Reverse Mathematics

Steel forcing

INDEC

Implication

Consistency

Summary

Steel forcing in reverse mathematics

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Outline

Steel forcing provides a powerful method for constructing models of certain axioms in reverse mathematics. Used initially to separate the strengths of axioms. More recently used to discover the strength, relative to standard axioms, of INDEC, the first non-logical statement shown to be a theorem of hyperarithmetic analysis.

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Primary reference, Simpson [1999].

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Work over a weak base system. Various standard axioms provide strengthening. Given a theorem Φ, find, ideally, a standard axiom A so that, over the base system:

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Conceptually similar to consistency proofs in set theory. But concerned mainly with theorems of analysis (second order number theory).

Reverse mathematics, continued

Some theorems addressed by reverse mathematics:

- Heine-Borel theorem on $[0, 1]$.
- Sequential completeness of $\mathbb{R}$.
- Bolzano–Weierstrass theorem.
- The perfect set theorem.
- Open determinacy.
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Reverse mathematics measures how much of this extra strength is needed for each theorem.

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With choice of K, powerful way to produce models of hyperarithmetic analysis.

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Theorem (Steel [1977, 1978])

Δ^1_1 comprehension does not imply Σ^1_1 choice.

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Weak Σ^1_1 choice holds in M_K.

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On the other hand.... let φ be a Σ^1_1 statement with parameters in $M[T, F]$, and a *unique* witness in M_K. This witness must belong to $M[T, F]$.

Weak Σ^1_1 choice holds in M_K.

Theorem (Van Wesep [1977])

Weak Σ^1_1 choice does not imply Δ^1_1 comprehension.

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Most importantly Montalbán discovered the first "natural" theorem of hyperarithmetic analysis. The next topic describes the theorem.

Basic Definitions

Work throughout with countable linear orders.

Definition

- A linear order $(U; <_U)$ is scattered if it does not embed $\mathbb{Q}$.
- A gap in U is a partition of U into sets L and R, closed leftward and rightward respectively.
- A gap $\langle L, R\rangle$ is a decomposition of U if U does not embed into L, and does not embed into R.
- U is **in**decomposable if, for every gap $\langle L, R\rangle$, U embeds into either L or R.

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Note

If U is scattered, it cannot embed into both L and R.

Jullien's Indecomposability Theorem

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Used classically for classifying linear orders. More recently by Montalbán working on strength of Fraïssé's conjecture.

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Proof of INDEC, continued

For each a, U embeds into either L_a or R_a, not both.

$$R^* = \{a \mid U \text{ embeds into } L_a\},$$

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Fix $a \in R^*$ (possible since $R^* \neq \emptyset$). Let $b = \sigma(a) \in L^*$.

Then $\mathrm{range}(\sigma^2)$ is to the left of b. So U embeds into L_b. Since $b \in L^*$, U also embeds into R_b. Contradiction.

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INDEC is thus a theorem of hyperarithmetic analysis. It is the first "natural" example of such a theorem.

Questions

1. Δ^0_1 comprehension.
2. Weak König lemma.
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4. Jump iteration: Jump iteration: Turing jumps exist, and existing iterations can be continued.
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Does it imply Δ^1_1 comprehension? can it be proved from weak Σ^1_1 choice?

From INDEC

Theorem

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Suppose $(\forall n)(\exists! x)\varphi(n, x)$, arithmetic φ.

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Construct, in RCA$_*$, a linear order $(U; <_U)$ so that:

1. U is scattered.
2. $L^* = \{a \mid U$ embeds into $R_a\}$ and $R^* = \{a \mid U$ embeds into $L_a\}$ form a non-trivial gap in U.
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By INDEC, $\langle L^*, R^*\rangle$ exists, hence $\langle y_n \mid n < \omega\rangle$ exists.

From INDEC, continued

General tactic

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Can only get scatteredness indirectly:

From uniqueness of $\langle y_n \mid n < \omega \rangle$ get U has only countably many branches. Hence $<_U$ is scattered. □

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The branches of T in M_K are those in K. No infinite sequence of branches in M_K.

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h is Δ^1_1 over M_K, so Δ^1_1 comprehension fails. Use homogeneity, scatteredness, and properties of Steel forcing, to argue INDEC holds.

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Used Steel forcing. Many other recent uses, see Montalbán [2006], [2008].

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