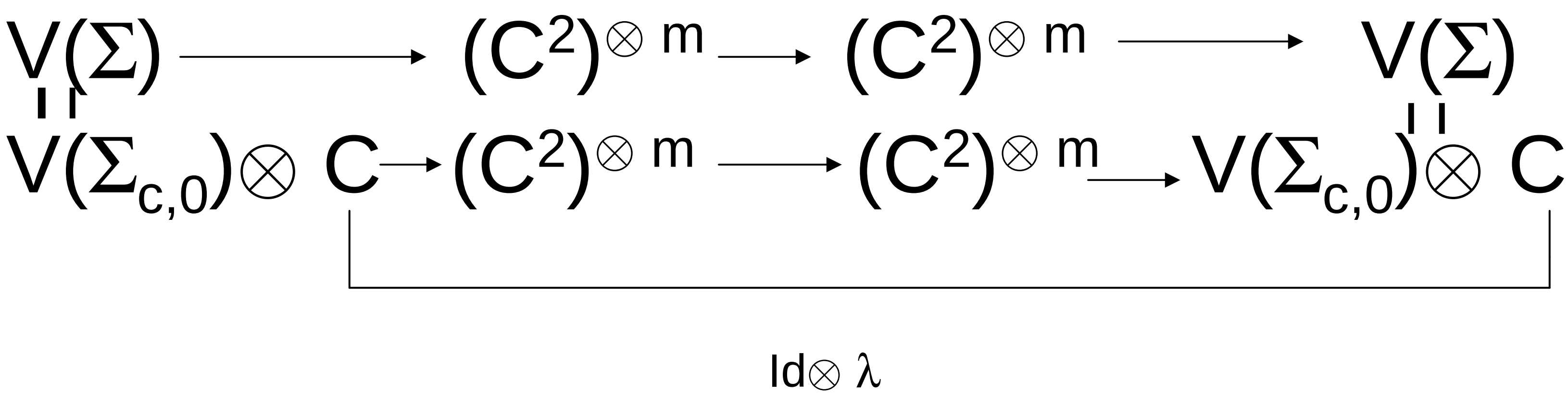
Fault tolerance of TQFTs

A pair $(V, (\mathbb{C}^2)^{\otimes n})$ is a (k, n)-code if for every k-local operator O_k, the following composition is a scalar multiple of id_V:

$$V \to (\mathbb{C}^2)^{\otimes n} \xrightarrow{O_k} (\mathbb{C}^2)^{\otimes n} \to V$$

Disk Axiom=Fault tolerance

Given a TQFT, a triangulated surface (Σ,Δ), and a model Hamiltonian for $V(\Sigma)$. Let k=disk radius (any $\leq$ k edges are inside disks), then $V(\Sigma)$ is a k-code.



Conjecture:

Fix the number of particle types, there are essentially only finitely many TQFTs.

Analogues:

1. (E. Landau) Finitely many finite groups with a fixed number of irreps
2. (L. Bieberbach) Finitely crytallographic groups in each dimension n, n=3 230 crytals

Axioms for Ribbon Category

Underlying categories of $\mathcal{C}$ will always be isomorphic to a category of a finite-set indexed vector spaces:

Objects: 2-vectors $(V_0, V_1, \ldots, V_m)$
Morphisms: f.d. vector spaces $\oplus_i \mathrm{Hom}(V_i, W_i)$

An object X is simple if Mor(X,X)=C

Tensor Category

Given a $\mathfrak{C}$, a tensor product on $\mathfrak{C}$ is a bi-functor $\otimes$:$\mathfrak{C}\times\mathfrak{C}\rightarrow \mathfrak{C}$ such that

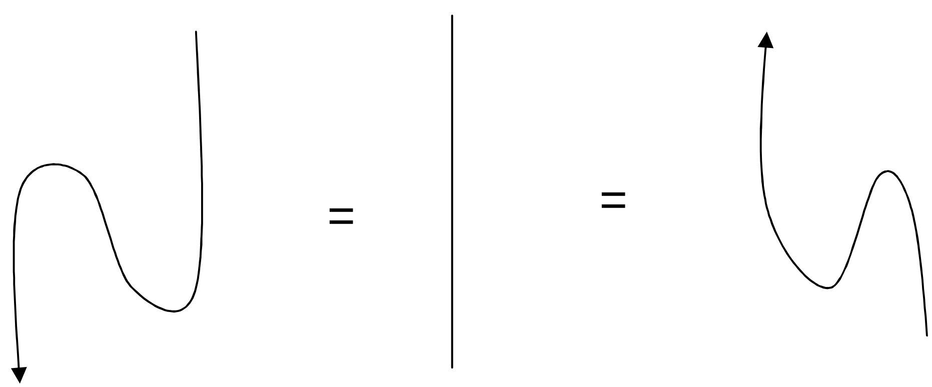
1. **For any U,V,W, there exists a natural isomorphism $\alpha_{U,V,W}$: $(U\otimes V)\otimes W\rightarrow U\otimes(V\otimes W)$**
2. **There exists a unit, denoted by 1,**

satisfying the pentagon and the triangle equations.

Rigidity

Given X in $\mathcal{C}$, X* is a right dual of X if there exist b_X: $1 \to X \otimes X^*$

and d_X: $X^* \otimes X \to 1$ such that

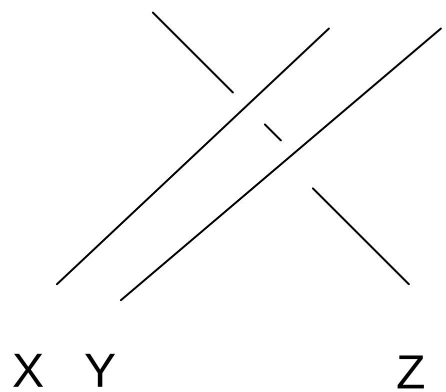


Braidings

For any X,Y, there exists a natural iso:

$c_{X,Y}$: $X\otimes Y \rightarrow Y\otimes X$ such that

$c_{X\otimes Y,Z}=id_X\otimes c_{Y,Z}\cdot c_{X,Z}\otimes id_Y$

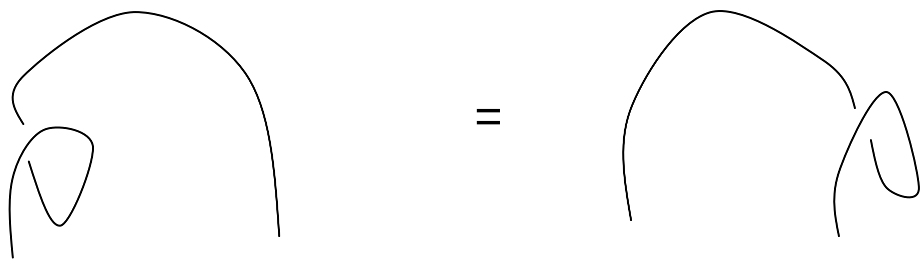


Similarly for inverse

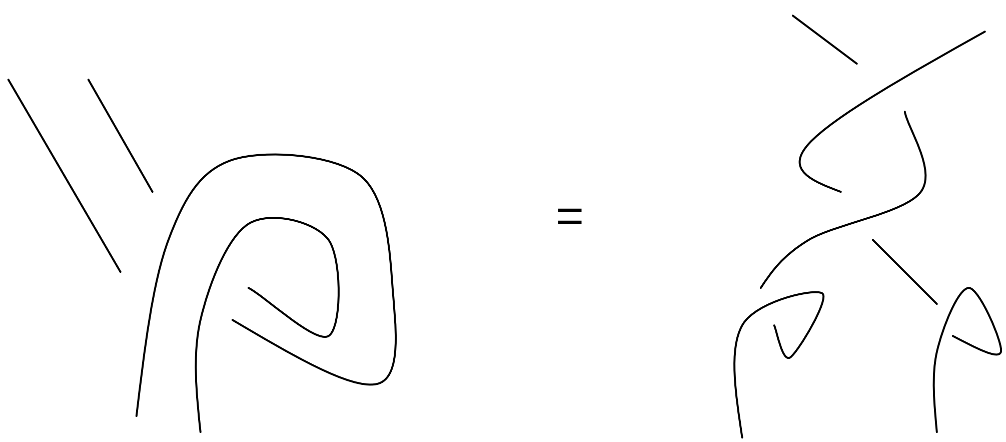
Twists

For any X, there is a natural isomorphism θ_X: X$\rightarrow$ X such that

1.



2.



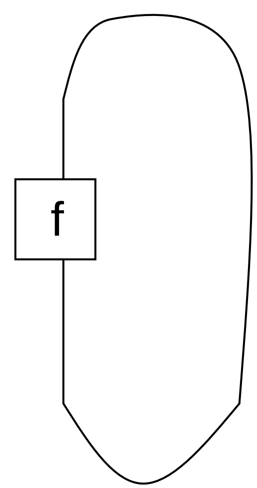
Ribbon Category

A ribbon category is a rigid, braided tensor category with twists.

We have X**=X, a trace on Mor(X,X):

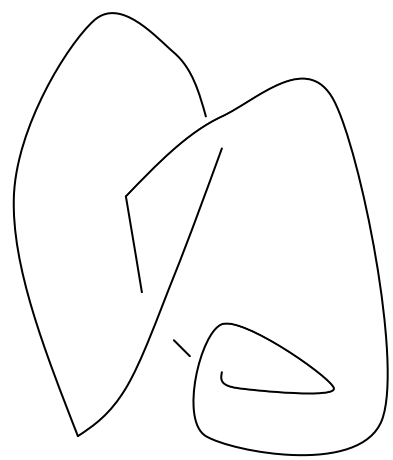
f$\in$ Mor(X,X),   Tr(f)$\in$ Mor(1,1)=C

Tr(f)=



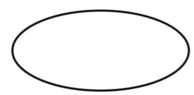
Link Invariant

Each ribbon category defines invariant for links, tangles, and they are quantum amplitudes for certain physical processes.

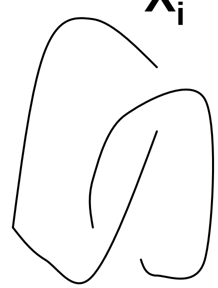


Modularity

Let $\{X_i\}$ be representatives of simple types

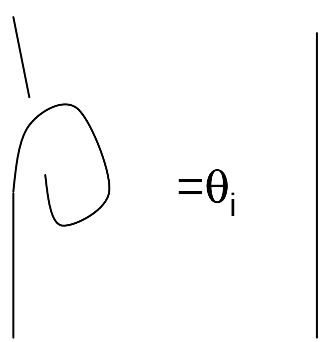
$d_i=$ 

Define $D^2=\sum_i d_i^2$

Then $s_{ij}=1/D$ 

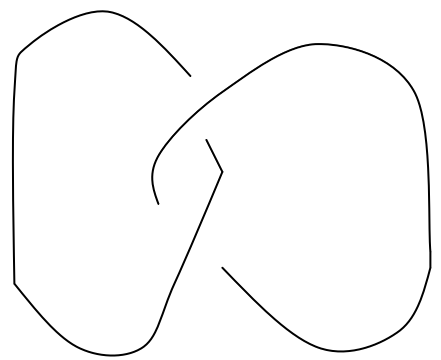
$S=(s_{ij})$ is the S-matrix and is symmetric, if it is non-singular then it is unitary.

Rep of SL(2,Z)



X_i

$s_{ij} = 1/D$



s→ S=(s_{ij})

Projective

t→ T=($\theta_i \delta_{ij}$)

Classification of MTCs

How to parameterize a MTC?

- Fusion rules: $\{N_{a,b}^{c}\}$, $a \otimes b = \oplus N_{a,b}^{c}\, c$

  $\mathbf{N_{a,b}^{c} = \dim V_{a,b,c^*}}$

- S-matrix: a symmetric unitary matrix

- T-matrix: always finite order (Vafa's thm)

Ocneanu Rigidity

Fix the fusion rule of a rigid tensor category (fusion category), then there are only finitely many isomorphism classes of such categories.

So the finiteness conjecture is reduced to the finiteness of fusion rules when the number of simple types is fixed.

Verlinder Formulae

Let $N_i=(N_{ij}^k)$, then N_i can be simultaneously diagonalized by the S-matrix. Furthermore, the eigenvalues of N_i are given by s_{ia}/s_{0a}, i.e. if

$D_i=(s_{ia}/s_{0a}\delta_{ab})$

Then $N_i=SD_i\,S$

Write out the (jk)-entry of $N_{i,}$ Verlinder formulas:

$$N_{ij}^k=\sum_{l=0}^{n-1} s_{il}s_{jl}s_{kl}/s_{0l}.$$

Hence finiteness conjecture is reduced to the finiteness of S-matrices.

Rep of SL(2, Z)

“Theorem“

The image of a rep of SL(2, Z) from a MTC is always a finite group, and the kernel is a always congruence subgroup of SL(2, Z).

Fix the number of simple types, there are only finitely many reps of SL(2, Z) from MTCs.

Galois Theory

Given a modular tensor category with twists θ_i, S-matrix s_{ij}, and global dimension=D

Consider the number field $L=Q(\theta_i, s_{ij}, D)$, then

Gal(L/Q) is abelian.

Conjecture: If the number of simple types is n, then the degree $[Q(\theta_i, s_{ij}, D):Q]$ is bounded by a polynomial of n.

G-equivariant MTC

Given a finite group G, G-equivariant modular tensor categories are defined by Turaev, and have been used to study orbifold rational conformal theory.

This theory should be useful for both the classification of MTCs and multi-layer quantum systems.

Low Rank Classification: Fusion Rules

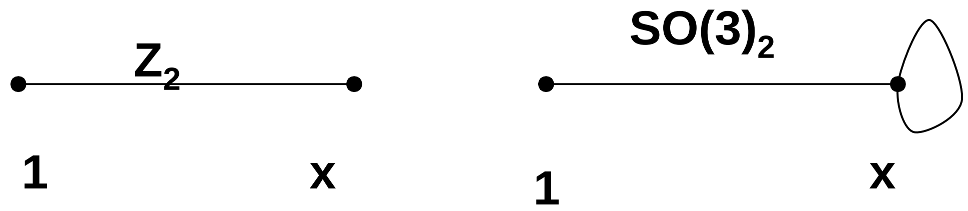
Consider self-dual (a=a*), indecomposable (not $C \otimes C$) modular tensor categories. If C is generated by a single object X, the principle graph is a connected graph G such that:

1. Each vertex is a simple type
2. For each vertex Y, $X \otimes Y = \oplus Z$, where Z is connected to Y by a single edge.

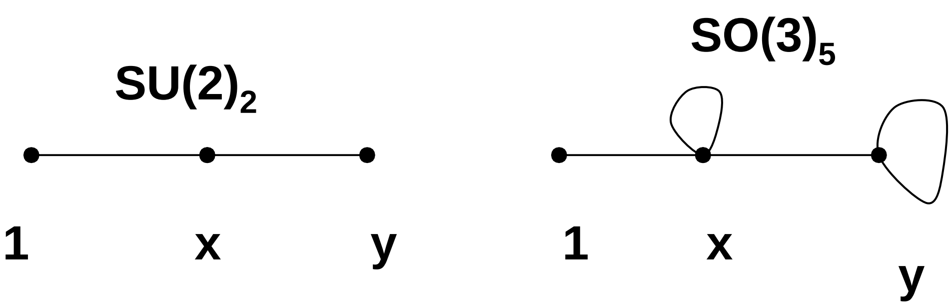
Ranks=1,2,3,4
Rowell, Stong, W.

Rank=1: •

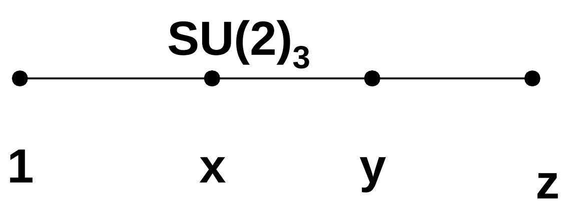
Rank=2:



Rank=3:

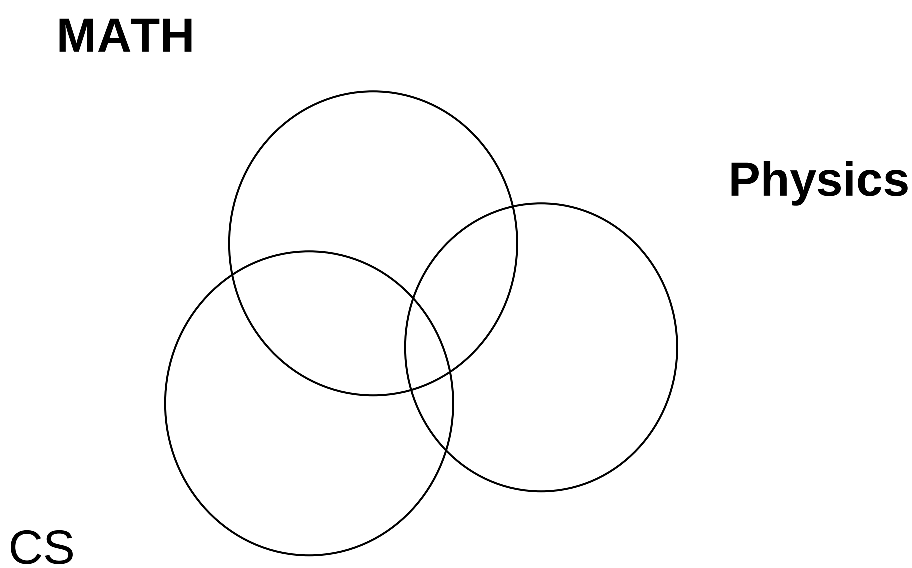


Rank=4



I consider it quite possible that physics cannot be based on the field concept, that is, on continuous structures. Then nothing remains of my entire castle in the air, including the theory of gravitation, but also nothing of the rest of modern physics

Einstein, 1954



Thanks and Good Luck!