INNER MODELS AND ULTRAFILTERS IN $\mathrm{L}(\mathbb{R})$.

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Abstract. We present a characterization of supercompactness measures for ω_1 in $\mathrm{L}(\mathbb{R})$, and of countable products of such measures, using inner models. We give two applications of this characterization, the first obtaining the consistency of $\boldsymbol{\delta}^1_3 = \omega_2$ with $\mathsf{ZFC}+\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$, and the second proving the uniqueness of the supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ in $\mathrm{L}(\mathbb{R})$ for $\lambda > \boldsymbol{\delta}^2_1$.

Starting with the work of Steel [13] it became clear that there is a deep connection between inner model theory, particularly for minimal inner models with ω Woodin cardinals, and the study of $\mathrm{L}(\mathbb{R})$ under determinacy. The connection centers on the discovery that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is a fine structural inner model, in the exact sense of the notion developed previously through the work of [3, 7, 8] up to Θ,[1] and in the more general sense of [14], that puts iteration strategies as well as extenders into the models, at Θ. To be precise, $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is the direct limit of all countable, iterable inner models with ω Woodin cardinals, together with an iteration strategy for this direct limit. This fact was used extensively by Woodin and Steel, among other things to study measures and ultrafilters: Steel used the directed system to show that for every regular $\kappa < \Theta$, the ω–club filter over κ is an ultrafilter in $\mathrm{L}(\mathbb{R})$. Woodin used the system to show that ω_1 is $<\Theta$–supercompact in $\mathrm{L}(\mathbb{R})$ (meaning that there is a sequence $\langle \mu_\lambda \mid \lambda < \Theta \rangle \in \mathrm{L}(\mathbb{R})$ so that μ_λ is a supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ for each λ) and huge to κ for each measurable κ below the largest Suslin cardinal.

Here we expand the connection in two ways. We use the directed system to obtain an ultrafilter over $[\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1}$, and to prove the uniqueness of the supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ for $\lambda > \boldsymbol{\delta}^2_1$.

Recall that ω_1 is λ–**supercompact** if there is a fine, normal, countably complete ultrafilter over $\mathcal{P}_{\omega_1}(\lambda)$. The ultrafilter, or more precisely its characteristic function, is a **supercompactness measure** over $\mathcal{P}_{\omega_1}(\lambda)$. Solovay derived the existence of such a measure from the determinacy of infinite games on ordinals below λ. He defined a filter $\mathcal{F}_\lambda$, essentially the club filter over $\mathcal{P}_{\omega_1}(\lambda)$, and used determinacy for infinite games on ordinals to show that it is an ultrafilter, and hence a supercompactness measure. For λ up to the largest Suslin cardinal, Harrington–Kechris [4] proved the determinacy of the ordinal games relevant to Solovay's argument from AD. In $\mathrm{L}(\mathbb{R})$, $\boldsymbol{\delta}^2_1$ is the largest Suslin cardinal, and it followed therefore that in $\mathrm{L}(\mathbb{R})$ under AD, ω_1 is λ–supercompact for each $\lambda < \boldsymbol{\delta}^2_1$.

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[1] By Θ and $\boldsymbol{\delta}^2_1$ here and throughout the paper we mean $\Theta^{\mathrm{L}(\mathbb{R})}$ and $(\boldsymbol{\delta}^2_1)^{\mathrm{L}(\mathbb{R})}$.

With a very elegant argument that uses just $\mathsf{ZF}+\mathsf{DC}_\omega$ Woodin [16] proved that any supercompactness ultrafilter $\mathcal{F}^*$ over $\mathcal{P}_{\omega_1}(\lambda)$ must contain Solovay's filter $\mathcal{F}_\lambda$. If $\mathcal{F}_\lambda$ is itself an ultrafilter over $\mathcal{P}_{\omega_1}(\lambda)$ then it follows that $\mathcal{F}^* = \mathcal{F}_\lambda$. Thus, using Harrington–Kechris [4], $\mathcal{F}_\lambda$ is the unique supercompactness ultrafilter over $\mathcal{P}_{\omega_1}(\lambda)$ in $\mathrm{L}(\mathbb{R})$, for $\lambda < \boldsymbol{\delta}^2_1$.

In Section 4 we use the directed system of inner models to construct a supercompactness ultrafilter over $\mathcal{P}_{\omega_1}(\lambda)$ in $\mathrm{L}(\mathbb{R})$, and adapt the argument of Woodin [16] to apply to this ultrafilter instead of Solovay's ultrafilter $\mathcal{F}_\lambda$. The construction works for all $\lambda < \Theta$, and we therefore obtain:

THEOREM. *For each $\lambda < \Theta^{\mathrm{L}(\mathbb{R})}$ there is a unique supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ in* $\mathrm{L}(\mathbb{R})$.

This extends the uniqueness proved in Woodin [16]. (Existence had already been known up to Θ, see the first paragraph above.)

Let U denote the set $\mathcal{P}_{\omega_1}(\omega_\omega)$. In Section 3 we use the directed system to obtain an ultrafilter over $[U]^{<\omega_1}$, that is over the set of increasing countable sequences of countable subsets of ω_ω. We prove for this ultrafilter the same kind of boundedness and uniformization results that Becker [2] proves for the supercompactness ultrafilter over U, and use it to force over $\mathrm{L}(\mathbb{R})$ to collapse ω_ω to ω_1 without adding reals and without collapsing $\omega_{\omega+1}$. The resulting extension satisfies $\boldsymbol{\delta}^1_3 = \omega_2$. Further forcing as in Steel–Van Wesep [15] and Woodin [17] adds the axiom of choice, leading to:

THEOREM (Neeman, Woodin). $\mathsf{ZFC}+\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ *is consistent with* $\boldsymbol{\delta}^1_3 = \omega_2$.

Similar results hold with higher levels of complexity, and in fact every Suslin cardinal of $\mathrm{L}(\mathbb{R})$ can be collapsed to ω_1 without adding reals and without collapsing its successor. These results are due independently to Woodin and the author. It is not known whether similar uses of ultrafilters can be made to force over $\mathrm{L}(\mathbb{R})$ and collapse ordinals to ω_2, rather than to ω_1. A positive answer would be very interesting, as it could lead to a model in which $\boldsymbol{\delta}^1_3 = \omega_3$.

In Section 2 we use the directed system to obtain an ultrafilter over ω_1, and an ultrafilter over $[\omega_1]^{<\omega_1}$. The results there are not new, but we prove them in a way that very easily carries over to the settings of the later sections. The section can be read without knowledge of fine structure, and its point is to illustrate the main ideas that come up repeatedly in the constructions of ultrafilters in Sections 3 and 4.

§1. Preliminaries. We work throughout the paper under the following large cardinal assumption:

$$\text{For every real } u,\ M^\sharp_\omega(u) \text{ exists.}$$

$M^\sharp_\omega(u)$ is the minimal iterable fine structural inner model over u with a sharp for ω Woodin cardinals. Our use of the large cardinals of $M^\sharp_\omega(u)$, and of consequences of its fine structural properties, is explained below.

REMARK 1.1. Our large cardinal assumption is slightly more than follows from determinacy in $\mathrm{L}(\mathbb{R})$. But all the proofs in the paper can be refined to only use the models that can be obtained from determinacy in $\mathrm{L}(\mathbb{R})$, and our results are therefore theorems of $\mathsf{ZFC}+\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$.

By a model with a sharp for ω Woodin cardinals we mean a model of the form $M = (\mathrm{V}^M_\delta)^\sharp$ where δ, which we denote $\delta(M)$, is the supremum of ω Woodin cardinals of M. We often confuse between the countable model with the sharp and the class model obtained by iterating the sharp through the ordinals, that is $\mathrm{L}(\mathrm{V}^M_\delta)$.

Let H be $\mathrm{col}(\omega, {<}\delta)$–generic over M. Define $R^*[H]$ to be $\bigcup_{\alpha<\delta} \mathbb{R}^{M[H\restriction\alpha]}$. We refer to $\mathrm{L}(R^*[H])$ as the **symmetric collapse** of M induced by H, and to $R^*[H]$—which is equal to $\mathbb{R}^{\mathrm{L}(R^*[H])}$—as the **reals in the symmetric collapse** of M induced by H. A set of reals R can be **realized as the reals of a symmetric collapse** of M if it is equal to $R^*[H]$ for some H which is $\mathrm{col}(\omega, {<}\delta)$–generic over M. Our main use of the Woodin cardinals of M comes in through the following theorem. For a proof of the theorem see Neeman [9].

Theorem 1.2 (Steel, Woodin). *Let M be an iterable countable model with a sharp for ω Woodin cardinals. Then there exists, in a generic extension of* V *collapsing $\mathbb{R}$ to ω, a model P with an elementary embedding $\pi\colon M \to P$, so that $\mathbb{R}^{\mathrm{V}}$ can be realized as the reals of a symmetric collapse of P.*

Moreover, given any ordinal $\tau < \delta(M)$, one can arrange that $\mathrm{crit}(\pi) > \tau$.

The embedding π in Theorem 1.2 sends Silver indiscernible for M to Silver indiscernibles for P, which, using the fact that $\mathbb{R} = \mathbb{R}^{\mathrm{V}}$ can be realized as the reals of a symmetric collapse of P, are Silver indiscernibles for $\mathbb{R}$. The following is therefore a direct consequence of Theorem 1.2:

Corollary 1.3. *Let M be an iterable countable model with a sharp for ω Woodin cardinals. Let x belong to $M\|\,\delta(M)$ and let $\alpha_1 < \cdots < \alpha_k$ be Silver indiscernibles for M. Let $\alpha_1^* < \cdots < \alpha_k^*$ be Silver indiscernibles for $\mathbb{R}$. Let $\varphi(v, u_1, \ldots, u_k)$ be a formula.*

Suppose that $\varphi[x, \alpha_1, \ldots, \alpha_k]$ is forced to hold in all symmetric collapses of M. Then $\varphi[x, \alpha_1^*, \ldots, \alpha_k^*]$ holds in $\mathrm{L}(\mathbb{R})$.

Note that, using the homogeneity of the poset $\mathrm{col}(\omega, {<}\delta)$, $\varphi[x, \alpha_1, \ldots, \alpha_k]$ is forced to hold in all symmetric collapses of M iff it is forced to hold in some symmetric collapses of M.

The remaining results in this section summarize our use of fine structure. They are not needed for the material in Section 2. Every model below is fine structural over a real. Recall that a **tree order** is an order T on an ordinal α so that:

1. T is a suborder of $<\restriction\alpha$.
2. For each $\eta < \alpha$, the set $\{\xi \mid \xi\ T\ \eta\}$ is linearly ordered by T.
3. For each ξ so that $\xi + 1 < \alpha$, the ordinal $\xi + 1$ is a successor in T.
4. For each limit ordinal $\lambda < \alpha$, the set $\{\xi \mid \xi\ T\ \lambda\}$ is cofinal in λ.

Definition 1.4. A (maximal, almost normal, fine structural) **iteration tree** $\mathcal{T}$ of length α on a model M consists of a tree order T on α and a sequence $\langle E_\xi \mid \xi+1 < \alpha\rangle$, so that the following conditions hold (for some $\langle M_\xi, j_{\zeta,\xi} \mid \zeta\ T\ \xi < \alpha\rangle$, which is determined uniquely by the conditions):

1. $M_0 = M$.
2. For each ξ so that $\xi + 1 < \alpha$, E_ξ is an extender on the sequence of M_ξ.

3. $M_{\xi+1} = \text{Ult}(\bar{M}, E_\xi)$ and $j_{\zeta,\xi+1}\colon \bar{M} \to M_{\xi+1}$ is the ultrapower embedding, where ζ is the T–predecessor of $\xi+1$ and $\bar{M}$ is the largest initial segment of M_ζ to which E_ξ can be applied.
4. For limit $\lambda < \alpha$, M_λ is the direct limit of the system $\langle M_\zeta, j_{\zeta,\xi} \mid \zeta\ T\ \xi\ T\ \lambda\rangle$, and $j_{\zeta,\lambda}\colon M_\zeta \to M_\lambda$ for (all sufficiently large) $\zeta\ T\ \lambda$ are the direct limit embeddings.
5. The remaining embeddings $j_{\zeta,\xi}$ for $\zeta\ T\ \xi < \alpha$ are obtained by composition.
6. The sequence $\langle \text{lh}(E_\xi) \mid \xi+1 < \alpha\rangle$ is increasing on every interval $[\beta,\gamma]$ with β the T–predecessor of $\gamma+1$.

The iteration tree $\mathcal{T}$ is **normal** if the entire sequence $\langle \text{lh}(E_\xi) \mid \xi+1 < \alpha\rangle$ is increasing.

Abusing notation slightly we write $\langle M_\xi, E_\xi, j_{\zeta,\xi} \mid \zeta\ T\ \xi < \alpha\rangle$ for the iteration tree $\mathcal{T}$. There is a double abuse here: if α is a successor then $E_{\alpha-1}$ is not defined; and M_ξ and $j_{\zeta,\xi}$ are not part of the tree $\mathcal{T}$, but rather are determined by $\mathcal{T}$ and M. The ultrapowers in item (3) are fine structural, see Andretta–Neeman–Steel [1] for more details. If $\bar{M}$ is a strict initial segment of M_ζ then $j_{\zeta,\xi+1}$ does not embed the entire model M_ζ into M_ξ, and we say that $\xi+1$ is a **drop point** of $\mathcal{T}$. An iteration tree $\mathcal{T}$ of successor length $\beta+1$, with a final model M_β, is **proper** if there are no drop points on the branch $[0,\beta]_{\mathcal{T}}$. In this case the tree gives rise to an embedding $j_{0,\beta}\colon M \to M_\beta$.

The creation of an iteration tree on M is usually divided into two tasks, one involving the choice of E_ξ and the T-predecessor of $\xi+1$, and the other involving the choice of the T–branch $[0,\lambda]_T = \{\zeta \mid \zeta\ T\ \lambda\}$ for each limit λ. Choices for the former are usually dictated very directly by circumstances. Choices for the latter generally are not directly dictated, and affect the wellfoundedness of the models along the tree. A mechanism for making these choices is called an iteration strategy for M. More precisely: a potential iteration strategy is a partial function Σ which assigns cofinal branches to iteration trees of limit lengths. An iteration tree $\mathcal{T}$ is consistent with Σ if $[0,\lambda]_{\mathcal{T}} = \Sigma(\mathcal{T}\restriction\lambda)$ for every limit $\lambda < \text{lh}(\mathcal{T})$. Every model on an iteration tree on M consistent with Σ is a Σ–**iterate** of M. Σ is an ω_1+1 **iteration strategy** for M if:

1. The domain of Σ includes all trees of limit lengths $\leq \omega_1$, consistent with Σ, on Σ–iterates of M.
2. All Σ–iterates of M are wellfounded.

A countable model M is ω_1+1 **iterable** if there is an ω_1+1 iteration strategy for M. Henceforth we write iterable instead of ω_1+1 iterable, and similarly with iteration strategies. An iteration strategy Σ helps in the creation of iteration trees of lengths up to and including ω_1+1. This is enough for all uses on countable models.

Fact 1.5. Let M be an iterable countable model so that no $\alpha \leq \text{ON} \cap M$ is Woodin with respect to all functions definable over M with parameters, and so that no strict initial segment of M has a sharp for ω Woodin cardinals. ($M^\sharp_\omega(u)$, its iterates, and many of its initial segments all fall into this category.) Then M has a unique iteration strategy Σ_M, and this strategy is characterized by the condition that $\Sigma_M(\mathcal{T}) = b$ iff the direct limit of the models of $\mathcal{T}$ along b is iterable.

The following facts are immediate consequences of Fact 1.5:

FACT 1.6. Let M satisfy the assumptions of Fact 1.5. Let $\mathcal{T}$ be an iteration tree on M, consistent with Σ_M, leading to a final model M^*. Then an iteration tree $\mathcal{U}$ on M^* is consistent with Σ_{M^*} iff $\mathcal{T}^\frown\mathcal{U}$ is consistent with Σ_M.

FACT 1.7. Let $\bar{M}$ and M satisfy the assumptions of Fact 1.5, with $\bar{M} = M\|\tau$ for some cardinal τ of M. Then every iteration tree on $\bar{M}$ is also an iteration tree on M, and an iteration tree $\mathcal{T}$ on $\bar{M}$ is consistent with $\Sigma_{\bar{M}}$ iff it is consistent with Σ_M.

When M has a sharp for ω Woodin cardinals, Σ_M does not belong to L($\mathbb{R}$). But we shall see below that many restrictions of the strategy do.

η is a **cut point** of M if M has no extenders indexed at or above η with critical points below η. A proper iteration tree $\langle M_\xi, E_\xi, j_{\zeta,\xi} \mid \zeta\ T\ \xi < \alpha+1\rangle$ of successor length $\alpha+1$ on M **acts below** η if every element of M_α is definable in M_α with parameters in range$(j_{0,\alpha}) \cup j_{0,\alpha}(\eta)$. (Using the almost normality of $\mathcal{T}$ this implies that $\mathcal{T}$ is a tree on $M\|\eta'$ where η' is the next cut point of M above η.) The end model of such a tree is a **proper iterate of** M **acting below** η.

FACT 1.8. Let $M = M_\omega^\sharp(u)$. Let $\langle M_\xi, E_\xi, j_{\zeta,\xi} \mid \zeta\ T\ \xi < \alpha+1\rangle$ be a proper iteration tree on M, consistent with Σ_M, and acting below η. Let N be the final model of the tree. Then N is uniquely determined by (M and) $N\|j_{0,\alpha}(\eta)$.

Let η be a cut point of M. By $\Sigma\restriction\eta$ we mean the restriction of Σ to the smallest domain which includes all trees leading to proper countable Σ–iterates of M acting below η.

If M is a model of the form $(\mathrm{V}_\delta^M)^\sharp$ where δ is the supremum of ω Woodin cardinals of M then we use $\delta_0(M)$ to denote the first Woodin cardinal of M, and use $\kappa_0(M)$ to denote the first cardinal strong to $\delta_0(M)$ in M. If M does not reach a sharp for ω Woodin cardinals then we set $\kappa_0(M) = \mathrm{ON} \cap M$.

FACT 1.9. There is a function Γ, definable in L($\mathbb{R}$), so that for every model M satisfying the assumptions in Fact 1.5, and every iteration tree $\mathcal{T}$ in the domain of $\Sigma_M\restriction\kappa_0(M)$:

1. $\langle M, \mathcal{T}\rangle \in \mathrm{dom}(\Gamma)$.
2. $\Gamma(M, \mathcal{T}) = \Sigma_M(\mathcal{T})$.

In particular, $\Sigma_M\restriction\kappa_0(M)$ belongs to L($\mathbb{R}$), and is definable in L($\mathbb{R}$) from M.

FACT 1.10. Let M satisfy the assumptions in Fact 1.5. Call η **locally Woodin** in M if η is a cardinal of M and Woodin in L$(M\|\eta)$. Let η^* be the first measurable limit of locally Woodin cardinals of M. (η^* is much smaller than $\kappa_0(M)$, in fact smaller than the first cardinal τ of M which is Woodin in L$((M\|\tau)^\sharp)$.) Then $\Sigma_M\restriction\eta^*$ is $\Pi^1_2(M)$ in the codes, uniformly in M.

For $M = M_\omega^\sharp(u)$ the restriction $\Sigma_M\restriction\delta_0(M)$ does not belong to L($\mathbb{R}$). Facts 1.9 and 1.10 thus sit at the two extreme ends of the range of iteration strategies in L($\mathbb{R}$).

FACT 1.11. Let M satisfy the assumptions of Fact 1.5. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two proper countable iteration trees on M, consistent with Σ_M, leading to end models M_1^* and M_2^*, and embeddings $j_1 \colon M \to M_1^*$ and $j_2 \colon M \to M_2^*$. Suppose that $M_1^* = M_2^*$. Then $j_1 = j_2$.

Fact 1.11 is a consequence of the Dodd–Jensen lemma, which states that every iteration embedding from M to M^* is pointwise minimal among elementary embeddings from M into M^*. In our context it is due to Steel [12].

Fact 1.12. Let M satisfy the assumptions of Fact 1.5. Let N_1 and N_2 be proper, countable Σ_M–iterates of M. Then there exists a countable model P which is both a normal proper Σ_{N_1}–iterate of N_1 and a normal proper Σ_{N_2}–iterate of N_2. By Fact 1.6, P is a Σ_M–iterate of M. If η is a cut point of M and N_1 and N_2 are Σ_M–iterates of M acting below η, then so is P.

The same is true with the two models N_1 and N_2 replaced by countably many models, N_i, $i < \omega$.

$$\begin{tikzcd}
 & P & \\
N_1 \arrow[ur, "\pi_{N_1,P}"] & & N_2 \arrow[ul, "\pi_{N_2,P}"'] \\
 & M \arrow[ul, "\pi_{M,N_1}"] \arrow[ur, "\pi_{M,N_2}"'] &
\end{tikzcd}$$

Diagram 1. Fact 1.12

Each of $\pi_{N_1,P} \circ \pi_{M,N_1}$ and $\pi_{N_2,P} \circ \pi_{M,N_2}$ is a Σ_M–iteration embedding. By Fact 1.11 the two embeddings are equal, and therefore Diagram 1 commutes.

Fact 1.12 is a consequence of the comparison argument, a central tool in inner model theory and one of the most important uses of fine structure. The fact is also central to this paper. It allows defining the directed systems $\mathcal{D}^{(\eta)}$ below, and these systems are the basis for the entire paper.

For each countable model M let Γ_M be the function defined by $\Gamma_M(\mathcal{T}) = b$ iff $\mathcal{T}$ is an iteration tree of countable limit length on M, $b = \Gamma(M, \mathcal{T})$, and b is a cofinal branch through $\mathcal{T}$. Γ here is taken from Fact 1.9.

Definition 1.13. M is **nice** to η if all proper Γ_M–iterates of M acting below η are wellfounded, and Facts 1.6, 1.7, 1.11, and 1.12 hold for trees on M acting below η with Σ replaced by Γ.

If M satisfies the assumptions of Fact 1.5 and $\eta \leq \kappa_0(M)$ then $\Gamma_M \restriction \eta = \Sigma_M \restriction \eta$ by Fact 1.9, and M is therefore nice to η. Definition 1.13 in this case abstracts the properties of $\Sigma_M \restriction \eta$ needed for Definition 1.14 below. Notice that $L(\mathbb{R})$ can tell which models M are nice to η and identify Γ_M for these models, but it cannot tell which models are iterable or identify Σ_M.

Definition 1.14. Let M be fine structural over u, let η be a cut point of M, and suppose that M is nice to η. Define $\mathcal{D}^{(\eta)}$ to be the set of proper, countable Γ_M–iterates of M acting below η. For $N, N^* \in \mathcal{D}^{(\eta)}$ with N^* a Γ_N–iterate of N, define $\pi^{(\eta)}_{N,N^*}$ to be the unique iteration embedding. Uniqueness follows from Fact 1.11, or more precisely the inclusion of this fact in Definition 1.13. Define $\mathcal{S}^{(\eta)}$ to be the system $\langle N, \pi_{N,N^*} \mid N, N^* \in \mathcal{D}^{(\eta)}$ and N^* is a Γ_N–iterate of $N \rangle$.

This system is directed, because of the inclusion of Fact 1.12 in Definition 1.13. Let $M_\infty^{(\eta)}$ be the direct limit of $\mathcal{S}^{(\eta)}$, and let $\pi_{N,\infty}^{(\eta)}$ for $N \in \mathcal{D}^{(\eta)}$ be the direct limit embeddings.

Fact 1.15. For $M = M_\omega^\sharp(u)$ and $\eta = \kappa_0(M)$, the direct limit $M_\infty^{(\eta)}$ agrees with $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}(u)$ up to $\boldsymbol{\delta}_1^2$, and $\pi_{M,\infty}^{(\eta)}(\eta) = \boldsymbol{\delta}_1^2$.

The fact is due to Steel [13] and [11, §8], where it was used to settle various classical questions concerning $\mathrm{L}(\mathbb{R})$ under AD. Among other things Steel showed that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ satisfies the GCH, and that for $\kappa < \Theta$ there is no sequence of κ^+ distinct subsets of κ in $\mathrm{L}(\mathbb{R})$.

To reach beyond $\boldsymbol{\delta}_1^2$ one has to take $\eta = \delta_0(M)$ and replace Γ_M by Σ_M in Definition 1.14 above. The directed system in this case does not belong to $\mathrm{L}(\mathbb{R})$. By approximating the system inside $\mathrm{L}(\mathbb{R})$ Woodin (see [14]) proved that its direct limit agrees with $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}(u)$ up to Θ, and that $\pi_{M,\infty}^{(\delta_0(M))}(\delta_0(M)) = \Theta$. Woodin also characterized cardinals of $\mathrm{L}(\mathbb{R})$ in terms of their properties in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}(u) = M_\infty^{(\delta_0(M))}(u)$. Among other things he showed that:

Fact 1.16. ω_ω is equal to the first locally Woodin cardinal of $M_\infty^{(\delta_0(M))}$.

Remark 1.17. Let τ be a cardinal cut point of M so that $M \| \tau$ is nice. Then $\pi_{M,\infty}^{(\eta)} \restriction \tau$ and $M_\infty^{(\eta)} \| \sup \pi_{M,\infty}^{(\eta)}{}''\tau$ depend only on (η and) $M \| \tau$. In fact they are equal to $\pi_{N,\infty}^{(\mu)}$ and $N_\infty^{(\mu)}$ respectively, where $N = M \| \tau$ and $\mu = \min(\tau, \eta)$.

§2. An ultrafilter over ω_1. Let M be a countable model with, as least, a measurable cardinal.

An iteration tree $\mathcal{T}$ on M is **linear** if the T–predecessor of $\xi + 1$ is ξ for all ξ. Models on linear iteration trees on M are **linear iterates** of M. M is **linearly iterable** if all its countable linear iterates are wellfounded.

Let $a(M)$ be the first measurable cardinal of M. Assuming that M is linearly iterable define $C_M = \{a(P) \mid P \text{ is a countable linear iterate of } M\}$. Notice that M is countable and all iterations here are countable, so that $C_M \subset \omega_1^{\mathrm{V}}$.

Claim 2.1. *Let M_i, $i < \omega$, be countable linearly iterable models, each with at least a measurable cardinal. Then there is a countable linearly iterable model M^* with a measurable cardinal so that $C_{M^*} \subset \bigcap_{i<\omega} C_{M_i}$.*

Proof. Let u be a real coding the sequence $\langle M_i \mid i < \omega \rangle$, and, using the large cardinal assumption in Section 1, take M^* to be an iterable model with a measurable cardinal and with $u \in M^*$. $\dashv$

Claim 2.1 shows that the collection $\{C_M \mid M \text{ is a countable linearly iterable model with a measurable cardinal}\}$ generates a countably complete filter over ω_1. Let $\mathcal{F}$ denote this filter. The filter is defined from operations on countable models which can be identified in $\mathrm{L}(\mathbb{R})$. Thus $\mathcal{F} \in \mathrm{L}(\mathbb{R})$.

Claim 2.2. *$\mathcal{F}$ is an ultrafilter in* $\mathrm{L}(\mathbb{R})$.

Proof. Let $X \subset \omega_1$ belong to $\mathrm{L}(\mathbb{R})$. Every set in $\mathrm{L}(\mathbb{R})$ is definable from a real and finitely many indiscernibles for $\mathbb{R}$. Fix then some real u and indiscernibles

$\alpha_1^* < \cdots < \alpha_k^*$ for $\mathbb{R}$ so that $\xi \in X$ iff $\mathrm{L}(\mathbb{R}) \models \varphi[\xi, u, \alpha_1^*, \ldots, \alpha_k^*]$. Let $M = M_\omega^\sharp(u)$ and let $\alpha_1 < \cdots < \alpha_k$ be indiscernibles for M.

Suppose first that $\varphi[a(M), u, \alpha_1, \ldots, \alpha_k]$ is forced to hold in all symmetric collapses of M. We show that $C_M \subset X$. Let $\xi \in C_M$, and fix a countable linear iterate P of M so that $\xi = a(P)$. Let $j\colon M \to P$ be the iteration embedding. By elementarity, $\varphi[a(P), u, j(\alpha_1), \ldots, j(\alpha_k)]$ is forced to hold in all symmetric collapses of P. Since $a(P) = \xi < \delta(P)$ and $j(\alpha_1), \ldots, j(\alpha_k)$ are Silver indiscernibles for P, it follows from Corollary 1.3 that $\mathrm{L}(\mathbb{R}) \models \varphi[\xi, u, \alpha_1^*, \ldots, \alpha_k^*]$, and hence $\xi \in X$.

Suppose next that $\varphi[a(M), u, \alpha_1, \ldots, \alpha_k]$ is not forced to hold in all symmetric collapses of M. Then by the homogeneity of $\mathrm{col}(\omega, {<}\delta(M))$, the formula is forced to fail in all the symmetric collapses, and an argument similar to the one in the previous paragraph shows that $C_M \subset \omega_1 - X$. ⊣

It is easy to check that in fact $\mathcal{F}$ is the club filter on ω_1. Steel [11, Theorem 8.26] had already used the directed system of inner models to show that the club filter on ω_1 is an ultrafilter in $\mathrm{L}(\mathbb{R})$ (and in any case this is a well known consequence of AD). The point here is not to prove that the club filter is an ultrafilter, but rather to construct an ultrafilter over a set U through a very general scheme: assigning to each iterable model M some $a(M) \in U$, setting $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$, and considering the filter generated by the sets C_M. This scheme easily adapts to produce ultrafilters over many sets U, as we see below and in the next section.

For a countable, linearly iterable M with a measurable limit of measurable cardinals, let $\kappa(M)$ be the first measurable limit of measurable cardinals in M, let $\alpha(M)$ be the order type of the set of measurable cardinals of M below $\kappa(M)$, and let $\langle \tau_\xi(M) \mid \xi < \alpha(M)\rangle$ enumerate the measurable cardinals of M below $\kappa(M)$ in increasing order.

Set $a^*(M) = \langle \tau_\xi(M) \mid \xi < \alpha(M)\rangle$ and set $C_M^* = \{a^*(P) \mid P \text{ is a countable linear iterate of } M\}$. C_M^* is then a subset of $[\omega_1]^{<\omega_1}$. The proofs of Claims 2.1 and 2.2 carry over to these new settings, and show that the collection $\{C_M \mid M \text{ is a countable linearly iterable model with a measurable limit of measurable cardinals}\}$ generates a countably complete ultrafilter $\mathcal{F}^*$ over $[\omega_1]^{<\omega_1}$ in $\mathrm{L}(\mathbb{R})$.

Claim 2.3. *Let $f\colon [\omega_1]^{<\omega_1} \to {\omega_2}^{\mathrm{L}(\mathbb{R})}$ belong to $\mathrm{L}(\mathbb{R})$. Then there is $X \in \mathcal{F}^*$ so that $f \restriction X$ is bounded below ω_2.*

By ω_2, and more generally ω_ξ, here and below we mean ${\omega_\xi}^{\mathrm{L}(\mathbb{R})}$.

Proof of Claim 2.3. For each real y let y^+ be the first Silver indiscernible for y above ω_1. The ordinals y^+, $y \in \mathbb{R}$, are cofinal in $\omega_2 = {\omega_2}^{\mathrm{L}(\mathbb{R})}$. A real x **codes** an ordinal $\eta < \omega_2$ if x has the form $y^\sharp$ and $\eta = y^+$. Let $U = \{\langle s, x\rangle \mid s \in [\omega_1]^{<\omega_1} \text{ and } x \text{ codes an ordinal above } f(s)\}$. Let u, $k < \omega$, and φ be such $\langle s, x\rangle \in U$ iff $\mathrm{L}(\mathbb{R}) \models \varphi[x, s, u, \alpha_1^*, \ldots, \alpha_k^*]$ where $\alpha_1^* < \cdots < \alpha_k^*$ are Silver indiscernibles for $\mathbb{R}$. Let $M = M_\omega^\sharp(u)$. Let $\alpha_1 < \cdots < \alpha_k$ be Silver indiscernibles for M. Define $A \subset \mathbb{R}$ by setting $x \in A$ iff there is a countable linear iterate P of M, with iteration embedding j, and a symmetric collapse of P in which $\varphi[x, a^*(P), u, j(\alpha_1), \ldots, j(\alpha_k)]$ holds true. Note that:

1. A is $\mathbf{\Sigma}^1_2$.

2. $x \in A$ implies that x codes an ordinal. (In fact x codes an ordinal above $f(s)$ for some $s \in C^*_M$.)
3. For every $s \in C^*_M$ there is $x \in A$ which codes an ordinal above $f(s)$.

The second and third conditions are immediate consequences of Corollary 1.3. The first follows from the definition of A noting that the set of countable linear iteration trees on M is $\Pi^1_1(M)$ in the codes. From conditions (1) and (2) it follows that the set of ordinals coded by reals in A is bounded below ω_2. From this and condition (3) it follow that $f \restriction C^*_M$ is bounded below ω_2. $\dashv$

Definition 2.4. Call $X \subset [\omega_1]^{<\omega_1}$ **nice** if:

1. X belongs to $\mathcal{F}^*$.
2. X is closed, meaning that if $s \in \omega_1^{<\omega_1}$ has limit length, and $s \restriction \alpha \in X$ for cofinally many $\alpha < \mathrm{lh}(s)$, then $s \in X$.
3. For each $s \in X$, $\{r \mid s^\frown r \in X\}$ belongs to $\mathcal{F}^*$.

Claim 2.5. *Each C^*_M is nice.*

Proof. C^*_M belongs to $\mathcal{F}^*$ by definition, and condition (2) can be proved by composing iterations.

As for condition (3), let $s \in C^*_M$ and let Q be a linear iterate of M so that $s = a(Q)$. Let $\mu \in Q$ be a measure on $\kappa(Q)$ and let $Q^* = \mathrm{Ult}(Q, \mu)$. Let h be $\mathrm{col}(\omega, \kappa(Q))$–generic over Q^*. Then $\kappa(Q^*)$ is still a measurable limit of measurable cardinals in $Q^*[h]$, and by composing iterations of $Q^*[h]$ with the iteration leading from M to Q it is easy to check that $r \in C^*_{Q^*[h]} \Rightarrow s^\frown r \in C^*_M$, and therefore $\{r \mid s^\frown r \in C^*_M\} \in \mathcal{F}^*$. $\dashv$

Let $\mathbb{P}$ be the poset consisting of pairs $\langle s, X\rangle$ where $s \in [\omega_1]^{<\omega_1}$, $X \subset [\omega_1]^{<\omega_1}$, and $\{r \mid s^\frown r \in X\}$ is nice, ordered by the condition $\langle s', X'\rangle \leq \langle s, X\rangle$ iff s' extends s, $X' \subset X$, and $s' \in X$. $\mathbb{P}$ belongs to $\mathrm{L}(\mathbb{R})$.

Definition 2.4 is such that if X is nice and $s \in X$, then $\{r \mid s^\frown r \in X\}$ is nice. From this, the countable closure in Definition 2.4, and the countable closure of $\mathcal{F}^*$, it follows that $\mathbb{P}$ is countably closed. Forcing with $\mathbb{P}$ over $\mathrm{L}(\mathbb{R})$ adds a **fast club** in ω_1, that is a club C which is contained in any club subset of ω_1 in $\mathrm{L}(\mathbb{R})$. By countable closure the forcing does not collapse ω_1. Moreover, using Claim 2.3 one can show that forcing with $\mathbb{P}$ over $\mathrm{L}(\mathbb{R})$ does not collapse ω_2 either.

In the next section we use a similar forcing to add, over $\mathrm{L}(\mathbb{R})$, a fast club in $\mathcal{P}_{\omega_1}(\omega_\omega)$, thereby collapsing ω_ω to ω_1, without collapsing ω_1 and without collapsing $\omega_{\omega+1}$.

§3. An ultrafilter over $\mathcal{P}_{\omega_1}(\omega_\omega)$. Recall that η is **locally Woodin** in M if η is a cardinal of M and η is Woodin in $\mathrm{L}(M\| \eta)$. Let K be the set of countable models M which are nice to their first locally Woodin cardinal (see Definition 1.13). For $M \in K$ let $\eta(M)$ be the least locally Woodin cardinal of M, and define:

1. $a(M) = \pi_{M,\infty}{}''\eta(M)$.
2. $C_M = \{a(P) \mid P$ is a proper, countable, Γ_M–iterate of M acting below $\eta(M)\}$.

$\pi_{M,\infty}$ here is the direct limit embedding of the system $\mathcal{S}^{(\eta(M))}$ of Definition 1.14. We omit the superscript $(\eta(M))$ throughout. All iterates of each model M are proper, countable, and acting below $\eta(M)$. We omit mention of this below.

K is not empty, and indeed the models $M_\omega^\sharp(u)$, $u \in \mathbb{R}$, are all in K. K, $M \mapsto a(M)$, and $M \mapsto C_M$ all belong to $\mathrm{L}(\mathbb{R})$. $\pi_{M,\infty}(\eta(M))$ is equal to ω_ω, see Fact 1.16. C_M is therefore a subset of $\mathcal{P}_{\omega_1}(\omega_\omega)$. Our plan is to show that the sets C_M, $M \in K$, generate a supercompactness measure over $\mathcal{P}_{\omega_1}(\omega_\omega)$ in $\mathrm{L}(\mathbb{R})$.

Claim 3.1. *Let $M \in K$. Then:*

1. *If $\xi \in a(M)$ then $\xi \in a(P)$ for every Γ_M–iterate P of M.*
2. *For every $\xi \in \omega_\omega$ there is an iterate M^* of M so that $a \in C_{M^*} \Longrightarrow \xi \in a$.*
3. *For every countable $a \subset \omega_\omega$ there is an iterate P of M so that $a \subset a(P)$.*

Proof. Condition (1) follows immediately from the commutativity of the system $\mathcal{S}$. For condition (2), fix $\xi \in \omega_\omega$ and using the fact that $\pi_{M,\infty}(\eta(M)) = \omega_\omega$ find a Γ_M–iterate M^* of M so that $\xi \in \pi_{M^*,\infty}{}''\eta(M^*)$. Then by condition (1), $\xi \in a(P)$ for every Γ_{M^*}–iterate P of M^*. Finally, for condition (3), fix for each $\xi \in a$ a Γ_M–iterate M_ξ of M so that $\xi \in a(M_\xi)$, and using Fact 1.12 find P which is a Γ_{M_ξ}–iterate of M_ξ for each $\xi \in a$. ⊣

Lemma 3.2. *Let $M \in K$. Let $u \in \mathbb{R}$ be Turing above a real coding M. Let $P = M_\omega^\sharp(u)$. Then $a(P) \in C_M$.*

Proof. Using condition (3) of Claim 3.1 construct sequences $\langle P_n \mid n < \omega\rangle$ and $\langle M_n \mid n < \omega\rangle$ so that:

(i) $P_0 = P$ and P_{n+1} is an iterate of P_n for each n, and similarly for M.
(ii) $a(P_{n+1}) \supset a(M_n)$, and $a(M_{n+1}) \supset a(P_{n+1})$.

Let P^* be the direct limit of the system $\langle P_n, \pi_{P_n,P_m} \mid n < m < \omega\rangle$, and define M^* similarly. Then by condition (ii), $\pi_{P^*,\infty}{}''\eta(P^*) = \pi_{M^*,\infty}{}''\eta(M^*)$, in other words $a(P^*) = a(M^*)$. Let $\bar{P} = P \| \eta(P)^+$ and let $\bar{P}^* = P^* \| \eta(P^*)^+$. By Remark 1.17, $a(\bar{P}) = a(P)$ and $a(\bar{P}^*) = a(P^*)$.

Let φ be a formula so that $\mathrm{L}(\mathbb{R}) \models \varphi[M, Q]$ iff there is a Γ_M–iterate M^* of M so that $a(M^*) = a(Q)$. Such a formula exists since the various directed systems are definable in $\mathrm{L}(\mathbb{R})$. We showed that $\mathrm{L}(\mathbb{R}) \models \varphi[M, \bar{P}^*]$. By Corollary 1.3, $\varphi[M, \bar{P}^*]$ holds in a symmetric collapse of P^*. Using the elementarity of the iteration embedding π_{P,P^*}, $\varphi[M, \bar{P}]$ holds in a symmetric collapse of P. (Note that M is coded by a real in P, and is not moved by any of the embeddings.) Using Corollary 1.3, $\varphi[M, \bar{P}]$ holds in $\mathrm{L}(\mathbb{R})$, and therefore $a(P) = a(\bar{P}) \in C_M$. ⊣

Corollary 3.3. *The collection C_M, $M \in K$, has the countable intersection property.*

Proof. Let M_n, $n < \omega$, belong to K. Let u be a real which codes the sequence $\langle M_n \mid n < \omega\rangle$, and let $P = M_\omega^\sharp(u)$. By the previous lemma, $a(P) \in \bigcap_{n<\omega} a(M_n)$. ⊣

Let $\mathcal{F}$ be the filter over $\mathcal{P}_{\omega_1}(\omega_\omega)$ generated by the sets C_M, $M \in K$.

Claim 3.4. *$\mathcal{F}$ is an ultrafilter in* $\mathrm{L}(\mathbb{R})$.

Proof. Similar to the proof of Claim 2.2. ⊣

CLAIM 3.5. *$\mathcal{F}$ is normal in* L(ℝ).

PROOF. Let $f\colon \mathcal{P}_{\omega_1}(\omega_\omega) \to \omega_\omega$ be a function so that $f(a) \in a$ for each $a \neq \emptyset$. We have to show that f is constant on a set in $\mathcal{F}$.

Fix a real u, $k < \omega$, and a formula ψ, so that

$$f(a) = \beta \text{ iff } \mathrm{L}(\mathbb{R}) \models \psi[a, \beta, u, \alpha_1^*, \dots, \alpha_k^*]$$

where $\alpha_1^* < \cdots < \alpha_k^*$ are Silver indiscernibles for ℝ. For simplicity assume that $k = 0$, so that the indiscernibles can be omitted below. For convenience below we use $\bar{M}$ to abbreviate $M\|\,\eta(M)^+$, and similarly with P.

Let $M = M_\omega^\sharp(u)$. Since $f(a(M)) \in a(M) = \pi_{M,\infty}{}''\eta(M)$, there is $\gamma < \eta(M)$ so that $f(a(M)) = \pi_{M,\infty}(\gamma)$. By Remark 1.17 it follows that $\psi[a(\bar{M}), \pi_{\bar{M},\infty}(\gamma), u]$ holds in L(ℝ). This can be written as a statement $\varphi[\bar{M}, \gamma, u]$ about $\bar{M}$, γ, and u. By Corollary 1.3 this statement holds in a symmetric collapses of M.

Fix now a Γ_M–iterate P of M. By the elementarity of the embedding $\pi_{P,M}$, $\varphi[\bar{P}, \pi_{M,P}(\gamma), u]$ holds in a symmetric collapse of P. Using Corollary 1.3 it follows that $f(\bar{P}) = \pi_{\bar{P},\infty}(\pi_{M,P}(\gamma))$. By Remark 1.17 then $f(a(P)) = \pi_{P,\infty}(\pi_{M,P}(\gamma))$, hence $f(a(P)) = \pi_{M,\infty}(\gamma) = a(M)$.

We showed that $f(a(P)) = f(a(M))$ for each Γ_M–iterate P of M, in other words $f(a) = f(a(M))$ for each $a \in C_M$. Thus f is constant on C_M. ⊣

COROLLARY 3.6. *$\mathcal{F}$ is a supercompactness measure over $\mathcal{P}_{\omega_1}(\omega_\omega)$ in* L(ℝ). ⊣

The existence of a supercompactness measure over $\mathcal{P}_{\omega_1}(\omega_\omega)$ in L(ℝ) under AD was proved by Harrington–Kechris [4]. Becker [2] showed that the measure is unique, and analyzed it more deeply, obtaining boundedness, coding, and uniformization results on measure one sets for functions on $\mathcal{P}_{\omega_1}(\omega_\omega)$. One can of course obtain the same results from the inner models construction given above, but we leave this as a pleasant exercise for the reader. Here instead we pass to the space $[\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$. With just a small modification to the construction we get an ultrafilter over this space, and we shall prove boundedness and uniformization results for this ultrafilter.

Let K^* be the set of countable models M which are nice to their first measurable limit of locally Woodin cardinals. For $M \in K^*$ let $\eta^*(M)$ be the first measurable limit of locally Woodin cardinals of M, let $\alpha(M)$ be the order type of the set of locally Woodin cardinals of M below $\eta^*(M)$, and let $\langle \eta_\xi(M) \mid \xi < \alpha(M)\rangle$ enumerate the locally Woodin cardinals of M below $\eta^*(M)$ in increasing order.

For each $\beta < \alpha(M)$ let $\nu_\beta = \sup_{\xi<\beta} \eta_\xi$ and let g_β be $\mathrm{col}(\omega, {\nu_\beta}^+)$–generic over M. $M[g_\xi]$ is fine structural over a real coding $M\|\,{\nu_\beta}^+$, and η_β is the first locally Woodin cardinal of $M[g_\beta]$. Define $a_\beta(M) = a(M[g_\beta])$, that is $a_\beta(M) = \pi^{(\eta_\beta)}_{M[g_\beta],\infty}{}''\eta_\beta$. (The definition is independent of the particular choice of the generic g_β.) Since η_β is the first locally Woodin cardinal of $M[g_\beta]$, $a_\beta(M)$ is a subset of ω_ω. Set now:

1. $a^*(M) = \langle a_\beta(M) \mid \beta < \alpha(M)\rangle$.
2. $C_M^* = \{a^*(P) \mid P$ is a proper, countable, Γ_M–iterate of M acting below $\eta^*(M)\}$.

LEMMA 3.7. *Let $\mathcal{F}^*$ be the filter generated by the collection C_M^*, $M \in K^*$. Then $\mathcal{F}^*$ is an ultrafilter in* L(ℝ) *and has the countable intersection property.*

$\mathcal{F}^*$ concentrates on increasing countable sequences of elements of $\mathcal{P}_{\omega_1}(\omega_\omega)$, and every measure one set has sequences of arbitrary countable length. (We say that $\mathcal{F}^*$ concentrates on **long sequences in** *$[\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$.)*

Proof. An argument similar to the one in the proof of Lemma 3.2 shows that the collection C^*_M, $M \in K^*$, has the countable intersection property. A straightforward adaptation of the proof of Claim 2.2 shows that $\mathcal{F}^*$ is an ultrafilter in $\mathrm{L}(\mathbb{R})$. For every $M \in K^*$ and every $\alpha < \omega_1$ it is easy to produce a Γ_M–iterate P of M so that $a^*(P)$ is increasing and has length $> \alpha$. From this and the fact that $\mathcal{F}^*$ is an ultrafilter it follows that $\mathcal{F}^*$ concentrates on long sequences in $[\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$. ⊣

Lemma 3.8 (Boundedness). *Let $f\colon [\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1} \to \omega_{\omega+1}$ belong to $\mathrm{L}(\mathbb{R})$. Then there is $X \in \mathcal{F}^*$ so that $f \restriction X$ is bounded below $\omega_{\omega+1}$.*

Proof. Similar to the proof of Claim 2.3, but using $\mathbf{\Sigma}^1_3$ boundedness (recall that $\omega_{\omega+1} = \boldsymbol{\delta}^1_3$ in $\mathrm{L}(\mathbb{R})$) and Fact 1.10 as replacements for the use of $\mathbf{\Sigma}^1_2$ boundedness and condition (1) in the proof of Claim 2.3. ⊣

Let WO be the set of wellorders of ω. We use ot to mean "order type." For $a^* = \langle a_\xi \mid \xi < \alpha \rangle$ we write $\mathrm{ot}(a^*)$ for $\sup_{\xi<\alpha} \mathrm{ot}(a_\xi)$.

Lemma 3.9 (Uniformization). *Let $F\colon [\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1} \to \mathcal{P}(\mathbb{R}) - \{\emptyset\}$ belong to $\mathrm{L}(\mathbb{R})$. Then there is $X \in \mathcal{F}^*$ and a function $f\colon X \times \mathrm{WO} \to \mathbb{R}$ in $\mathrm{L}(\mathbb{R})$ so that $f(a^*, w) \in F(a)$ for all $a^* \in X$ and $w \in \mathrm{WO}$ with $\mathrm{ot}(w) \geq \mathrm{ot}(a^*)$.*

Proof. Fix a real u, $k < \omega$, and a formula φ so that $x \in F(a^*)$ iff $\mathrm{L}(\mathbb{R}) \models \varphi[u, x, a^*, \alpha^*_1, \ldots, \alpha^*_k]$ where $\alpha^*_1 < \ldots \alpha^*_k$ are Silver indiscernibles for $\mathbb{R}$. For simplicity suppose that $k = 0$ so that the indiscernibles can be omitted below.

Let $M = M^\sharp_\omega(u)$ and let $X = C^*_M$. Let $\vec{e}$ enumerate M in order type ω.

Let $a^* \in C^*_M$ and let P be a Γ_M–iterate of M acting below $\eta^*(M)$ so that $a^* = a^*(P)$. From the fact that $a^*(P) = a^*$ it follows that $\eta^*(P) = \mathrm{ot}(a^*)$. Since the iteration tree leading to P acts below $\eta^*(M)$, every element of P is definable in P from parameters in $\mathrm{range}(\pi_{M,P}) \cup \eta^*(P)$. One can therefore produce an enumeration of P definably from $\vec{e}$ and any $w \in \mathrm{WO}$ with $\mathrm{ot}(w) \geq \mathrm{ot}(a^*)$. From an enumeration of P in turn one can definably produce a symmetric collapse of P and an enumeration of the reals in that symmetric collapse.

A short argument using Corollary 1.3 shows that every symmetric collapse of P has a real x which belongs to $F(a^*(P))$. Using the enumerations of the previous paragraph one can therefore produce a function h in $\mathrm{L}(\mathbb{R})$, so that $h(P, w)$ is defined and belongs to $F(a^*(P))$ whenever P is a Γ_M–iterate of M acting below $\eta^*(M)$.

From Fact 1.8 it follows that there is a function $g \in \mathrm{L}(\mathbb{R})$ which assigns to each $a^* \in C^*_M$ an iterate P of M so that $a^*(P) = a^*$. Set $f(a^*, w) = h(g(a^*), w)$. Then $f(a^*, w) \in F(a^*)$ whenever $a^* \in C^*_M$ and $\mathrm{ot}(w) \geq \mathrm{ot}(a^*)$. ⊣

Adapting definitions from Section 2, call $X \subset [\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$ **nice** if: X belongs to $\mathcal{F}^*$, X is closed, and for each $s \in X$, $\{r \mid s^\frown r \in X\}$ belongs to $\mathcal{F}^*$. The proof of Claim 2.5 carries over to the current context, showing that each C^*_M is nice.

Let $\mathbb{P}$ be the poset consisting of pairs $\langle s, X \rangle$ where $s \in [\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$, $X \subset [\mathcal{P}_{\omega_1}(\omega_\omega)]^{<\omega_1}$, and $\{r \mid s^\frown r \in X\}$ is nice, ordered by the condition $\langle s', X' \rangle \leq \langle s, X \rangle$ iff s' extends s, $X' \subset X$, and $s' \in X$. $\mathbb{P}$ belongs to $\mathrm{L}(\mathbb{R})$.

To prevent confusion below let λ denote ${\omega_\omega}^{\mathrm{L}(\mathbb{R})}$. Let H be $\mathbb{P}$–generic over $\mathrm{L}(\mathbb{R})$. Let $b = \bigcup\{s \mid (\exists X)\langle s, X\rangle \in H\}$. b has the form $\langle b_\xi \mid \xi < \omega_1\rangle$, with $b_\xi \in \mathcal{P}_{\omega_1}(\lambda)$ for each ξ.

Lemma 3.10. $\mathrm{L}(\mathbb{R})[H] \models$ *"$\boldsymbol{\delta}^1_3 = \omega_2$."*

Proof. $\mathbb{P}$ is ω–closed, so that ${\omega_1}^{\mathrm{L}(\mathbb{R})} = {\omega_1}^{\mathrm{L}(\mathbb{R})[H]}$ and $(\boldsymbol{\delta}^1_3)^{\mathrm{L}(\mathbb{R})} = (\boldsymbol{\delta}^1_3)^{\mathrm{L}(\mathbb{R})[H]}$. By the genericity of H, for every countable $a \subset \lambda$ there is ξ so that $b_\xi \supset a$. It follows that $\lambda = \bigcup_{\xi<\omega_1} b_\xi$ and hence $\mathrm{card}(\lambda) = \omega_1$ in $\mathrm{L}(\mathbb{R})[H]$. Using Lemma 3.8 one can prove that ${\omega_{\omega+1}}^{\mathrm{L}(\mathbb{R})}$, which is equal to $\boldsymbol{\delta}^1_3$, is not collapsed by H. So ${\omega_{\omega+1}}^{\mathrm{L}(\mathbb{R})}$ is equal to ${\omega_2}^{\mathrm{L}(\mathbb{R})[H]}$. ⊣

Next we force over $\mathrm{L}(\mathbb{R})[H]$ to add the axiom of choice, while preserving the fact that $\boldsymbol{\delta}^1_3 = \omega_2$. We follow the route established by Steel–Van Wesep [15] and Woodin [17]: forcing first to add codes for countable ordinals, arguing that the resulting model satisfies DC_{ω_1}, and forcing over it to enumerate the reals in order type ω_2.

Let G be $\mathrm{col}(\omega, {<}\omega_1)$–generic over $\mathrm{L}(\mathbb{R})[H]$. $\mathrm{col}(\omega, {<}\omega_1)$ is c.c.c., so that no cardinals are collapsed in the move from $\mathrm{L}(\mathbb{R})[H]$ to $\mathrm{L}(\mathbb{R})[H][G]$. Of course G adds reals, but by Neeman–Zapletal [10] there is an elementary embedding $\Psi\colon \mathrm{L}(\mathbb{R}) \to \mathrm{L}(\hat{\mathbb{R}})$, where $\hat{\mathbb{R}}$ are the reals of $\mathrm{L}(\mathbb{R})[H][G]$, with $\Psi\restriction\mathrm{ON} = \mathrm{id}$. It follows that $(\boldsymbol{\delta}^1_3)^{\mathrm{L}(\mathbb{R})} = (\boldsymbol{\delta}^1_3)^{\mathrm{L}(\hat{\mathbb{R}})}$, and since no cardinals are collapsed by the addition of G, $\mathrm{L}(\mathbb{R})[H][G]$ satisfies $\boldsymbol{\delta}^1_3 = \omega_2$.

Claim 3.11. *Let $X \subset H(\omega_1)$ and let $F\colon X \to \mathcal{P}(\mathbb{R}) - \{\emptyset\}$ be a function so that the relation $y \in F(x)$ is $\Sigma_1(\mathbb{R} \cup \{\mathbb{R}\})$ over $\mathrm{L}(\mathbb{R})$. Then there is, in $\mathrm{L}(\mathbb{R})[G]$, a function $f\colon \hat{X} \to \hat{\mathbb{R}}$ so that $f(x) \in \hat{F}(x)$ for all $x \in \hat{X}$.*

As a general matter of notation, here and below, we use $\hat{Z}$ for $Z \in \mathrm{L}(\mathbb{R})$ to denote $\Psi(Z)$. If Z is defined in $\mathrm{L}(\mathbb{R})$ from reals and ordinals $c_1, \ldots, c_k$, then $\hat{Z}$ is the object defined from the same reals and ordinals and in the same manner, but in $\mathrm{L}(\hat{\mathbb{R}})$.

Proof of Claim 3.11. If X were a subset of $\mathbb{R}$, the conclusion of the claim would follow simply from $\boldsymbol{\Sigma}^2_1$ uniformization in $\mathrm{L}(\mathbb{R})$, proved by Martin–Steel [5]. Using the generic G one can obtain a function which codes elements of $H(\omega_1)$ by reals, and reduce the general case to the case of $X \subset \mathbb{R}$. ⊣

Claim 3.12. *Let $k \in \mathrm{L}(\mathbb{R})[G]$ be a function assigning to each $s \in [\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1}$ a set $k(s) \subset [\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1}$ so that $\{r \mid s^\frown r \in k(s)\}$ is nice, where $[\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1}$ and "nice" are both interpreted in $\mathrm{L}(\hat{\mathbb{R}}) = \mathrm{L}(\mathbb{R}^{\mathrm{L}(\mathbb{R})[G]})$. Then there is some condition $\langle s, X\rangle \in H$ so that $X \subset k(s)$.*

Proof. Immediate from the genericity of $H \times G$ and the observation that any set in the ultrafilter $\hat{\mathcal{F}}^*$ can be refined to a set in $\mathcal{F}^*$. ⊣

Definition 3.13. Let $\dot{A}$ be a $\mathbb{P} \times \mathrm{col}(\omega, {<}\omega_1)$–name. Let $\langle s, X\rangle \in \mathbb{P}$, let $\alpha < \omega_1$, and let $g = \langle g_\xi \mid \xi < \alpha\rangle$ with $g_\xi\colon \omega \to \xi$ for each ξ. We write $\langle s, X, g\rangle \Vdash x \in \dot{A}$ just in case that $x \in \dot{A}[\dot{E} \times (g^\frown \dot{F})]$ is forced by $\langle s, X, \emptyset\rangle$ in the forcing to add a generic $E \times F$ for $\mathbb{P} \times \mathrm{col}(\omega, [\alpha, \omega_1))$.

DC_{ω_1} **for reals** is the statement "every countably closed tree $T \subset \mathbb{R}^{<\omega_1}$ with no terminal nodes has a branch of length ω_1."

LEMMA 3.14. *For every sufficiently closed ordinal ν, $\mathrm{L}_\nu(\mathbb{R})[H][G]$ satisfies DC_{ω_1} for reals.*

By "sufficiently closed" in Lemma 3.14 we mean that $\mathrm{L}_\nu(\mathbb{R})$ satisfies some large enough fragment of ZF+DC+AD. For $s \in [\mathcal{P}_{\omega_1}(\lambda)]^{\omega_1}$ and M a countable model in K^* let $X_{s,M}$ be the set $\{s^\frown r \mid r \in C^*_M\}$. Every condition $\langle s, X\rangle \in \mathbb{P}$ can be refined to a condition of the form $\langle s, X_{s,M}\rangle$. All conditions of the form $\langle s, X_{s,M}\rangle$ belong to $\mathrm{L}_\nu(\mathbb{R})$, as they are definable from reals using a directed system which by Fact 1.10 has low projective complexity, and $\mathrm{L}_\nu(\mathbb{R})$ is correct about projective statements. Certainly all the conditions in $\mathrm{col}(\omega, {<}\omega_1)$ belong to $\mathrm{L}_\nu(\mathbb{R})$. It follows therefore that $H \times G$ is $\mathbb{P} \times \mathrm{col}(\omega, {<}\omega_1)$–generic over $\mathrm{L}_\nu(\mathbb{R})$.

PROOF OF LEMMA 3.14. Suppose the lemma fails. Its statement, or more precisely an equivalent statement using the forcing language, is $\Pi_1(\mathbb{R} \cup \{\mathbb{R}\})$ over $\mathrm{L}(\mathbb{R})$. Picking a minimal witness to its failure we can find ν, a tree T in $\mathrm{L}_\nu(\mathbb{R})[H][G]$, and a name $\dot{T} \in \mathrm{L}_\nu(\mathbb{R})$ for T, so that the relation $\langle s, X, g\rangle \Vdash x \in \dot{T}$ is $\Sigma_1(\mathbb{R} \cup \{\mathbb{R}\})$ over $\mathrm{L}(\mathbb{R})$.

Let U be the set of sequences $g = \langle g_\xi \mid \xi < \alpha\rangle$ where $\alpha < \omega_1$ and $g_\xi \colon \omega \to \xi$ for each $\xi < \alpha$. Let F be the function which assigns to each $\langle g, s, t\rangle \in U \times [\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1} \times \mathbb{R}^{<\omega_1}$ the set of pairs $\langle M, y\rangle \in K^* \times \mathbb{R}$ so that $\langle s, X_{s,M}, g\rangle \Vdash t^\frown\langle y\rangle \in \dot{T}$. By Claim 3.11 there is a function $f \in \mathrm{L}(\mathbb{R})[G]$ so that $\hat{F}(g,s,t) \neq \emptyset \implies f(g,s,t) \in \hat{F}(g,s,t)$. Using f, G, H, and Claim 3.12 it is not hard to construct a branch $Y = \langle y_\xi \mid \xi < \omega_1\rangle$ through T.

We showed so far that T has a branch Y of length ω_1 in $\mathrm{L}(\mathbb{R})[H][G]$. It remains to see that Y belongs to $\mathrm{L}_\nu(\mathbb{R})[H][G]$. Let $\dot{Y} \in \mathrm{L}(\mathbb{R})$ be a name for Y. Suppose for contradiction that there is no name $\dot{Z} \in \mathrm{L}_\nu(\mathbb{R})$ so that $\dot{Z}[H][G] = \dot{Y}[H][G]$. Suppose for simplicity that this fact is forced by the empty condition.

The relation $\langle s, X, q\rangle \Vdash \dot{Y}_{\check{\xi}}(\check{n}) = \check{m}$ is definable over $\mathrm{L}(\mathbb{R})$ from a real and finitely many Silver indiscernibles. Suppose for notational simplicity that the number of indiscernibles needed is zero. Fix a real u and a formula φ so that $\langle s, X_{s,N}, q\rangle \Vdash \dot{Y}_{\check{\xi}}(\check{n}) = \check{m}$ iff $\mathrm{L}(\mathbb{R}) \models \varphi[u, s, N, q, \xi, n, m]$. Let $M = M^\sharp_\omega(u)$.

Without loss of generality we may assume that $\langle \emptyset, C^*_M\rangle \in H$. Using Corollary 1.3 one can see that $Y_\xi(n) = m$ iff there are q, P, and N so that:

1. q is a condition in $\mathrm{col}(\omega, {<}\omega_1)$, P is a Γ_M–iterate of M acting below $\eta^*(M)$, and N is a countable model which belongs to a symmetric collapse of P.
2. The statements "$N \in K^*$" and $\varphi[u, a^*(P), N, q, \xi, n, m]$ hold in that symmetric collapse.
3. $q \in G$ and $\langle a^*(P), X_{a^*(P),N}\rangle \in H$.

Condition (2) is absolute between $\mathrm{L}(\mathbb{R})$ and $\mathrm{L}_\nu(\mathbb{R})$ since it is first order over a countable structure. Condition (1) and the function $\langle s, N\rangle \mapsto X_{s,N}$ are absolute since $\mathrm{L}_\nu(\mathbb{R})$ is correct about projective statements on reals and Γ_M is $\boldsymbol{\Pi}^1_2$ by Fact 1.10. Using the three conditions one can therefore identify Y inside $\mathrm{L}_\nu(\mathbb{R})[H][G]$. ⊣

COROLLARY 3.15. $\mathrm{L}(\mathbb{R})[H][G]$ *satisfies* DC_{ω_1} *for reals.* ⊣

REMARK 3.16. Our main interest is in Corollary 3.15. But the statement of the corollary is not $\Pi_1(\mathbb{R}\cup\{\mathbb{R}\})$ over $L(\mathbb{R})$. We had to prove the stronger Lemma 3.14 since its statement *is* $\Pi_1(\mathbb{R}\cup\{\mathbb{R}\})$, and its failure can be subjected to an application of Claim 3.11. This approach follows Woodin [17], who used it to prove DC_{ω_1} for reals in $L(\mathbb{R})[G]$.

Since every element of $L(\mathbb{R})[H][G]$ is definable in this model from H, G, ordinals, and real parameters, it follows from Corollary 3.15 that the full axiom DC_{ω_1} holds in $L(\mathbb{R})[H][G]$. Let $\mathbb{Q}$ be the poset $\mathrm{col}(\omega_2, \hat{\mathbb{R}})$ as computed in $L(\mathbb{R})[H][G]$. Let F be $\mathbb{Q}$–generic over $L(\mathbb{R})[H][G]$. $\mathbb{Q}$ is ω_1 closed. Using DC_{ω_1} in $L(\mathbb{R})[H][G]$, the standard forcing proof shows that the addition of F does not add reals and does not collapse ω_2. Thus $L(\mathbb{R})[H][G][F]$ continues to satisfy "$\boldsymbol{\delta}^1_3 = \omega_2$." Since the reals can be wellordered in $L(\mathbb{R})[H][G][F]$, and since every element of the model is definable from H, G, F, and ordinal and real parameters, $L(\mathbb{R})[H][G][F]$ satisfies the axiom of choice. We have:

THEOREM 3.17 (Neeman, Woodin). $\mathsf{ZFC}+\mathsf{AD}^{L(\mathbb{R})}$ *is consistent with* $\boldsymbol{\delta}^1_3 = \omega_2$. $\dashv$

Similar results hold for $\boldsymbol{\delta}^1_n$, $n > 3$, and for larger cardinals up to $\boldsymbol{\delta}^2_1$. In fact every Suslin cardinal of $L(\mathbb{R})$ can be collapsed to ω_1 without collapsing its successor, by an argument which for the most part is similar to the one above collapsing ω_ω. These results, and Theorem 3.17, are due independently to Woodin and the author.

§4. Uniqueness. For $\lambda \le \boldsymbol{\delta}^2_1$ the construction in Section 3 generalizes, using Facts 1.9 and 1.15, to produce a supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$. Next we extend the construction to cover all λ between $\boldsymbol{\delta}^2_1$ and Θ, using a directed system, discovered by Woodin, that captures $\mathrm{HOD}^{L(\mathbb{R})}$ up to levels arbitrarily high in Θ. We begin by describing the system.

Let $k < \omega$ and let $a \in H(\omega_1)$ be transitive. Define $T_k(a)$ to be the type of k Silver indiscernibles in $M^\sharp_\omega(a)$ with parameters in a.

FACT 4.1 (Woodin). For each $k < \omega$, the operation $a \mapsto T_k(a)$ belongs to $L(\mathbb{R})$, and is in fact definable over $L(\mathbb{R})$ from $k+1$ Silver indiscernibles for $\mathbb{R}$.

Let $W = \{$countable $M \mid M$ has a sharp for ω Woodin cardinals and no strict initial segment of M has a sharp for ω Woodin cardinals$\}$. Let $M \in W$. Recall that $\delta_0(M)$ is the first Woodin cardinal of M, and $\kappa_0(M)$ is the first cardinal strong to $\delta_0(M)$ in M. Suppose that M is nice to $\kappa_0(M)$ (see Definition 1.13). Let $\mathcal{T} = \langle M_\xi, E_\xi, j_{\zeta,\xi} \mid \zeta \ T \ \xi < \alpha+1\rangle$ be a proper iteration tree of length $\alpha+1$, leading to a final model M_α. Suppose that $\mathcal{T}$ is normal, meaning that $\langle \mathrm{lh}(E_\xi) \mid \xi < \alpha\rangle$ is increasing, and that α is a limit. Let $\delta(\mathcal{T})$ denote $\sup_{\xi<\alpha} \mathrm{lh}(E_\xi)$ and let $\Delta(\mathcal{T})$ denote $\bigcup_{\xi<\alpha} M_\xi \| \mathrm{lh}(E_\xi)$. $\Delta(\mathcal{T})$ is an initial segment of M_α, independent of the final branch $[0,\alpha]_{\mathcal{T}}$ used by the tree. We refer to it as the **lined-up part** of $\mathcal{T}$.

DEFINITION 4.2. $\mathcal{T}$ is k–**correct** if either:

1. $\mathcal{T}$ acts below $\kappa_0(M)$ and is consistent with Γ_M; or

2. All strict initial segments of $\mathcal{T}$ are consistent with Γ_M, $j_{0,\alpha}(\delta_0(M)) = \delta(\mathcal{T})$, and $j_{0,\alpha}$ sends $T_k(M\| \delta_0(M))$ to $T_k(\Delta(\mathcal{T}))$.

If condition (2) holds then we say that $\mathcal{T}$ is **full** and that the branch $[0,\alpha]_{\mathcal{T}}$ is k–**correct**.

For $P \in W$ define $\rho_k(M)$ to be the supremum of the ordinals $\alpha < \delta_0(P)$ which are definable from k indiscernibles (and no other parameters) in $M_\omega^\sharp(P\| \delta_0(P))$. $\rho_k(P)$ can be determined from knowledge of $T_k(P\| \delta_0(P))$, so the operation $P \mapsto \rho_k(P)$ belongs to $\mathrm{L}(\mathbb{R})$.

Fact 4.3. Let $M \in W$ and suppose that M is iterable. Let $\mathcal{T}$ be a full iteration tree on M. Let $\alpha + 1 = \mathrm{lh}(\mathcal{T})$. Then:

1. $\mathcal{T}\restriction\alpha$ is consistent with Σ_M.
2. Let $b = \Sigma_M(\mathcal{T}\restriction\alpha)$, let M_b be the direct limit of the models of $\mathcal{T}$ along b, and let $j_b\colon M \to M_b$ be the direct limit embedding. Then j_b and $j_{0,\alpha}$ agree to $\rho_k(M)$.

Condition (1) in Fact 4.3 follows from condition (2) in Definition 4.2, the fact that $\Sigma_M\restriction\kappa_0(M) = \Gamma_M\restriction\kappa_0(M)$, and the fact that every strict initial segment of $\mathcal{T}$ can be extended to a tree which acts below $\kappa_0(M)$. Condition (2) in Fact 4.3 is a condition of agreement between the "true" branch $\Sigma_M(\mathcal{T}\restriction\alpha)$, and the k–correct branch $[0,\alpha]_{\mathcal{T}}$ picked by $\mathcal{T}$. It is due to Woodin, using the investigation of iteration trees with distinct cofinal branches in Martin–Steel [6].

Remark 4.4. The ordinals $\rho_k(M)$, $k < \omega$, are cofinal in $\delta_0(M)$. It follows from this and condition (2) in Fact 4.3 that $\Sigma_M(\mathcal{T}\restriction\alpha)$ is the only cofinal branch through $\mathcal{T}\restriction\alpha$ which is k–correct for all k.

Definition 4.5. A k–**correct iteration sequence** on $M \in W$ is a sequence $\langle M_n, \mathcal{T}_n \mid n < \omega\rangle$ so that $M_0 = M$, and for each $n < \omega$, $\mathcal{T}_n$ is a k–correct iteration tree on M_n with final model M_{n+1}. Each of the models M_n is a k–**correct iterate** of M. The direct limit M_ω of the sequence is k–**wellfounded** if its membership relation is wellfounded on $\bigcup_{n<\omega} j_{n,\omega}{}''\rho_k(M_n)$.

M is k–**iterable** if:

1. Every k–correct iteration tree $\mathcal{T}$ of countable limit length on a k–iterate of M has a cofinal branch b so that $\mathcal{T}^\frown b$ is k–correct.
2. The direct limit of every k–correct iteration sequence on M is k–wellfounded.

Let $M \in W$ be k–iterable. Let $\mathcal{D} = \{\langle Q, x\rangle \mid Q$ is a k–iterate of M and $x \in Q\| \rho_k(Q)\}$. Define an equivalence relation $\sim$ on $\mathcal{D}$ by setting $\langle Q_1, x_1\rangle \sim \langle Q_2, x_2\rangle$ iff there are k–correct iteration trees $\mathcal{U}_1$ on Q_1 and $\mathcal{U}_2$ on Q_2, leading to final models Q_1^* and Q_2^*, and embeddings $j_1\colon Q_1 \to Q_1^*$ and $j_2\colon Q_2 \to Q_2^*$, so that $j_1(x_1) = j_2(x_2)$. Define a relation R on $\mathcal{D}$ similarly but replacing the final condition $j_1(x_1) = j_2(x_2)$ by $j_1(x_1) \in j_2(x_2)$.

Define M_∞^k to be the transitive collapse of the structure $(\mathcal{D}/{\sim}; R)$, and for every $\langle Q, x\rangle \in \mathcal{D}$ define $\pi^k_{Q,\infty}(x)$ to be the equivalence class of $\langle Q, x\rangle$. Parallels of Facts 1.11 and 1.12 for k–correct iterations show that these definitions make sense. If M is iterable then, using Fact 4.3, M_∞^k agrees with the limit $M_\infty^{(\delta_0(M))}$ of the "true" directed system mentioned at the end of Section 1, up to

$\pi^{(\delta_0(M))}_{M,\infty}(\rho_k(M))$. But the true directed system does not belong to $\mathrm{L}(\mathbb{R})$, while the k–correct system does. The k–correct systems were defined by Woodin, who used them to identify $\mathsf{HOD}^{\mathrm{L}(\mathbb{R})}$ as the join of a fine structural model with a restriction of its iteration strategy. Among other things he showed:

FACT 4.6 (Woodin, see Steel [14]). M^k_∞ is an initial segment of $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}(u)$. Moreover there are ordinals ν_k, $k < \omega$, independent of M and cofinal in Θ, so that $\mathrm{ON} \cap M^k_\infty \geq \nu_k$ for each k.

Using Fact 4.6 we can generalize the construction in Section 3 to $\lambda > \boldsymbol{\delta}^2_1$.

Fix $\lambda < \Theta$. Fix k so that $\nu_k > \lambda$. Let K be the set of k–iterable models in W. For each $M \in K$ set $a(M) = \lambda \cap \pi^k_{M,\infty}{}''\rho_k(M)$. Define further:

1. $a(M_n \mid n < \omega) = \bigcup_{n<\omega} a(M_n)$.
2. $C_M = \{a(M_n \mid n < \omega) \mid \langle M_n, \mathcal{T}_n \mid n < \omega\rangle$ is a k–correct iteration sequence on $M\}$.

REMARK 4.7. Let $\langle M_n, \mathcal{T}_n \mid n < \omega\rangle$ be a k–correct iteration sequence on M. Let $j_{n,n+1} \colon M_n \to M_{n+1}$ be the embedding generated by the tree $\mathcal{T}_n$. Let M_ω be the direct limit of the system $\langle M_n, j_{n,n+1} \mid n < \omega\rangle$. If M is iterable and the trees $\mathcal{T}_n$ are all consistent with Σ_M, then $M_\omega \in K$ and $a(M_n \mid n < \omega) = a(M_\omega)$. Condition (1) in this case is an exact parallel of condition (1) at the start of Section 3. We need the more complicated condition here to allow for the case of trees which are k–correct but not according to Σ_M (remember that the former property can be recognized in $\mathrm{L}(\mathbb{R})$, but the latter cannot) and for M which are k–iterable but not iterable.

CLAIM 4.8. *Let $M \in K$. Let u be a real Turing above a real coding M and a real defining λ (from Silver indiscernibles for $\mathbb{R}$). Let $P = M^\sharp_\omega(u)$. Then $a(P)$ belongs to C_M.*

PROOF. By a back-and-forth argument construct k–correct iteration sequences $\langle M_n, \mathcal{T}_n \mid n < \omega\rangle$ and $\langle P_n, \mathcal{U}_n \mid n < \omega\rangle$ so that $a(M_n \mid n < \omega) = a(P_n \mid n < \omega)$. Without loss of generality suppose that the trees $\mathcal{U}_n$ are consistent with Σ_P. (If not, simply change their final branches to be the ones given by Σ_P. The change does not affect $a(P_n \mid n < \omega)$.) Let $j_{n,n+1} \colon P_n \to P_{n+1}$ be the embeddings generated by the trees $\mathcal{U}_n$. Let P^* be the direct limit of the chain $\langle P_n, j_{n,n+1} \mid n < \omega\rangle$. Let $j \colon P \to P^*$ be the direct limit embedding.

Since $a(M_n \mid n < \omega) = a(P^*)$, $a(P^*)$ belongs to C_M. P^* is a Σ_P–iterate of P and therefore iterable itself. An argument using Corollary 1.3, similar to the one in Claim 3.2, deduces $a(P) \in C_M$ from $a(P^*) \in C_M$. ⊣

COROLLARY 4.9. *The collection C_M, $M \in K$, has the countable intersection property.* ⊣

Let $\mathcal{F}$ be the filter generated by the sets C_M.

CLAIM 4.10. *$\mathcal{F}$ is a supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ in* $\mathrm{L}(\mathbb{R})$.

PROOF. Similar to the proofs in Section 3, using the same switch from a k–correct iteration to a Σ_P–iteration that was used in the proof of Claim 4.8. (These switches can be made so long as P is iterable.) ⊣

Remark 4.11. The proof of Claim 4.10 in fact shows that the collection $\{C_P \mid P \in W$ and P is iterable$\}$ generates $\mathcal{F}$. We used the larger collection $\{C_M \mid M \in K\}$ in the definitions since k–iterability can be recognized in $\mathrm{L}(\mathbb{R})$, while iterability cannot.

Lemma 4.12. *Let μ be a supercompactness measure over $\mathcal{P}_{\omega_1}(\lambda)$ in $\mathrm{L}(\mathbb{R})$. Then $\mu = \mathcal{F}$.*

Proof. We adapt the proof of Woodin [16] to apply to $\mathcal{F}$. Suppose that $\mu \neq \mathcal{F}$, and fix some $A \subset \mathcal{P}_{\omega_1}(\lambda)$ on which the two differ. Switching to the complement of A if necessary we may assume that $\mu(A) = 1$ and $A \notin \mathcal{F}$. There is then some k–correct M so that $A \cap C_M = \emptyset$.

For each $x \in \mathcal{P}_{\omega_1}(\lambda)$ define the game G_x to be played according to Diagram 4 and the following rules:

1. (Rule for I.) $\alpha_n \in x$.
2. (Rule for II.) $\langle M_n, \mathcal{T}_n \mid n < \omega\rangle$ is a k–correct iteration sequence on M.
3. (Rule for II.) $\alpha_n \in a(M_n)$ and $a(M_n) \subset x$.

Infinite runs of G_x are won by player II.

I	α_1		α_2		$\cdots$	
II		$\mathcal{T}_0, M_1$		$\mathcal{T}_1, M_2$		$\cdots$

Diagram 2. The game G_a

Claim 4.13. *If* II *has a winning quasi-strategy in G_x then $x \in C_M$.*

Proof. Using II's winning quasi-strategy and DC_ω construct an infinite play $\langle \alpha_{n+1}, \mathcal{T}_n, M_{n+1} \mid n < \omega\rangle$ subject to the rules of G_x with $\{\alpha_n \mid 1 \leq n < \omega\} = x$. Then $a(M_n \mid n < \omega) = x$ and therefore $x \in C_M$. ⊣

Player I is the open player in G_x, and her moves come from a wellordered set. Thus, in $\mathrm{L}(\mathbb{R})$, either I has a winning strategy in G_x or else II has a winning quasi-strategy in the game. Since $A \cap C_M = \emptyset$, it follows from the last claim that I has a winning strategy in G_x for every $x \in A$. As I is the open player and her moves come from a wellordered set, she has a canonical winning strategy definable from G_x. Thus there is a function $x \mapsto \sigma_x$ in $\mathrm{L}(\mathbb{R})$ so that for each $x \in A$, σ_x is a winning strategy for I in G_x.

By a **position** we mean a sequence $p = \langle \alpha_1, \mathcal{T}_0, M_1, \ldots, \alpha_l, \mathcal{T}_{l-1}, M_l\rangle$ so that $\alpha_n \in a(M_n)$ for each $n \leq l$. For x so that p is consistent with σ_x define $f_p(x) = \sigma_x(p)$. The rules of G_x require $\alpha_{l+1} \in x$, so f_p is a regressive function.

Set $A_0 = A$. Using the normality of the supercompactness measure μ construct a sequence $\langle \alpha_{n+1}, A_{n+1}, \mathcal{T}_n, M_{n+1} \mid n < \omega\rangle$ so that for each $l < \omega$:

1. For every $x \in A_l$, $p_l = \langle \alpha_1, \mathcal{T}_0, M_1, \ldots, \alpha_l, \mathcal{T}_{l-1}, M_l\rangle$ is consistent with σ_x.
2. $A_{l+1} \subset A_l$, $\mu(A_{l+1}) = 1$, and f_{p_l} is constant on A_{l+1}.
3. α_{l+1} is the constant value taken by f_{p_l} on A_{l+1}.
4. $\mathcal{T}_l$ is a k–correct iteration tree on M_l, leading to a final model M_{l+1} so that $\alpha_{l+1} \in a(M_{l+1})$.

Let $A = \bigcap_{n<\omega} A_n$ and let $a = \bigcup_{n<\omega} a(M_n)$. Since μ is ω–complete, $\mu(A) = 1$. Since μ is fine, there is $x \in A$ with $x \supset a$. Fix such an x. By condition (3), condition (4), and the definition of the functions f_p, the sequence $\langle \alpha_{n+1}, \mathcal{T}_n, M_{n+1} \mid n < \omega \rangle$ is an infinite play of G_x consistent with σ_x. But this is a contradiction as σ_x is a winning strategy for player I, the open player in G_x. ⊣

COROLLARY 4.14. *For each* $\lambda < \Theta^{\mathrm{L}(\mathbb{R})}$ *there is a unique supercompactness measure over* $\mathcal{P}_{\omega_1}(\lambda)$ *in* $\mathrm{L}(\mathbb{R})$. ⊣

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