Anyonic Quantum Computing

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A Story of Anyons

In a kingdom far far away, there are delicate peaceful **little 2D beings, named anyons by Wilczek.**

They emerged from some underlying "vacuum" with an energy gap. Each anyon carries a locally conserved type and has an anti-particle (dual type).

They can be fused and braided consistently, and can be created only in pairs.

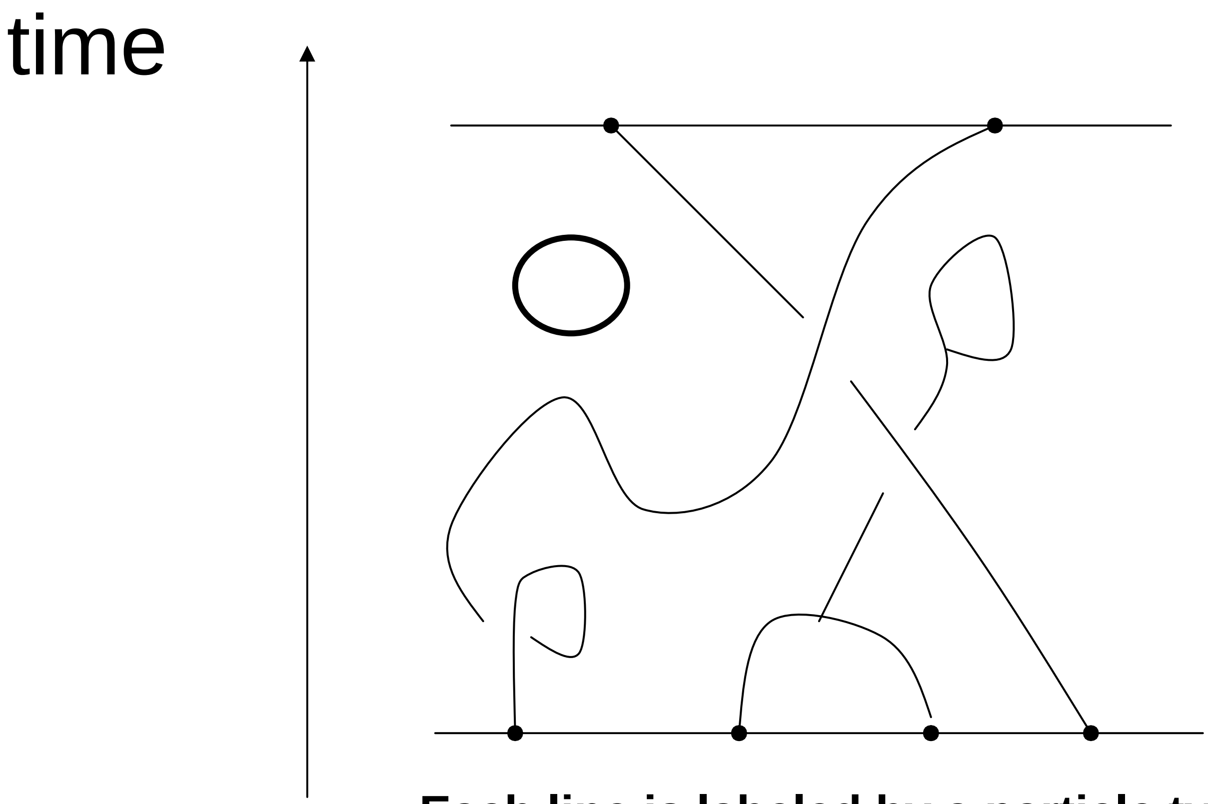
Some kingdoms are governed by the Chern-Simons theory whose cousin is named rational conformal field theory, and are employed by Microsoft to do quantum computing.

Anyons are quasi-particles (=particles) which are neither fermions nor bosons, and exist in the fractional quantum Hall liquids, and can carry fractional charges like e/3.

1. **Jones anyonic systems (SU(2)-CS)?**

2. **How to define anyonic quantum computers and simulate the quantum circuit model (QCM) efficiently?**

Trajectories of Anyons



Each line is labeled by a particle type
Move forward=particle, move backward=anti-partilce
Special trajectories are braids and links.

Operator Valued Invariant

At time t_0, t_1, the ground states of n anyons with fixed positions are Hilbert spaces

The movements of anyons induce linear maps from V_0 to V_1. This is an operator valued invariant of the trajectories. Restricting to the braid groups leads to braid groups rep. We usually also consider formal linear sums of trajectories.

Jones-Kauffman SU(2)-Theory

Pictorial formulation of the Jones/SU(2)-Chern-Simons theory by Kauffman (different for some levels).

Fix $r \geq 3$, and $k=r-2$, called the level

The particle types are $L=\{0,1,\ldots,r-2\}$ and each is its own dual, $a^*=a$

Fusion rules:

$$a \otimes b = \oplus\, c \text{ such that}$$

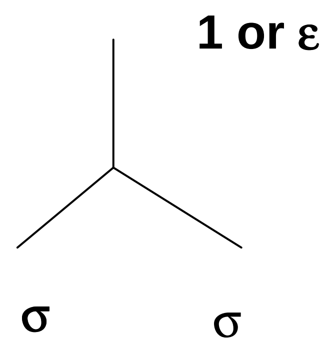
1) $a+b+c$ even 2) $a+b \geq c$, $b+c \geq a$, $c+a \geq b$

3) $a+b+c \leq 2(r-2)$

Level k=r-2=2 (FQHE at ν=5/2)

Particle types are {0,1,2} or {1,σ,ε}

Fusion rules:



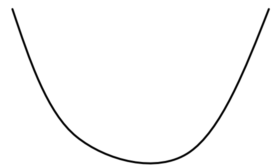
$\sigma^2 \cong 1+\varepsilon$, $\varepsilon^2 \cong 1$, $\sigma\varepsilon \cong \varepsilon\sigma \cong \sigma$

Moore-Read Conjecture:

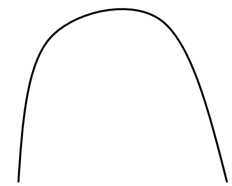
1. **σ and ε arise as quasi-particles in ν=5/2 FQH liquids and σ has charge e/4**
2. **σ is a non-abelian anyon**

Kauffman Bracket

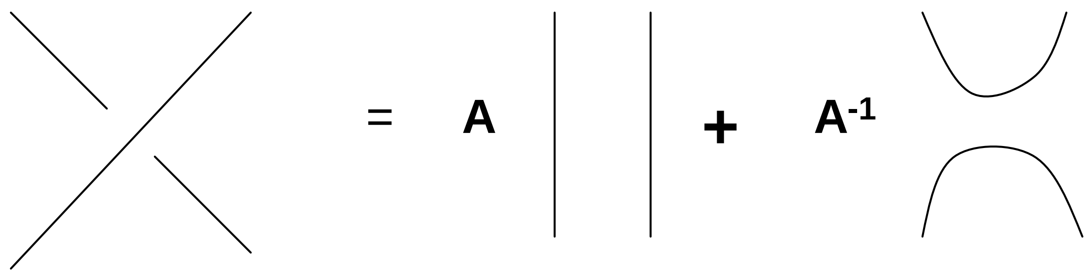
Creation



Annihilation



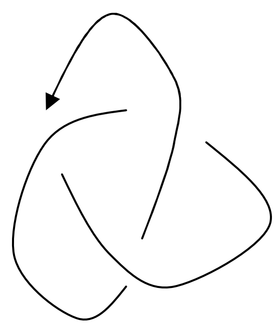
Kauffman Bracket

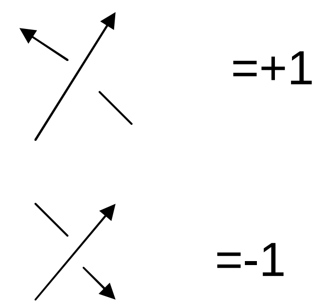


all curves are labeled by 1 if not labeled, $d=-A^2-A^{-2}$

Jones polynomial of knots

Knots: trajectories of single particles

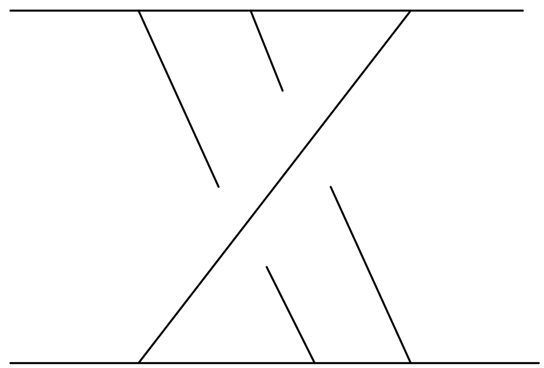




Resolve each crossing using the Kauffman bracket (2^n terms if n is the number of crossings), each term is a collection of simple loops, and set it to $d^{\#\text{ of loops}}$. The sum is the Jones polynomial if multiplied by $A^{-3\text{ writhe}}$

Jones Rep of Braid Groups

Given a braid:



Resolve each crossing to get 2^n terms. Each term has n disjoint arcs connecting the top n and bottom n points multiplied by a power of A. This diagram is an element of the Temperley-Lieb algebra.

Temperley-Lieb algebra

$TL_n(d)$=the vector space spanned by n disjoint arcs in a rectangle connecting n points at the bottom and n points at the top up to isotopy. Define multiplication on diagrams as vertical stacking, and set any loop=d, $TL_n(d)$ is a semi-simple algebra.

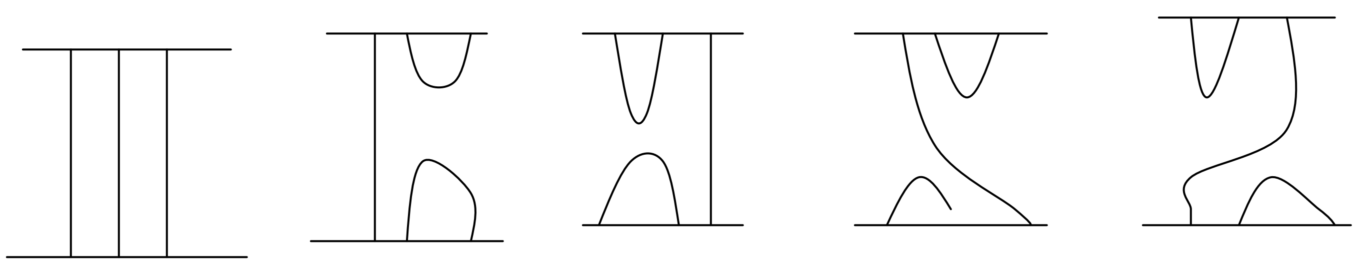
This leads to the generic Jones Representation

Not known faithful, but its generalization BMW is.

Examples

n=2:

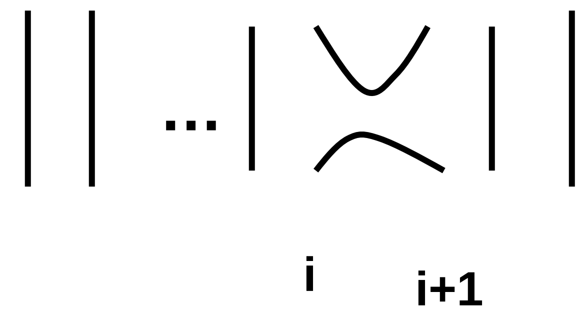
n=3

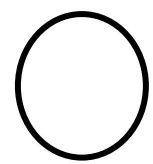


Temperley-Lieb Algebras

$TL_n(d)$ is the f.d. algebra generated by 1, $U_1,\dots,U_{n-1}$

$U_i=$



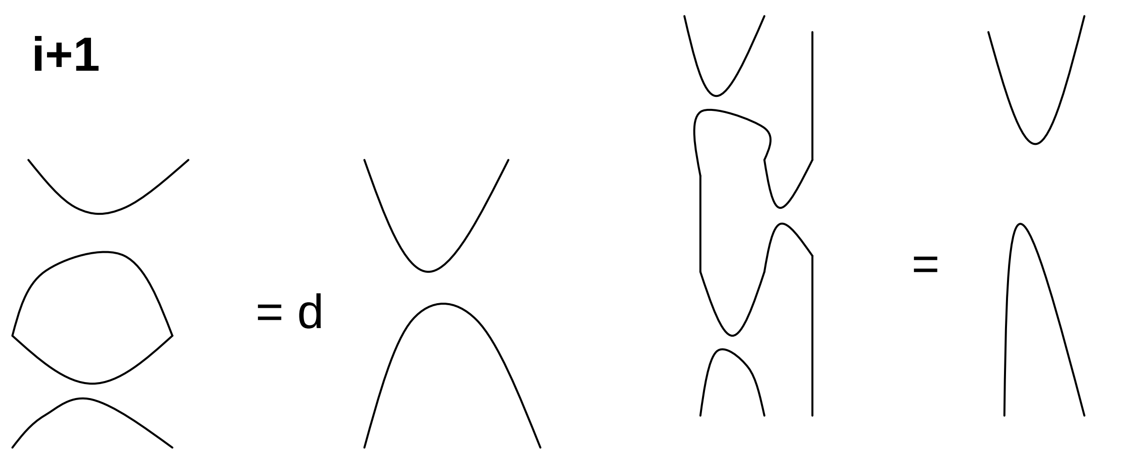


= **d**

$U_i^2=dU_i$

$U_i U_{i\pm 1} U_i=U_i$

$U_iU_j=U_jU_i \quad |i-j|\geq 2$



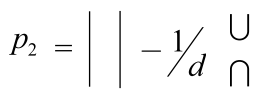
Jones-Wenzl Projectors

There exists a unique non-zero element p_n in $TL_n(d)$ such that

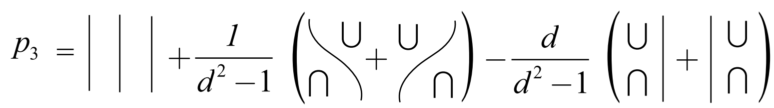
1. $p_n^2 = p_n$,

2. $p_n U_i = U_i p_n = 0$, $i = 1, 2, \ldots, n-1$

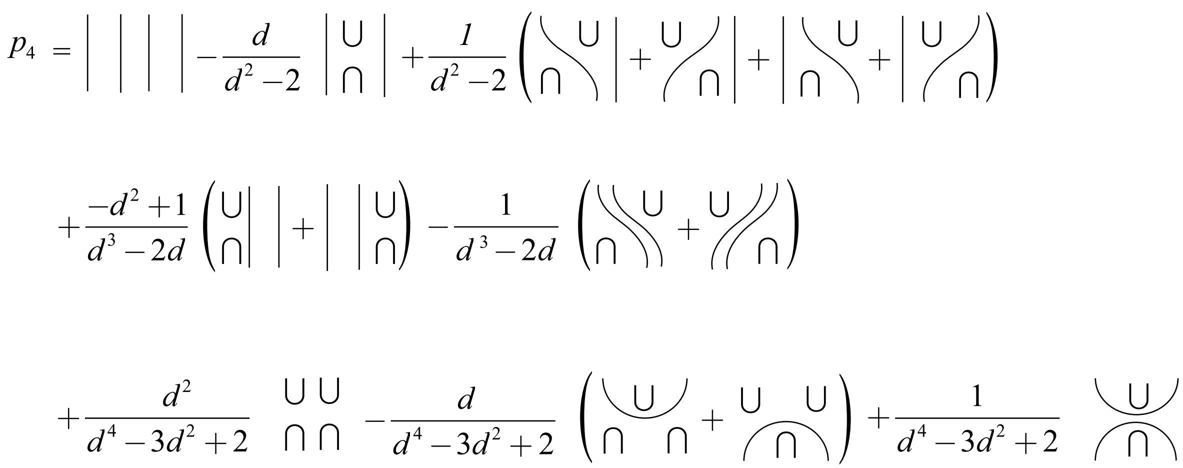
p_n is the Jones-Wenzl idempotent



, p_2 generates a proper radical for $d = 1, -1$;



p_3 genrerates aproprer radical for $d = \pm\sqrt{2}$, and $d = 0$;



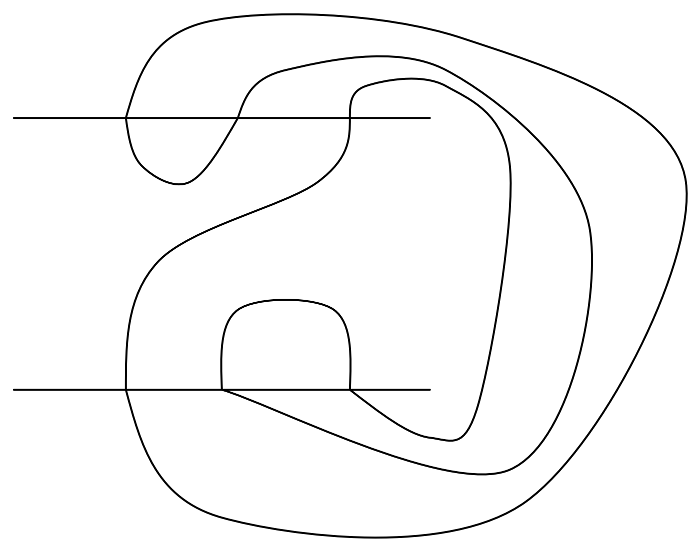
p_4 genrerates a proper radical for $d = \pm\left(\frac{1+\sqrt{5}}{2}\right), \pm\left(\frac{1-\sqrt{5}}{2}\right)$.

Markov Trace

The Markov trace: braid closure and $d^{\#\text{ loops}}$

$$TL_n(d) \xrightarrow{Tr} C(d)$$

$Tr(p_n)=\Delta_n$, Chebyshev polynomial



UNITARITY

To be relevant to quantum physics, and quantum computing, we need a unitary theory. So we will specialize d to complex numbers. It turns out only when $d=2\cos\pi/r$ that we will get a unitary theory which requires $q=A^4$ to be ONLY $e^{\pm 2\pi i/r}$.

This is also exactly the values that Witten explained using TQFTs.

Jones algebras

Let d be a non-zero complex number

- **If $d = -A^{2} - A^{-2}$ where A is not some root of unity, then $TL_n(d)$ is still semi-simple.**
- **If $d = -A^{2} - A^{-2}$ for some root of unity A, then $TL_n(d)$ is not semi-simple. If A is a primitive 4rth root of unity, then $J_n(A) =: TL_n(d)/(p_{r-1})$ is semi-simple---Jones algebras.**
- **When r=4, Jones algebras=Clifford algebras**

Jones Rep of Braid Groups

Jones algebras are f.d. semi-simple algebras, so decompose into a direct sum of matrix algebras.

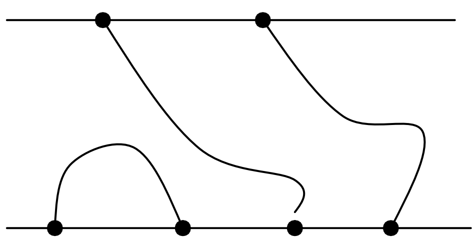
Kauffman Bracket

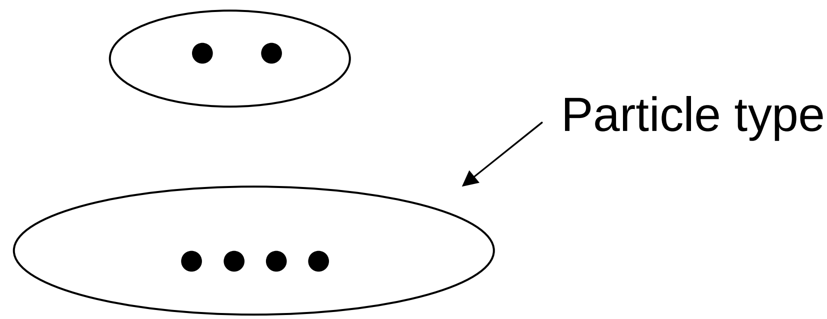
$$C[B_n] \longrightarrow J_n(d)$$

Kauffman bracket is an algebra homomorphism.

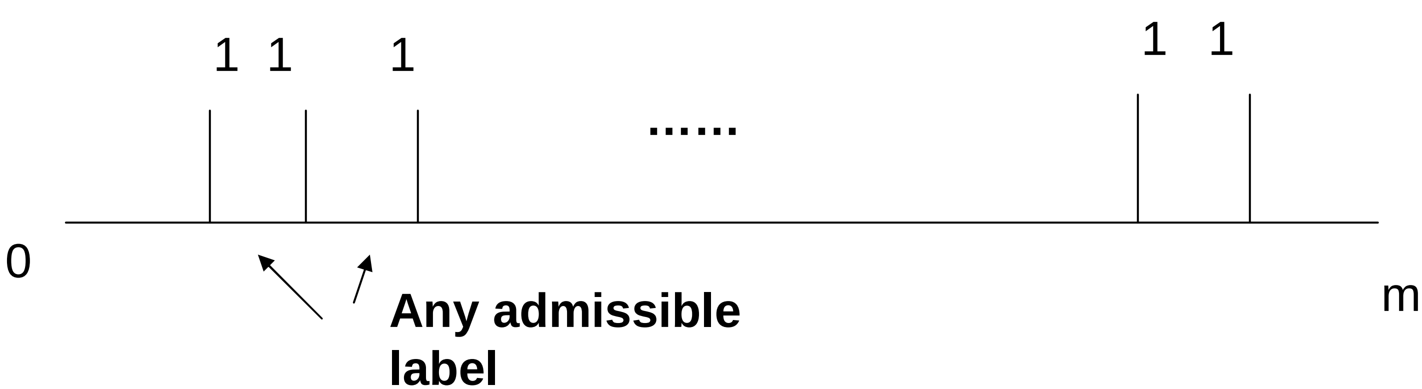
How to find this decomposition into matrix algebras?

Given n particles in a disk at fixed positions, there will be a Hilbert space to describe the ground states of this quantum system. Any TL algebra element induces a linear map from this Hilbert space to the later Hilbert spaces. How to find a basis for such Hilbert spaces?





Fix a level k=r-2, a basis for the Hilbert space of n internal particles labeled by 1, and the outside boundary labeled by m can be chosen as follows:

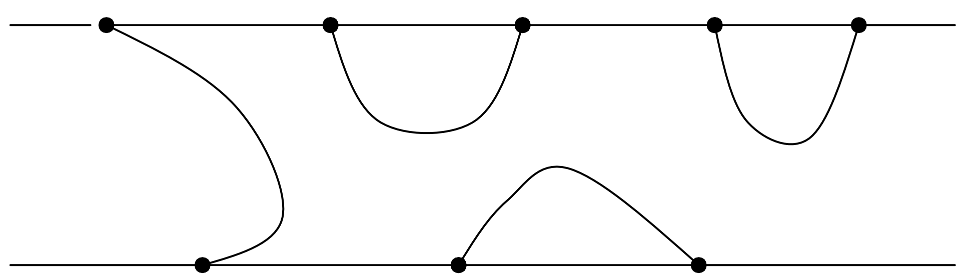


TL Categories

$\{TL_n(d)\}$ can be generalized to a category

Objects: particles in an interval

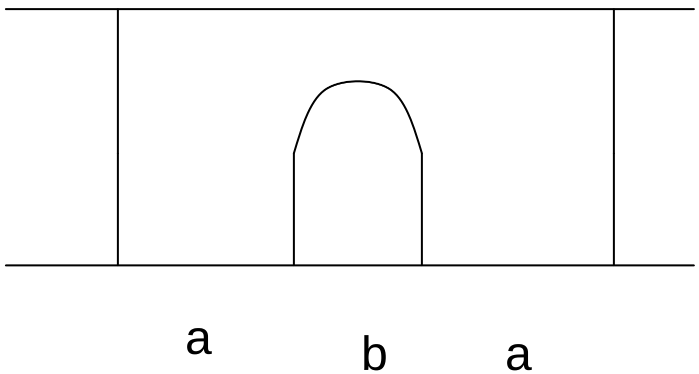
Morphisms: Disjoint arcs connecting objects



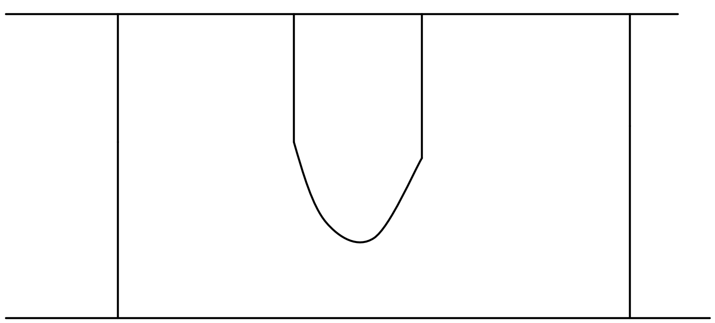
and their formal sums. Each basis diagram is a composition of creations/annihilations and identities.

To get a rep of the TL category, it suffices to have a rep of the creation and annihilation operators.

On basis, let $x_{b,a}$, $y_{b,a}$ be the matrix entries for the creation/annihilation



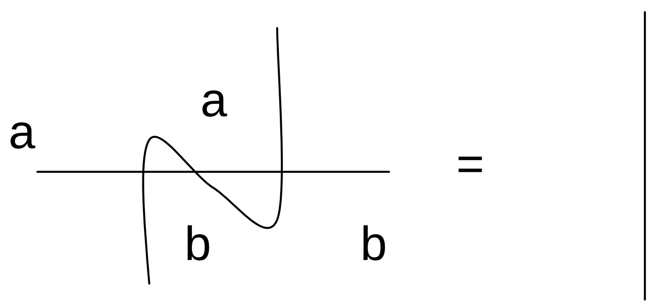
$=x_{a,b}$



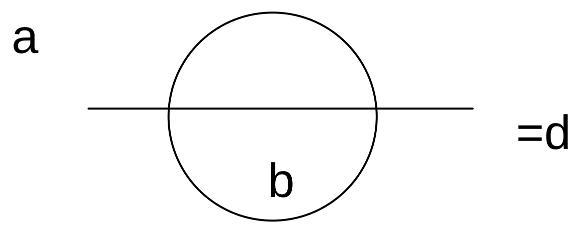
$=y_{a,b}$

Constrants:

d-isotopy invariant:



$$\mathbf{y_{b,a}x_{a,b}=1}$$



$$\mathbf{\sum_b y_{a,b}x_{a,b}=d}$$

Solutions

By fusion rules, b=a$\pm$ 1

Suffice to solve:

$$x_{a,a+1}/x_{a+1,a}+x_{a,a-1}/x_{a-1,a}=d$$

a=0: $x_{0,1}/x_{1,0}=d=\Delta_1$, $\Delta_0=1$

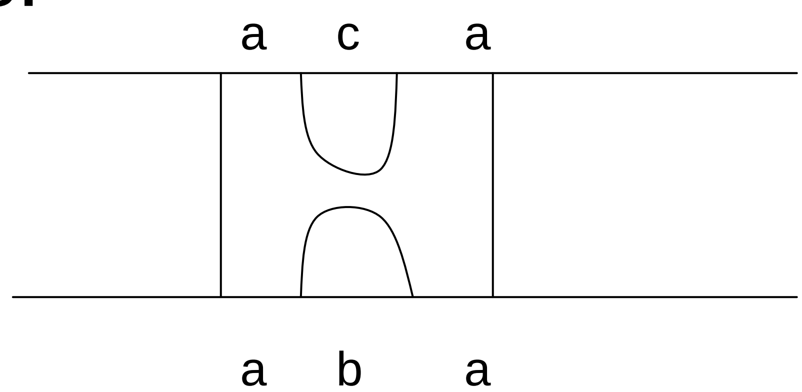
(Chebyshev Polys) $\Delta_n=d\Delta_{n-1}-\Delta_{n-2}$

a=1: $x_{1,2}/x_{2,1}=d-1/d=\Delta_2/\Delta_1$

Claim: $x_{a,a+1}/x_{a+1,a}=\Delta_{a+1}/\Delta_a$

Rep Matrix of U_i

On basis:



a=0, b=1, then U_i=d on this basis.

a$\neq$ 0, then U_i are 2$\times$ 2 blocks

2×2 Block

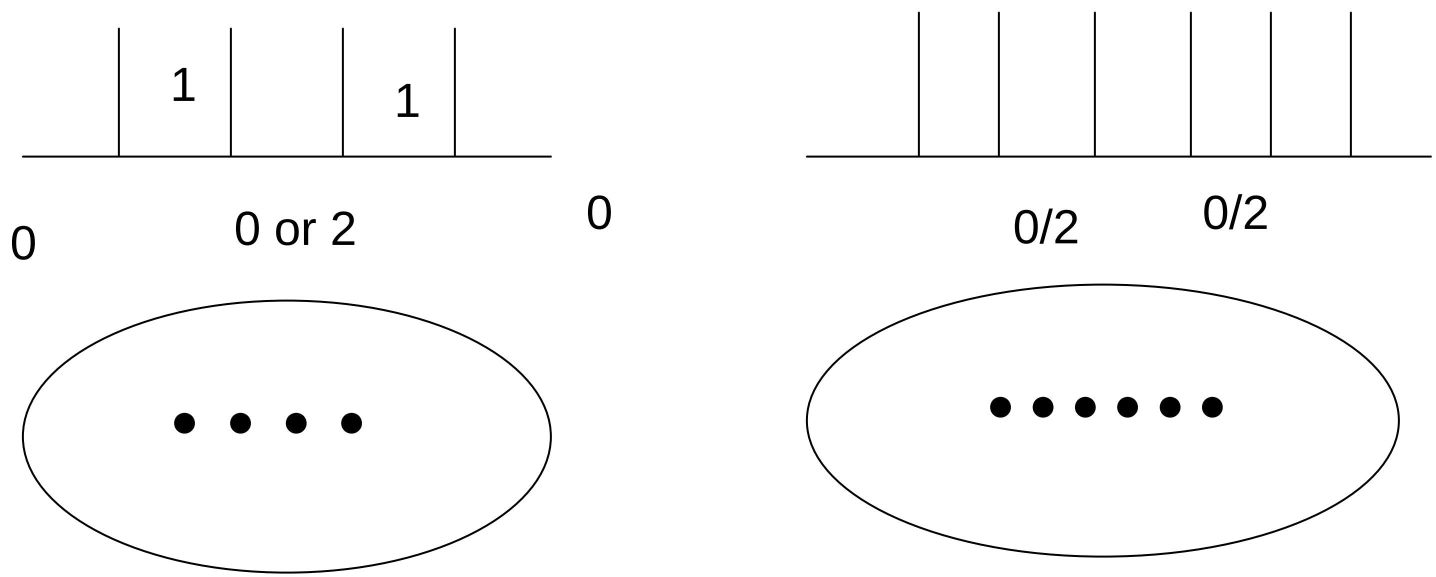
The 2×2 block is

$$\begin{pmatrix} x_{a,a+1}/x_{a+1,a} & x_{a,a+1}/x_{a-1,a} \\ x_{a,a-1}/x_{a+1,a} & x_{a,a-1}/x_{a-1,a} \end{pmatrix}$$

To be unitary, this block needs to be normalized to a real, symmetric projector which requires A to satisfy $q=A^4=e^{\pm 2\pi i/r}$

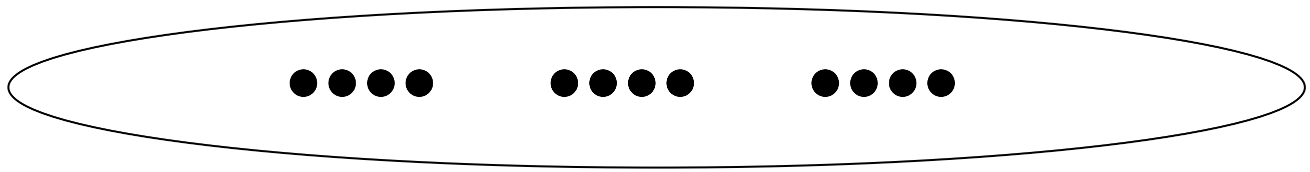
Anyonic Quantum Computer based on CS level=2

The ground states of 4 σ's in a disk is C^2---a qubit.
6 σ's in a disk is C^4---2 qubits.
For 1-qubit gates, ρ_{CS4}: $B_4 \to U(2)$
For 2-qubits gates, ρ_{CS4}: $B_6 \to U(4)$



For n qubits, consider the 2n+2 punctured disk D_{2n+2} and

$$\rho_{CS4}: B_{2n+2} \to U(2^n)$$



Given a quantum circuit on n qubits:

$$U_L: (C^2)^{\otimes n} \to (C^2)^{\otimes n}$$

Ideally to find a braid $b \in B_{2n+2}$ so that the following diagram commutes: Impossible for r=4, Almost r=5

$$\begin{array}{ccc} (C^2)^{\otimes n} & \to & V(D_{2n+2}) \\ U_L \downarrow & & \downarrow \rho_{CS4}(b) \\ (C^2)^{\otimes n} & \to & V(D_{2n+2}) \end{array}$$

In 1981, Jones proved that $\rho_{r,m}(B_n)$ is infinite

if $r \neq 1,2,3,4,6$, $n \geq 3$ or $r=10$, $n \geq 4$,
and asked:

What are the closed images of $\rho_{r,m}(B_n)$?

The Density Theorem (FLW):

Always contain SU if $r \neq 1,2,3,4,6$, $n \geq 3$ or $r=10$, $n \geq 4$.

Others are finite groups which can be identified.

Universality of Anyonic Quantum Computers:

To be universal, it suffices to approximate every unitary matrix up to a phase.

$$\rho_r(B_4) \supseteq PSU \text{ and } \rho_r(B_6) \supseteq PSU$$

True if $r \neq 1,2,3,4,6$ by the density theorem.

Then by Kitaev-Solvay Theorem that each unitary matrix can be approximated efficiently.

Properties of Jones Reps

1. $\rho_{CSr}(\sigma_i)$ **has two eigenvalues -1,q, where** $q=e^{\pm 2\pi i/r}$
2. **all** σ_i**'s are conjugate with each other**

Consider an irreducible sector $\rho_{r,m}$**, and let G=closure of** $\rho_{r,m}(B_n)$**, then G has a faithful rep V with the following two eigenvalue property.**

The Two Eigenvalue Problem

Two eigenvalue property:

(G,V) is a pair: V is a faithful irrep of G and G is compact Lie group generated by a single conjugacy class g with two distinct eigenvales whose ratio not +1 or -1. Find all such pairs.

Theorem (FLW): all such pairs can be classified

Recent Work: N eigenvalue problem (LRW)

Solutions to 2-eigenvalue Problem

(G,V) is a pair with the 2-eigevalue property. Let G_1 be the universal covering of $[G_0,G_0]$, where G_0 identity component of G.

If G is positive dimension modulo center, then (G_1, V) also a pair with the 2-eigenvalue property with highest weight ω, and (G_1, ω) is one of the following:

1) $(SU(l+1), \omega_i)$ for some $l \geq 1$, and $1 \leq i \leq l$
2) $(Spin(2l+1), \omega_l)$ for some $l \geq 2$
3) $(Sp(2l), \omega_1)$ for some $l \geq 3$
4) $(Spin(2l), \omega_i)$ for some $l \geq 4$ and $i=1,l-1,l$,

where ω_i is the i-th fundamental rep.

Other Applications

1. Distribution of Jones polynomial values at $q=e^{\pm 2\pi i/r}$. What are the induced measures on the plane?
2. Bordewich-Freedman-Lovasz-Welsh introduced a new approximate scheme inspired by the algorithms for approximating Jones polynomial. But their main goal is still open.

Quantum Algorithms

Approximation of Jones polynomial is a radically new quantum algorithm (No Fourier transform is used, see an exposition by Aharonov-Jones-Landau).

The interesting open question is:

Can we approximate other partition functions of the statistical mechanics models such as the Potts models?

How to Classify Anyon Kingdoms

Anyon systems are modeled by ribbon categories, which are rigid, braided, and balanced tensor categories. Ribbon category is the algebraic data for a topological quantum field theory. We will see how such tensor categories can be classified.

Conjecture:

Fixed the number of particle types, there are only finitely ribbon categories.