HIERARCHIES OF FORCING AXIOMS I

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ABSTRACT. We prove new upper bound theorems on the consistency strengths of SPFA(θ), SPFA(θ-linked) and SPFA(θ^+-cc). Our results are in terms of (θ, Γ)-subcompactness, which is a new large cardinal notion that combines the ideas behind subcompactness and Γ-indescribability. Our upper bound for SPFA($\mathfrak{c}$-linked) has a corresponding lower bound, which is due to Neeman and appears in his follow-up to this paper. As a corollary, SPFA($\mathfrak{c}$-linked) and PFA($\mathfrak{c}$-linked) are each equiconsistent with the existence of a Σ^2_1-indescribable cardinal. Our upper bound for SPFA($\mathfrak{c}$-c.c.) is a Σ^2_2-indescribable cardinal, which is consistent with $V = L$. Our upper bound for SPFA($\mathfrak{c}^+$-linked) is a cardinal κ that is (κ^+, Σ^2_1)-subcompact, which is strictly weaker than κ^+-supercompact. The axiom MM($\mathfrak{c}$) is a consequence of SPFA($\mathfrak{c}^+$-linked) by a slight refinement of a theorem of Shelah. Our upper bound for SPFA($\mathfrak{c}^{++}$-c.c.) is a cardinal κ that is (κ^+, Σ^2_2)-subcompact, which is also strictly weaker than κ^+-supercompact.

1. Getting started

To better understand the Semi-proper Forcing Axiom (SPFA) we break up SPFA into SPFA($\mathcal{C}$) for various classes $\mathcal{C}$, where SPFA($\mathcal{C}$) is the statement that for all $\mathbb{P} \in \mathcal{C}$, if $\mathbb{P}$ is semi-proper poset and $\mathcal{D}$ is a family of maximal antichains of $\mathbb{P}$ such that $|\mathcal{D}| = \aleph_1$, then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$. We treat notation regarding the Proper Forcing Axiom (PFA) and Martin's Maximum (MM) similarly.

Definition 1. A poset $\mathbb{P} = (P, <_{\mathbb{P}})$ is *θ-linked* iff there is a function $\ell : P \to \theta$ such that if $\ell(p) = \ell(q)$, then p and q are compatible in $\mathbb{P}$.

Observe that if $\mathbb{P}$ has cardinality θ, then $\mathbb{P}$ is θ-linked, and if $\mathbb{P}$ is θ-linked, then $\mathbb{P}$ is θ^+-cc. These three classes of posets are the most relevant to this paper and we abbreviate the corresponding semi-proper forcing axioms SPFA(θ^+-cc), SPFA(θ-linked) and SPFA(θ).

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Definition 2. Let λ be a cardinal such that $2^{<\lambda} = \lambda$. Then λ is Σ^2_1-*indescribable* iff for all $Q \subseteq H_\lambda$ and first order formulas $\varphi(x)$, if there exists $B \subseteq H_{\lambda^+}$ such that

$$(H_{\lambda^+}, B) \models \varphi(Q),$$

then there are $\kappa < \lambda$ and $A \subseteq H_{\kappa^+}$ such that

$$(H_{\kappa^+}, A) \models \varphi(Q \cap H_\kappa).$$

Definition 3. Let $\mathbb{I}$ be a forcing iteration of length λ. Then $\mathbb{I}$ is a *semi-proper* iteration iff $\mathbb{I}$ is a revised countable support iteration and

$$\Vdash_{\mathbb{I}\restriction\kappa} \mathbb{I}_\kappa \text{ is a semi-proper poset}$$

for all $\kappa < \lambda$.

Definition 4. Let λ be a regular cardinal and $\mathbb{I}$ be a forcing iteration. Then $\mathbb{I}$ is *amenable to* H_λ iff $\mathbb{I}$ has length $\leq \lambda$ and

$$\Vdash_{\mathbb{I}\restriction\kappa} \mathbb{I}_\kappa \in H_\lambda$$

for all $\kappa <$ the length of $\mathbb{I}$.

Theorem 5. *Let λ be a Σ^2_1-indescribable cardinal and assume that $2^\lambda = \lambda^+$. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \text{SPFA}(\mathfrak{c}\text{-linked}).$$

Moreover,

$$V^{\mathbb{I}} \models \lambda = \mathfrak{c} = \aleph_2.$$

That $\mathfrak{c} = \aleph_2$ follows from PFA($\mathfrak{c}$) by a theorem of Todorcevic and Velickovic (See [2] and [15]). Theorem 5 is one half of an equiconsistency, the other half of which is due to Neeman [8], who showed that if PFA($\mathfrak{c}$-linked) holds, then ω_2^V is a Σ^2_1-indescribable cardinal in L.

Corollary 6. *The following three theories are equiconsistent.*

$$\text{ZFC} + \text{SPFA}(\mathfrak{c}\text{-linked})$$

$$\text{ZFC} + \text{PFA}(\mathfrak{c}\text{-linked})$$

$$\text{ZFC} + \textit{There is a } \Sigma^2_1\textit{-indescribable cardinal.}$$

The consistency strengths of PFA($\mathfrak{c}$) and SPFA($\mathfrak{c}$) are somewhere between a Π^1_1 indescribable (weakly compact) cardinal and a Σ^1_2 indescribable cardinal. The upper bound follows from Corollary 6. The lower bound combines results of Jensen, Todorcevic and Velickovic on the cardinality of the continuum and the failure of $\square(\omega_2)$. It would be interesting and seemingly within the reach of current techniques to know the consistency strengths of PFA($\mathfrak{c}$) and SPFA($\mathfrak{c}$) exactly.

Recall from Schimmerling-Zeman [10] that a cardinal λ is *subcompact* iff for all $Q \subseteq H_{\lambda^+}$, there exists $\kappa < \lambda$, $P \subseteq H_{\kappa^+}$ and an elementary embedding $\pi : (H_{\kappa^+}, P) \to (H_{\lambda^+}, Q)$ with $\mathrm{crit}(\pi) = \kappa$. For comparison, if λ is subcompact, then λ is weakly compact (but not necessarily measurable) and the class of superstrong cardinals is stationary in λ. By Burke [3], if λ is a subcompact cardinal, then $\square_\lambda$ fails.

Definition 7. Let $\lambda \leq \theta$ be cardinals. Then λ is (θ, Σ^2_1)-*subcompact* iff for all $Q \subseteq H_\theta$ and first order formulas $\varphi(x)$, if there exists $B \subseteq H_{\theta^+}$ such that

$$(H_{\theta^+}, B) \models \varphi(Q),$$

then there exist $\kappa \leq \eta < \lambda$, $P \subseteq H_\eta$ and $A \subseteq H_{\eta^+}$ such that

$$(H_{\eta^+}, A) \models \varphi(P)$$

and there exists an elementary embedding

$$\pi : (H_\eta, P) \to (H_\theta, Q)$$

with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ and $\pi(\kappa) = \lambda$. We also say that $[\lambda, \theta]$ is a Σ^2_1-*indescribable* interval of cardinals in this case.

Literally, to say that $\pi(\kappa) = \lambda$ does not make sense if $\kappa = \eta$, which would be the case iff $\lambda = \theta$. In this case, we interpret the final clause of Definition 7 as saying that the identity map from $(H_\kappa, Q \cap H_\kappa)$ to (H_λ, Q) is an elementary embedding. In other words, that

$$(H_\kappa, Q \cap H_\kappa) \prec (H_\lambda, Q).$$

Lemma 8. *λ is (λ, Σ^2_1)-subcompact iff λ is Σ^2_1-indescribable.*

Proof. The left to right direction is clear from the definitions. Assume that λ is Σ^2_1-indescribable. Let $Q \subseteq H_\lambda$ and $\varphi(x)$ be a first order formula. Suppose that there exists $B \subseteq H_{\lambda^+}$ such that $(H_{\lambda^+}, B) \models \varphi(Q)$. Let

$$T = \{(\ulcorner \psi \urcorner, c) \in \omega \times H_\lambda \mid (H_\lambda, Q) \models \psi(c)\}$$

and $\tau(Q, T)$ be the first order sentence that says: if λ is the largest cardinal, then T is the first order theory of (H_λ, Q). Then $H_{\lambda^+} \models \tau(Q, T)$. By Σ^2_1-indescribability, there are $\kappa < \lambda$ and $A \subseteq H_{\kappa^+}$ such that

$$(H_{\kappa^+}, A) \models \varphi(Q \cap H_\kappa) \wedge \tau(Q \cap H_\kappa, T \cap H_\kappa).$$

Then $(H_\kappa, Q \cap H_\kappa) \prec (H_\lambda, Q \cap H_\lambda)$, which is what is needed to witness (λ, Σ^2_1)-subcompactness. $\square$

Here are some comparisons with subcompact and supercompact cardinals. Perhaps the most interesting of these is Lemma 10(1), which implies that the least cardinal κ that is (κ^+, Σ^2_1)-subcompact is strictly less than the least cardinal λ that is λ^+-supercompact. In a nutshell, this is because κ being (κ^+, Σ^2_1)-subcompact can be witnessed by a family of superstrong extenders whereas λ being λ^+-supercompact can only be witnessed by an extender that is beyond superstrong, i.e., an extender with ultrafilters that concentrate on ordinals $> \lambda$.

While subcompact cardinals are not necessarily measurable, if λ is (λ^+, Σ^2_1)-subcompact, then λ is measurable assuming GCH as the next lemma shows.

Lemma 9. *Assume* GCH. *If λ is (λ^+, Σ^2_1)-subcompact, then λ is subcompact and for every $\nu < \lambda^{++}$, there is an extender E with* $\mathrm{crit}(E) = \lambda$ *and*

$$\sup(\{\xi \mid \xi \textit{ is a generator of } E\}) > \nu.$$

Proof. The fact that λ is subcompact is obvious from the definitions and does not use GCH. Let $\varphi(\lambda)$ be the assertion that there exists $\nu < \lambda^{++}$ such that for every extender E with critical point λ, the generators of E are bounded by ν. Then $\varphi(\lambda)$ is absolute for $H_{\lambda^{++}}$. For contradiction, suppose that $\varphi(\lambda)$ holds. Pick $\kappa < \lambda$ and $\pi : H_{\kappa^+} \to H_{\lambda^+}$ such that $\mathrm{crit}(\pi) = \kappa$ and $\varphi(\kappa)$ holds in $H_{\kappa^{++}}$. Let E be the superstrong extender derived from π. That is,

$$E = \{(a, X) \mid a \in [\lambda]^{<\omega}, X \subseteq [\kappa]^{|a|} \text{ and } a \in \pi(X)\}.$$

Then the generators of E are unbounded in λ. In particular, $\varphi(\kappa)$ fails in $H_{\kappa^{++}}$. □

Lemma 10. *Assume* GCH.

(1) *If λ is λ^+-supercompact, then the set of $\kappa < \lambda$ such that κ is (κ^+, Σ^2_1)-subcompact is stationary in λ.*
(2) *If λ is λ^{++}-supercompact, then λ is (λ^+, Σ^2_1)-subcompact .*

Proof. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \lambda$ and ${}^{\lambda^+}M \subseteq M$. Observe that $H_{\lambda^{++}} = H^M_{\lambda^{++}}$. Let $\pi = j \restriction H_{\lambda^+}$. Then $\pi \in M$ by the closure of M under λ^+-sequences. Consider an arbitrary $Q \subseteq H_{\lambda^+}$. Then

$$\pi : (H_{\lambda^+}, Q) \to (H^M_{j(\lambda^+)}, j(Q))$$

is elementary. Consider an arbitrary $B \subseteq H_{\lambda^{++}}$ such that

$$(H_{\lambda^{++}}, B) \models \varphi(Q).$$

Then

$$(H^M_{j(\lambda^{++})}, j(B)) \models \varphi(j(Q)).$$

Assume that $B \in M$. Then, since j is elementary, there exist $\kappa < \lambda$, $P \subseteq H_{\kappa^+}$, $A \subseteq H_{\kappa^{++}}$ and an elementary embedding

$$\pi' : (H_{\kappa^+}, P) \to (H_{\lambda^+}, Q)$$

such that

$$(H_{\kappa^{++}}, A) \models \varphi(P).$$

Since $\pi' \in H_{\kappa^+} \subset M$, this shows that λ is (λ^+, Σ^2_1)-subcompact in M. Hence, in terms of the the normal measure derived from j, almost every $\kappa < \lambda$ is (κ^+, Σ^2_1)-subcompact. From this, (1) is immediate. For (2), notice that if $H^M_{\lambda^{+++}} = H_{\lambda^{+++}}$, then the fact that λ is (λ^+, Σ^2_1)-subcompact in M is absolute to V. □

Lemma 11. *Assume* GCH*.*

(1) *If λ is $(\lambda^{++}, \Sigma^2_1)$-subcompact, then the set of $\kappa < \lambda$ such that κ is κ^+-supercompact is stationary in λ.*
(2) *If λ is $(\lambda^{+++}, \Sigma^2_1)$-subcompact, then λ is λ^+-supercompact.*

Proof. Let $\pi : H_{\kappa^{++}} \to H_{\lambda^{++}}$ be an elementary embedding such that $\mathrm{crit}(\pi) = \kappa$. One easily sees that the set of such κ is stationary in λ. Then κ is κ^+-supercompact as witnessed by

$$U = \{X \subseteq \mathcal{P}_\kappa(\kappa^+) \mid \pi[\kappa^+] \in \pi(X)\}.$$

Thus (1) holds. If, instead, $\pi : H_{\kappa^{+++}} \to H_{\lambda^{+++}}$, then λ is λ^+-supercompact as witnessed by $\pi(U)$. This gives (2). □

Here are two remarks about Lemmas 10 and 11. First, notice that the hypotheses are stronger than what is used in the proofs. Second, the lemmas generalize to θ-supercompactness and (θ, Σ^2_1)-subcompactness. These generalizations show that the two hierarchies are nested in a particular way.

The following is the main result of the paper. Its proof is given in Section 2.

Theorem 12. *Let $\lambda \leq \theta$ be cardinals such that $\theta^{<\lambda} = \theta$ and $2^\theta = \theta^+$. Suppose that λ is (θ, Σ^2_1)-subcompact. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \mathrm{SPFA}(\theta\text{-linked})$$

and

$$V^{\mathbb{I}} \models \lambda = \mathfrak{c} = \aleph_2.$$

Theorem 5 is the $\theta = \lambda$ case of Theorem 12. The following corollary is the $\theta = \lambda^+$ case.

Corollary 13. *Suppose that λ is a (λ^+, Σ^2_1)-subcompact cardinal and $2^{\lambda^+} = \lambda^{++}$. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \text{SPFA}(\mathfrak{c}^+\text{-linked}).$$

It is worth emphasizing that no degree of supercompactness is assumed in Corollary 13. As a partial converse, Neeman [8] showed that if V is an extender model and there is a proper poset $\mathbb{I} \subset H_\lambda$ such that

$$V^{\mathbb{I}} \models \lambda = \aleph_2 = \mathfrak{c} \text{ and PFA}(\mathfrak{c}^+\text{-linked}) \text{ holds,}$$

then λ is a (λ^+, Σ^2_1)-subcompact cardinal in V.

Conjecture 14. The theories

$$\text{ZFC} + \text{PFA}(\mathfrak{c}^+\text{-linked})$$

and

$$\text{ZFC} + \text{There is a cardinal } \lambda \text{ that is } (\lambda^+, \Sigma^2_1)\text{-subcompact}$$

are equiconsistent.

Steel showed that if PFA($\mathfrak{c}^+$) holds, then there is an inner model with infinitely many Woodin cardinals; Schimmerling [9] had shown earlier that for all $n < \omega$, there is an inner model with n Woodin cardinals.

Shelah [12] proved that SPFA implies MM. The following is a slight refinement, the proof of which can be found in Section 2.

Theorem 15. *Let η be a regular cardinal and $\mathbb{P}$ be a stationary preserving poset. Suppose that $\mathbb{P} \subseteq H_\eta$ and $\mathbb{P}$ is η-cc. Let $\theta = |H_\eta|^{\aleph_0}$. Assume* SPFA($\theta$-linked)*. Then $\mathbb{P}$ is semi-proper.*

Corollary 16. SPFA($\mathfrak{c}^+$-linked) *implies* MM($\mathfrak{c}$).

Thus Corollary 13 also provides a new upper bound on the consistency strength of MM($\mathfrak{c}$): again it is a cardinal λ that is (λ^+, Σ^2_1)-subcompact. Steel and Zoble showed that if MM($\mathfrak{c}$) holds, then there is an inner model with infinitely many Woodin cardinals. Woodin [16] had shown earlier that for every $n < \omega$, there is an inner model with n Woodin cardinals. Woodin [16] also gave a different sort of consistency proof of MM($\mathfrak{c}$) starting from the theory

$$\text{ZF} + \text{AD}_{\mathbb{R}} + \Theta \text{ is regular}$$

where

$$\Theta = \sup(\{\alpha \in \text{ON} \mid \text{ there is a surjection } f : \mathbb{R} \to \alpha\})$$

and using his forcing notion $\mathbb{P}_{\max}$.

The following corollary is the $\theta = \lambda^{++}$ case of Theorem 12.

Corollary 17. *Suppose that λ is a $(\lambda^{++}, \Sigma^2_1)$-subcompact cardinal and $2^{\lambda^{++}} = \lambda^{+++}$. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \text{SPFA}(\mathfrak{c}^{++}\text{-linked}).$$

As a lower bound, Jensen, Schimmerling, Schindler and Steel recently showed that PFA($\mathfrak{c}^{++}$) implies that there is a transitive proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.

The following is a corollary to the proof of Theorem 12. The proof can be found in Section 2.

Corollary 18. *Suppose that there is a proper class of cardinals θ such that $\theta^{<\lambda} = \theta$, $2^\theta = \theta^+$ and λ is a (θ, Σ^2_1)-subcompact cardinal. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that $V^{\mathbb{I}} \models$ SPFA.*

Using Corollary 18, one can recover Shelah's theorem that SPFA is consistent relative to the existence of a supercompact cardinal. (See [11], [5] or [4].) Our method of proof is not unrelated or simpler than Shelah's; rather, it is a more elaborate proof of a similar sort.

If we remove "semi-" and "revised" from Definition 3 then we arrive at the definition of a "proper iteration". We remark that Theorem 12 and its consequences, Theorem 5 and Corollaries 13, 17 and 18, remain true if we substitute "proper" for "semi-proper" and "PFA" for "SPFA" in their statements. This fact is relevant to [8].

2. The linkage hierarchy

In this section, after we develop a bit of general theory, we prove Theorem 12, Corollary 18 and Theorem 15. The following sort of fast function will be used in the proof of Theorem 12 in the definition of the semi-proper iteration $\mathbb{I}$.

Definition 19. Let $\lambda \leq \theta$ be cardinals and $f : \lambda \to \lambda$ be a cardinal valued function. Then f is (θ, Σ^2_1)-*fast* iff for all $Q \subseteq H_\theta$ and first order formulas $\varphi(x)$, if there exists $B \subseteq H_{\theta^+}$ such that

$$(H_{\theta^+}, B) \models \varphi(Q),$$

then there exist $\kappa \leq \eta \leq f(\kappa)$, $P \subseteq H_\eta$ and $A \subseteq H_{\eta^+}$ such that

$$(H_{\eta^+}, A) \models \varphi(P)$$

and there exists an elementary embedding

$$\pi : (H_\eta, P) \to (H_\theta, Q)$$

with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ and $\pi(\kappa) = \lambda$.

Clearly, if there is a (θ, Σ^2_1)-fast function $f : \lambda \to \lambda$, then λ is (θ, Σ^2_1)-subcompact. The following lemma is the converse.

Lemma 20. *Let $\lambda \leq \theta$ be cardinals. Suppose that λ is (θ, Σ^2_1)-subcompact. Then there is a function $f : \lambda \to \lambda$ such that f is (θ, Σ^2_1)-fast.*

Proof. By induction on θ. Assume that for all $\zeta < \theta$, there is a function $f : \lambda \to \lambda$ such that f is (ζ, Σ^2_1)-fast. For contradiction, suppose that there is no function $f : \lambda \to \lambda$ such that f is (θ, Σ^2_1)-fast.

Notice that $\theta >> \lambda$. For example, suppose that there are no inaccessible cardinals between λ and θ and λ is a (θ, Σ^2_1)-subcompact cardinal. Let $f(\kappa)$ be the least inaccessible cardinal strictly greater than κ. Then $f : \lambda \to \lambda$ is a (θ, Σ^2_1)-fast function. (The reader who is only interested in special cases such as $\theta = \lambda^+$ and its relationship to SPFA($\mathfrak{c}^+$-linked) has as much of Lemma 20 as he needs at this point.)

We digress to define a different kind of conjunction for Σ^2_1 statements. Consider an arbitrary cardinal η. Our first step towards defining Σ^2_1-conjunction is to consider a single Σ^2_1 statement. Let φ be a first order formula and $P \subseteq H_\eta$. Suppose that there exists $A \subseteq H_{\eta^+}$ such that

$$(H_{\eta^+}, A) \models \varphi(P).$$

This is equivalent to saying that there exist $A, T \subseteq H_{\eta^+}$ such that

$$T = \{(\ulcorner\psi\urcorner, p_1, \ldots, p_k) \mid (H_{\eta^+}, A) \models \psi(p_1, \ldots, p_k)\}$$

and

$$(\ulcorner\varphi\urcorner, P) \in T.$$

The first displayed formula says that T is the elementary diagram of (H_{η^+}, A). Uniformly in A and T, this can be expressed as a first order sentence over (H_{η^+}, A, T). This sentence, which we call Δ, says that for every Gödel number $\ulcorner\psi\urcorner$ of a first order formula $\psi(x_1, \ldots, x_k)$ and all parameters $p_1, \ldots, p_k \in H_{\eta^+}$,

- if ψ is $x_i \in x_j$, then
$$T(\ulcorner\psi\urcorner, \bar{p}) \iff p_i \in p_j,$$
- if ψ is $A(x_i)$, then
$$T(\ulcorner A(x_i)\urcorner, \bar{p}) \iff A(p_i),$$
- if ψ is $\neg\chi$, then
$$T(\ulcorner\psi\urcorner, \vec{p}) \iff \neg T(\ulcorner\chi\urcorner, \bar{p}),$$

- if ψ is $\chi \wedge \chi'$, then
$$T(\ulcorner\psi\urcorner, \bar{p}) \iff T(\ulcorner\chi\urcorner, \bar{p}) \wedge T(\ulcorner\chi'\urcorner, \bar{p})$$
and
- if ψ is $\exists x_i \chi$, then
$$T(\ulcorner\psi\urcorner, p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_k) \iff$$
$$\text{there exists } p_i \text{ such that } T(\ulcorner\chi\urcorner, \bar{p}).$$

Thus, the following Σ^2_1 statements about H_η are equivalent.

(i) There exists $A \subseteq H_{\eta^+}$ such that
$$(H_{\eta^+}, A) \models \varphi(P).$$

(ii) There exist $A, T \subseteq H_{\eta^+}$ such that
$$(H_{\eta^+}, A, T) \models \Delta \wedge T(\ulcorner\varphi\urcorner, P).$$

Now we say what this has to do with conjunctions. Consider arbitrary cardinals $\kappa < \eta$ and, for each $f : \kappa \to \kappa$, a first order formula φ_f and a parameter $P_f \subseteq H_\eta$. By what we just explained, the following statements are equivalent to each other.

(1) For every $f : \kappa \to \kappa$, there exists $A \subseteq H_{\eta^+}$ such that
$$(H_{\eta^+}, A) \models \varphi_f(P_f).$$

(2) For every $f : \kappa \to \kappa$, there exist $A, T \subseteq H_{\eta^+}$ such that
$$(H_{\eta^+}, A, T) \models \Delta \wedge T(\ulcorner\varphi_f\urcorner, P_f).$$

Assume that $\eta >> \kappa$. Let
$$P = \{(f, x) \mid x \in P_f\}.$$

For each function $f \in \kappa \to \kappa$, let φ^*_f be the result of replacing every occurrence of $A(x)$ in φ_f by $A(f, x)$. Let $\varphi(P)$ be the first order sentence
$$\Delta \wedge \forall f \in {}^\kappa\kappa \;\; T(\ulcorner\varphi^*_f\urcorner, P_f).$$

Then statements (1) and (2) above are equivalent to the following.

(3) There exist $A, T \subseteq H_{\eta^+}$ such that
$$(H_{\eta^+}, A, T) \models \varphi(P).$$

Notice that (3) is a Σ^2_1 statement about H_η. We call $\varphi(P)$ the Σ^2_1-conjunction of $\varphi_f(P_f)$ for $f \in {}^\kappa\kappa$.

Now we return to the main line of the proof of Lemma 20. For each $f : \lambda \to \lambda$, pick a witness $\varphi_f(Q_f)$ that f is not (θ, Σ^2_1)-fast. Consequently, for every $f : \lambda \to \lambda$ there exists $B \subseteq H_{\theta^+}$ such that
$$(H_{\theta^+}, B) \models \varphi_f(Q_f).$$

Let $\varphi(Q)$ be the Σ^2_1-conjunction of $\varphi_f(Q_f)$ for $f \in {}^\lambda\lambda$.

Define a function $z : \lambda \to \lambda$ as follows. For each $\kappa < \lambda$, let $z(\kappa)$ be the least $\zeta < \lambda$ such that there is no (ζ, Σ^2_1)-fast function at κ. If there is no such ζ, then set $z(\kappa) = \kappa$. Define another partial function $g : \lambda \to \lambda$ by setting

$$g(\kappa) = |H_{z(\kappa)^+}|.$$

Now apply the fact that λ is (θ, Σ^2_1)-subcompact to find

$$\kappa < \eta < \lambda,$$

$$P \subseteq H_\eta$$

and an elementary embedding

$$\pi : (H_\eta, P) \to (H_\theta, Q)$$

with $\text{crit}(\pi) = \kappa$ and $\pi(\kappa) = \lambda$ such that, for some $A, T \subseteq H_{\eta^+}$,

$$(H_{\eta^+}, A, T) \models \varphi(P).$$

By the definition that we gave of g, it is clear that $g \in \text{ran}(\pi)$. Hence $g = \pi(g \restriction \kappa)$. It follows that

$$\pi : (H_\eta, P_{g\restriction\kappa}) \to (H_\theta, Q_g)$$

is an elementary embedding and there exists $A \subseteq H_{\eta^+}$ such that

$$(H_{\eta^+}, A) \models \varphi_g(P_{g\restriction\kappa}).$$

To finish the proof of Lemma 20 with a contradiction, it is enough to show that $\eta \leq g(\kappa)$. This involves two claims.

First we claim that there is no $e : \kappa \to \kappa$ such that e is (η, Σ^2_1)-fast. For suppose otherwise. Let $f = \pi(e)$. It follows that

$$\pi : (H_\eta, P_e) \to (H_\theta, Q_f)$$

is an elementary embedding and there exists $A \subseteq H_{\eta^+}$ such that

$$(H_{\eta^+}, A) \models \varphi_f(P_e).$$

Apply the (η, Σ^2_1)-fastness of $e : \kappa \to \kappa$ to find

$$\bar\kappa < \bar\eta \leq e(\bar\kappa) < \kappa,$$

$$\bar P \subseteq H_{\bar\eta}$$

and an elementary embedding

$$\bar\pi : (H_{\bar\eta}, \bar P) \to (H_\eta, P_e)$$

with $\text{crit}(\bar\pi) = \bar\kappa$ and $\pi(\bar\kappa) = \kappa$ such that, for some $\bar A \subseteq H_{\bar\eta^+}$,

$$(H_{\bar\eta^+}, \bar A) \models \varphi_f(\bar P).$$

But

$$\pi \circ \bar\pi : (H_{\bar\eta}, \bar P) \to (H_\theta, Q_f)$$

is an elementary embedding and $f \upharpoonright \kappa = e$ hence $\bar{\eta} \leq f(\bar{\kappa})$. Taken together, these facts directly contradict our choice of $\varphi_f(Q_f)$. This proves our first claim.

The first claim tells us that $z(\kappa) \leq \eta$ and that $z(\kappa)$ is the least $\zeta < \lambda$ such that there is no (ζ, Σ^2_1)-fast function at κ.

Second we claim that if $\mathcal{P}(H_{\zeta^+}) \subseteq H_\eta$, then $z(\kappa) > \zeta$. This is clear from the fact that $\pi : H_\eta \to H_\theta$ is an elementary embedding and our induction hypothesis that for all $\zeta < \theta$, there is a function $f : \lambda \to \lambda$ such that f is (ζ, Σ^2_1)-fast.

By the two claims,

$$2^{|H_{z(\kappa)^+}|} > |H_\eta|.$$

Hence,

$$|H_{z(\kappa)^+}| \geq |H_\eta|.$$

Thus, $g(\kappa) \geq \eta$ as desired. □

The following result will be used in the proof of Theorem 12.

Lemma 21. *Let θ be a cardinal and*

$$\mathcal{C} = \{\mathbb{P} \mid |P| = 2^\theta \text{ and } \mathbb{P} \text{ is } \theta\text{-linked}\}.$$

Then SPFA$(\mathcal{C})$ *implies* SPFA$(\theta$-linked$)$.

Proof. Consider an arbitrary poset $\mathbb{Q}$ that is semi-proper and θ-linked. We may assume that $\mathbb{Q} = (\rho, <_\mathbb{Q})$ where $\rho > 2^\theta$. Let $\ell : \rho \to \theta$ witness that $\mathbb{Q}$ is θ-linked. Suppose that $\mathcal{D}$ is a family of maximal antichains of $\mathbb{Q}$ with $|\mathcal{D}| = \aleph_1$. Then $|\bigcup \mathcal{D}| \leq \theta$. Let $\sigma < \tau$ be regular cardinals much greater than ρ. Because $\mathbb{Q}$ is semi-proper, there exists a function

$$F : H_\sigma^{<\omega} \longrightarrow H_\sigma$$

such that if $M \subseteq H_\sigma$ is nonempty and closed under F, then

(1) $M \preccurlyeq H_\sigma$ and $\mathbb{Q} \in M$,
(2) if M is countable, then for all $r \in \mu \cap M$, there exists $s <_\mathbb{Q} r$ such that s is $(M, \mathbb{Q})$-semi-generic.

Let $X \preccurlyeq H_\tau$ with $F \in X$, $|X| = 2^\theta$, $\bigcup \mathcal{D} \subseteq X$ and ${}^\theta X \subseteq X$. Let $\mathbb{P} = \mathbb{Q} \upharpoonright X$. That is, $\mathbb{P} = (P, <_\mathbb{P})$ where $P = \rho \cap X$ and

$$r <_\mathbb{P} s \iff r <_\mathbb{Q} s$$

for all $r, s \in P$. Observe that $\mathbb{P}$ is θ-linked as witnessed by $\ell \upharpoonright P$.

If A is an antichain of $\mathbb{P}$, then A is an antichain of $\mathbb{Q}$ and $A \in X$ because ${}^\theta X \subseteq X$. Moreover, if A is an antichain of $\mathbb{P}$, then

$$A \text{ is maximal in } \mathbb{P} \iff A \text{ is maximal in } \mathbb{Q}.$$

This is because if A is not maximal in $\mathbb{Q}$, then there exists $r \in \rho - A$ such that $A \cup \{r\}$ is an antichain in $\mathbb{Q}$, but since $X \preccurlyeq H_\sigma$ and $A, \mathbb{Q} \in X$, there exists

$$s \in X \cap (\rho - A) = P - A$$

such that $A \cup \{s\}$ is an antichain in $\mathbb{Q}$, which implies that $A \cup \{s\}$ is an antichain in $\mathbb{P}$.

We claim that $\mathbb{P}$ is semi-proper. Let $N \preccurlyeq H_\tau$ be countable with $F, X \in N$. Suppose that $r \in P \cap N$. Let $M = N \cap H_\sigma \cap X$. Then $M \preccurlyeq X \preccurlyeq H_\sigma$ and M is non-empty, countable and closed under F. Since $r \in M$, there exists $s <_{\mathbb{Q}} r$ such that s is $(M, \mathbb{Q})$-semi-generic. Because $X \preccurlyeq H_\tau$ and $r, M, \mathbb{Q} \in X$, there exists $t <_{\mathbb{P}} r$ such that $t \in X$ and t is $(M, \mathbb{Q})$-semi-generic. A name for a countable ordinal is essentially a partition of a maximal antichain into ω_1 many pieces. Suppose that $c : A \to \omega_1$ is a partition of a maximal antichain of $\mathbb{P}$. Then $c \in X$ and c is a partition of a maximal antichain of $\mathbb{Q}$. Suppose that further that $c \in N$. Then $c \in M$. By semi-genericity,

$$t \Vdash_{\mathbb{Q}} \exists \alpha \in M \ \left(G \cap c^{-1}[\{\alpha\}] \cap M \neq \emptyset \right).$$

Using again that $X \preccurlyeq H_\sigma$ and $t, \mathbb{Q}, c, M \in X$, it is straightforward to verify that

$$t \Vdash_{\mathbb{P}} \exists \alpha \in M \ \left(G \cap c^{-1}[\{\alpha\}] \cap M \neq \emptyset \right).$$

In particular,

$$t \Vdash_{\mathbb{P}} \exists \alpha \in N \ \left(G \cap c^{-1}[\{\alpha\}] \cap N \neq \emptyset \right).$$

This shows that t is $(N, \mathbb{P})$-semi-generic and proves the claim.

To finish the proof of the lemma, note that if $A \in \mathcal{D}$, then $A \subseteq P$ and A is a maximal antichain of $\mathbb{P}$. So we may apply the hypothesis of the lemma to $\mathbb{P}$ to find a filter G on $\mathbb{P}$ such that $G \cap A \neq \emptyset$ for all $A \in \mathcal{D}$. Then

$$H = \{s \in \mu \mid r \leq_{\mathbb{Q}} s \text{ for some } r \in G\}$$

is a filter on $\mathbb{Q}$ and $H \cap A \neq \emptyset$ for all $A \in \mathcal{D}$. □

Definition 22. A function $F : H_{\theta^+}^{<\omega} \to H_{\theta^+}$ is a *witness that* $\mathbb{Q}$ *is semi-proper* iff $\mathbb{Q} \subseteq H_{\theta^+}$ is a θ^+-cc poset and for every non-empty countable $M \subseteq H_{\theta^+}$, if $F[M^{<\omega}] \subseteq M$, then for all $p \in M$, there exists $q <_{\mathbb{Q}} p$ such that q is $(M, \mathbb{Q})$-semi-generic.

Lemma 23. *Let $\mathbb{Q} \subseteq H_{\theta^+}$ be a θ^+-cc poset. Then $\mathbb{Q}$ is semi-proper iff there is a witness $F : H_{\theta^+}^{<\omega} \to H_{\theta^+}$ that $\mathbb{Q}$ is semi-proper.*

Proof. If A is a maximal antichain of $\mathbb{Q}$ and $c : A \to \omega_1$, then $c \in H_{\theta^+}$. The rest follows from the definitions. □

The proofs of Theorem 12 does not use a Laver function. In fact, we do not see how to construct a suitable Laver function without strengthening the large cardinal hypothesis. Instead we use the following established technique.

Definition 24. $\mathbb{U} = (U, <_{\mathbb{U}})$ is the proper class partial ordering

$$U = \{(p, \mathbb{P}) \mid \mathbb{P} \text{ is semi-proper and } p \in P\}$$

and

$$(p, \mathbb{P}) <_{\mathbb{U}} (q, \mathbb{Q}) \iff (\mathbb{P} = \mathbb{Q} \text{ and } p <_{\mathbb{P}} q)$$

In other words, $\mathbb{U}$ is the direct sum of all semi-proper partial orderings. This is called the *lottery sum* in Hamkins [7] where the same idea is used for related purposes.

Lemma 25. *For every cardinal η, $\mathbb{U} \cap H_\eta$ is a semi-proper poset.*

Proof. Let θ be a regular cardinal that is large enough so that for every semi-proper $\mathbb{P} \in H_\eta$, there is a witness $F_{\mathbb{P}}$ that $\mathbb{P}$ is semi-proper such that $F_{\mathbb{P}} \in H_\theta$. Consider a countable $M \prec H_\theta$ with

$$\langle F_{\mathbb{P}} \mid \mathbb{P} \in H_\eta \text{ and } \mathbb{P} \text{ is semi-proper} \rangle \in M.$$

Let $(p, \mathbb{P}) \in U \cap H_\eta \cap M$. Then $F_{\mathbb{P}} \in M$ so M is closed under $F_{\mathbb{P}}$. So, there is $q <_{\mathbb{P}} p$ such that q is $(M, \mathbb{P})$-semi-generic. Hence $(q, \mathbb{P}) <_{\mathbb{U}} (p, \mathbb{P})$ and $(q, \mathbb{P})$ is $(M, \mathbb{U} \cap H_\eta)$-semi-generic. □

Definition 26. Let λ be an inaccessible cardinal and $h : \lambda \to \lambda$ be a cardinal valued function. The *universal semi-proper iteration associated to h* is the revised countable support iteration $\mathbb{I}$ of length λ such that

$$\Vdash_{\mathbb{I} \restriction \kappa} \mathbb{I}_\kappa = \mathbb{U} \cap H_{h(\kappa)}$$

for all $\kappa < \lambda$.

Lemma 27. *Let λ be an inaccessible cardinal, $h : \lambda \to \lambda$ be an increasing cardinal valued function and $\mathbb{I}$ be the universal iteration associated to h. Then $\mathbb{I}$ is a semi-proper iteration that is amenable to H_λ and λ-cc. Moreover, if*

$$S = \{\kappa < \lambda \mid \kappa \textit{ is inaccessible and } h[\kappa] \subseteq \kappa\},$$

then

$$\mathbb{I} \cap H_\kappa = \mathbb{I} \restriction \kappa$$

for all $\kappa \in S$, hence

$$\mathbb{I} = \bigcup \{\mathbb{I} \restriction \kappa \mid \kappa \in S\}.$$

The proof of Lemma 27 is immediate from Lemma 25, Definition 26 and well-known theorems of Shelah on revised countable support iteration of semi-proper forcing. See [11] or [4].

If $\mathbb{P} = (P, <_{\mathbb{P}})$ and D is a sequence of subsets of P, then we write $(\mathbb{P}, D)$ for the structure $(P, <_{\mathbb{P}}, D_i)_{i \in \mathrm{dom}(D)}$. The following relation is used in the proof of Theorem 12.

Definition 28. We write $\tau : (\mathbb{P}, D) \sqsubseteq (\mathbb{Q}, E)$ iff $\mathbb{P}$ and $\mathbb{Q}$ are posets, D is a sequence of maximal antichains of $\mathbb{P}$, E is a sequence of maximal antichains of $\mathbb{Q}$ of the same length as D,

$$\tau : (\mathbb{P}, D) \to (\mathbb{Q}, E)$$

is an elementary embedding and $\bigcup E \subseteq \tau[P]$.

Lemma 29. *Suppose that $\tau : (\mathbb{P}, D) \sqsubseteq (\mathbb{Q}, E)$. Let G be a D-generic filter on $\mathbb{P}$ and*

$$H = \{q \in Q \mid \text{there exists } p \in G \text{ such that } \tau(p) \leq_{\mathbb{Q}} q\}.$$

Then H is an E-generic filter on $\mathbb{Q}$.

The proof of Lemma 29 is clear.

Lemma 30. *Suppose that $\tau : (\mathbb{P}, D) \sqsubseteq (\mathbb{Q}, E)$. Assume that for all $u, v \in \bigcup E$, there exists $\alpha < \mathrm{dom}(E)$ such that*

$$E_\alpha \subseteq \{w \in Q \mid (w \leq_{\mathbb{Q}} u, v) \text{ or } (w \perp_{\mathbb{Q}} u) \text{ or } (w \perp_{\mathbb{Q}} v)\}.$$

Let H be an E-generic filter on $\mathbb{Q}$ and $G = \tau^{-1}[H]$. Then G is a D-generic filter on $\mathbb{P}$.

Proof. The added assumption is used to show that if $p, q \in G$, then there exists $r \in G$ such that $r \leq_{\mathbb{P}} p, q$. Equivalently, if $\tau(p), \tau(q) \in H$, then there is $r \in P$ such that $\tau(r) \in H$ and $\tau(r) \leq_{\mathbb{Q}} \tau(p), \tau(q)$. Let $\alpha < \mathrm{dom}(E)$ be such that

$$E_\alpha \subseteq \{w \in Q \mid (w \leq_{\mathbb{Q}} \tau(p), \tau(q)) \text{ or } (w \perp_{\mathbb{Q}} \tau(p)) \text{ or } (r \perp_{\mathbb{Q}} \tau(q))\}.$$

Since H is E-generic, there is $w \in H \cap E_\alpha$. Since $\bigcup E \subseteq \tau[P]$, there is $r \in P$ such that $\tau(r) = w$. Since H is a filter, $\tau(p)$, $\tau(q)$ and $\tau(r)$ are pairwise compatible. Hence $\tau(r) \leq_{\mathbb{Q}} \tau(p), \tau(q)$. The rest is clear. □

Given an infinite cardinal length sequence $E = \langle E_\alpha \mid \alpha < |E| \rangle$ of maximal antichains of $\mathbb{Q}$, there is a cardinal length sequence

$$\overline{E} = \langle E_\alpha \mid \alpha < |\overline{E}| \rangle$$

of maximal antichains of $\mathbb{Q}$ such that

$$\{E_\alpha \mid \alpha < |E|\} \subseteq \{\overline{E}_\alpha \mid \alpha < |\overline{E}|\}$$

and $\overline{E}$ satisfies the assumption of Lemma 30. A natural operation produces the family of maximal antichains $\{\overline{E}_\alpha \mid \alpha < |\overline{E}|\}$ from E in ω-many steps and then we re-index so that $\mathrm{dom}(\overline{E}) = |\overline{E}|$. This operation is defined relative to the poset and a wellordering, both of which we suppress in the notation. Let us write

$$\tau : (\mathbb{P}, D, \overline{D}) \sqsubseteq (\mathbb{Q}, E, \overline{E})$$

iff $\tau : (\mathbb{P}, D) \sqsubseteq (\mathbb{Q}, E)$ and $\tau : (\mathbb{P}, \overline{D}) \sqsubseteq (\mathbb{Q}, \overline{E})$. We write

$$(\mathbb{P}, D, \overline{D}) \sqsubseteq (\mathbb{Q}, E, \overline{E})$$

to mean that there there exists τ such that $\tau : (\mathbb{P}, D, \overline{D}) \sqsubseteq (\mathbb{Q}, E, \overline{E})$.

We remark that Lemma 30 holds without the assumption on E if $\mathbb{Q}$ is closed under meets, that is, if for all $u, v \in Q$, there exists $w \in Q$ such that $w = u \wedge_Q v$. Observe that if $\mathbb{Q}$ is an infinite poset, then there is a poset $\mathbb{Q}'$ that is closed under meets such that $\mathbb{Q}$ is dense in $\mathbb{Q}'$ and $|Q| = |Q'|$. In fact, we could make $\mathbb{Q}'$ a Boolean algebra if we wanted. If, in addition, $\mathbb{Q}$ is semi-proper, then so is $\mathbb{Q}'$. The reader who is only interested in forcing axioms for posets that are closed under meets can strip away discussion of the operation $E \mapsto \overline{E}$ in what follows. Just after getting started with the proof of Theorem 12, in the second paragraph, we give an outline of what is to come.

Proof of Theorem 12

Assume that $\theta^{<\lambda} = \theta$ and $2^\theta = \theta^+$. Let $f : \lambda \to \lambda$ be a (θ, Σ^2_1)-fast function. We may assume that f is cardinal valued. Define $h : \lambda \to \lambda$ by $h(\kappa) = f(\kappa)^{++}$ for $\kappa < \lambda$. Let $\mathbb{I}$ be the universal iteration associated to h. For contradiction, suppose that there is a generic extension of V by $\mathbb{I}$ in which there are $\mathbb{Q}$ and E such that $\mathbb{Q}$ is a semi-proper and θ-linked poset and E is an ω_1-sequence of maximal antichains of $\mathbb{Q}$ but there is no E-generic filter on $\mathbb{Q}$. Work in such a generic extension. Let $\ell : Q \to \theta$ witness that $\mathbb{Q}$ is θ-linked. By Lemma 21, we may assume that $\mathbb{Q} \subseteq H_{\theta^+}$. Then, there is a function $F : H^{<\omega}_{\theta^+} \to H_{\theta^+}$ that witnesses that $\mathbb{Q}$ is semi-proper. Apply the operation described above to form $\overline{E}$. We may assume that $\mathrm{dom}(\overline{E}) = \theta$. Let $X \preccurlyeq (\mathbb{Q}, \overline{E})$ with $\bigcup \overline{E} \subseteq X$ and $|X| = \theta$. Let $\tau : \theta \to X$ be a bijection. Use τ to define a poset $\mathbb{P} = (\theta, <_\mathbb{P})$, an ω_1-sequence D of maximal antichains of $\mathbb{P}$ and a θ-sequence $\overline{D}$ of maximal antichains of $\mathbb{P}$ such that

$$\tau : (\mathbb{P}, D, \overline{D}) \simeq (\mathbb{Q} \restriction X, E, \overline{E})$$

and $\overline{D}$ is the closure of D relative to $\mathbb{P}$. Then $\tau : (\mathbb{P}, D, \overline{D}) \sqsubseteq (\mathbb{Q}, E, \overline{E})$. By Lemma 29, there is no D-generic filter on $\mathbb{P}$.

Before continuing, let us give a general description of the proof to come. The existence of $\mathbb{Q}$, E, ℓ and F as above is a Σ^2_1 assertion about $\mathbb{P}$ and D over H_θ in $V^{\mathbb{I}}$. Claim 31 below gives a version of this Σ^2_1 over H_θ assertion that holds in V. Using the assumption that $f : \lambda \to \lambda$ is a (θ, Σ^2_1)-fast function, we find $\kappa \leq \eta \leq f(\kappa) < \lambda$ and an elementary embedding

$$\pi : (H_\eta, \mathbb{I} \restriction \kappa, \mathbb{P}^*, D^*) \to (H_\theta, \mathbb{I}, \mathbb{P}, D)$$

with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ and $\pi(\kappa) = \lambda$. Moreover, the same Σ^2_1 assertion holds about $\mathbb{P}^*$ and D^* over H_η in $V^{\mathbb{I}\restriction\kappa}$. For the precise formulation, see Claim 32. This provides $\mathbb{I} \restriction \kappa$-names for a semi-proper η-linked poset $\mathbb{Q}^*$ and an ω_1-sequence E^* of maximal antichains of $\mathbb{Q}^*$ such that $(\mathbb{P}^*, D^*, \overline{D^*}) \sqsubseteq (\mathbb{Q}^*, E^*, \overline{E^*})$. The fact that $\mathbb{I}$ is a universal iteration plus a density argument reduces us to the case in which a $V^{\mathbb{I}\restriction\kappa}$-generic filter over $\mathbb{Q}^*$ is added at stage κ of the iteration. See Claim 33 below. From Lemma 30, it follows that a D^*-generic filter g^* over $\mathbb{P}^*$ is added at stage κ of the iteration. At the end of the argument, we use π to lift g^* to a D-generic filter g on $\mathbb{P}$. This contradicts the last sentence of the previous paragraph! Our outline is complete; now we return to the proof.

Work in V and treat the sets discussed in the first paragraph of the proof as $\mathbb{I}$-names. Let $u \in I$ force all the facts we listed in the previous paragraph; we will be specific about what we mean by this. We may assume that, as $\mathbb{I}$-names,

$$\mathbb{Q}, \ell, F \subseteq H_{\theta^+},$$

$$\overline{E}, \mathbb{P}, \overline{D} \in H_{\theta^+}$$

and

$$\mathbb{P}, \overline{D} \subseteq H_\theta.$$

To justify the assumptions about $\overline{E}$, $\mathbb{P}$ and $\overline{D}$, we use that $\theta^{<\lambda} = \theta$ and $\mathbb{I}$ is λ-cc. By $\overline{E}$ we mean the natural $\mathbb{I}$-name for the closure of E relative to $\mathbb{Q}$. Naturalness implies that if $E \in H_{\theta^+}$, then $\overline{E} \in H_{\theta^+}$ and vice-versa. Similarly, if $D \subseteq H_\theta$, then $\overline{D} \subseteq H_\theta$ and vice-versa for the closure relative to $\mathbb{P}$.

Let $\psi(\mathbb{P}, D, \mathbb{Q}, E, \ell, F)$ be the conjunction of the following sentences in the forcing language of $\mathbb{I}$.

(1) $\ell : Q \to \theta$ witnesses that $\mathbb{Q}$ is θ-linked.
(2) $F : H^{<\omega}_{\theta^+} \to H_{\theta^+}$ witnesses that $\mathbb{Q}$ is semi-proper.
(3) E is an ω_1-sequence of maximal antichains of $\mathbb{Q}$.
(4) $(\mathbb{P}, D, \overline{D}) \sqsubseteq (\mathbb{Q}, E, \overline{E})$
(5) There is no D-generic filter on $\mathbb{P}$.

We have fixed $u \in I$ precisely so that

$$u \Vdash_{\mathbb{I}} \psi(\mathbb{P}, D, \mathbb{Q}, E, \ell, F).$$

Let $\varphi(\xi, \lambda, h, \mathbb{I}, u, \theta, \mathbb{P}, D)$ be the statement that there the exists $N \subseteq H_{\theta^+}$ such that N is a wellfounded model of ZFC - P and, if M is the Mostowski collapse of N, then ${}^{\theta}M \subseteq M$ and

$$M \models \begin{array}{l} \text{there are } \mathbb{Q}^*,\ E^*,\ \ell^*, \text{ and } F^* \text{ such that} \\ \chi(\xi, \lambda, h, \mathbb{I}, u, \theta, \mathbb{P}, D, \mathbb{Q}^*, E^*, \ell^*, F^*) \end{array}$$

where $\chi(\cdots)$ is the following first order sentence:

- θ^+ exists and is the largest cardinal,
- $\xi < \lambda \leq \theta$,
- ${}^{<\lambda}\theta = \theta$,
- $h : \lambda \to \lambda$ is cardinal valued,
- $\mathbb{I}$ is the universal iteration associated to h,
- $u \in I$,
- $\mathbb{P}, \overline{D} \subseteq H_\theta$,
- $\mathbb{Q}^*, \ell^*, F^* \subseteq H_{\theta^+}$,
- $\overline{E^*}, \mathbb{P}, \overline{D} \in H_{\theta^+}$ and
- $u \Vdash_{\mathbb{I}} \psi(\mathbb{P}, D, \mathbb{Q}^*, E^*, \ell^*, F^*)$.

Claim 31. *For every $\xi < \lambda$, $\varphi(\xi, \lambda, h, \mathbb{I}, u, \theta, \mathbb{P}, D)$ is a true Σ^2_1 property of H_θ.*

Proof. Notice that the leading quantifier of $\varphi(\cdots)$ is $\exists N \subseteq H_{\theta^+}$ and that what follows this quantifier is first order over the structure (H_{θ^+}, N). Notice also that the every parameter of $\varphi(\cdots)$ is a subset of H_θ. This is what we mean by a Σ^2_1 statement about H_θ. To verify that $\varphi(\cdots)$ is true, let M be a transitive θ-closed Skolem hull of $\{\mathbb{Q}, f, F\}$ taken in $H_{\theta^{++}}$ and obtain N from M. $\square$

We cannot replace θ-linked with θ^+-cc in the proof of Claim 31 because the definition of θ^+-cc involves an additional universal quantifier over antichains of $\mathbb{Q}$, which are named by subsets of H_{θ^+}. Compare this with Theorem 38.

Claim 32. *For every $\xi < \lambda$ such that $u \in H_\xi$, there are κ, η, $\mathbb{P}^*$, D^*, π, $\mathbb{Q}^*$, E^*, ℓ^* and F^* such that*

$$\xi < \kappa \leq \eta \leq f(\kappa) < \lambda,$$

$$\pi : (H_\eta, \mathbb{I} \restriction \kappa, \mathbb{P}^*, D^*, \overline{D^*}) \to (H_\theta, \mathbb{I}, \mathbb{P}, D, \overline{D})$$

is an elementary embedding with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ *and* $\pi(\kappa) = \lambda$, *and*

$$\begin{array}{ll} u \Vdash_{\mathbb{I}\restriction\kappa} & \mathbb{Q}^* \subseteq H_{\eta^+}, \\ & \ell^* : Q^* \to \eta^+ \text{ witnesses that } \mathbb{Q}^* \text{ is } \theta\text{-linked}, \\ & F^* : H_{\eta^+}^{<\omega} \to H_{\eta^+} \text{ witnesses that } \mathbb{Q}^* \text{ is semi-proper}, \\ & (\mathbb{P}^*, D^*, \overline{D^*}) \sqsubseteq (\mathbb{Q}^*, E^*, \overline{E^*}). \end{array}$$

Proof. Immediate from the fact that $f : \lambda \to \lambda$ is a (θ, Σ^2_1)-fast function, Lemma 27, Claim 31 and the absoluteness of

$$\chi(\xi, \kappa, h \restriction \kappa, \mathbb{I} \restriction \kappa, u, \eta, \mathbb{P}^*, D^*, \mathbb{Q}^*, E^*, \ell^*, F^*)$$

from M^* to V if M^* is a transitive model of ZFC - P such that ${}^{\eta}M^* \subseteq M^*$. □

For each $\xi < \lambda$ such that $u \in H_\xi$, pick corresponding κ^*, η^*, $\mathbb{P}^*$, D^*, π^*, $\mathbb{Q}^*$ and E^* as in Claim 32 and call them κ_ξ, η_ξ, $\mathbb{P}_\xi$, D_ξ, π_ξ, $\mathbb{Q}_\xi$ and E_ξ. The next claim is immediate from Claim 32.

Claim 33. *Let* X *be the set of* $v \leq_{\mathbb{I}} u$ *for which there exist* $\xi < \lambda$ *and* $q \in H_{\eta_\xi^+}$ *such that* $u \in H_\xi$,

$$u \Vdash_{\mathbb{I}\restriction\kappa_\xi} q \in Q_\xi$$

and

$$v = (v \restriction \kappa_\xi) \cup \{(\kappa_\xi, (q, \mathbb{Q}_\xi))\}.$$

Then X *is dense below* u.

It follows from Claims 32 and 33 that

$$\begin{array}{ll} u \Vdash_{\mathbb{I}} & \text{there exists } \xi < \lambda \text{ such that } \{q \in Q_\xi \mid (q, \mathbb{Q}_\xi) \in I_{\kappa_\xi}\} \\ & \text{is a } V[G \restriction \kappa_\xi]\text{-generic filter on } \mathbb{Q}_\xi \end{array}$$

Let G be a V-generic filter on $\mathbb{I}$ with $u \in G$. Work in $V[G]$. Fix a corresponding $\xi < \lambda$ but let us revert to the notation of Claim 32. Thus we have $\kappa < \lambda$ such that $V[G \restriction \kappa + 1]$ is generic extension of $V[G \restriction \kappa]$ by $\mathbb{Q}^*_{G\restriction\kappa}$. In particular, there is an $\overline{E^*_{G\restriction\kappa}}$-generic filter on $\mathbb{Q}^*_{G\restriction\kappa}$. By Lemma 30, since

$$\left(\mathbb{P}^*_{G\restriction\kappa}, \overline{D^*_{G\restriction\kappa}}\right) \sqsubseteq \left(\mathbb{Q}^*_{G\restriction\kappa}, \overline{E^*_{G\restriction\kappa}}\right),$$

there is a $\overline{D^*_{G\restriction\kappa}}$-generic filter g^* on $\mathbb{P}^*_{G\restriction\kappa}$. For us, it only matters that g^* is $D^*_{G\restriction\kappa}$-generic. We have the elementary embedding

$$\pi : (H^V_\eta, \mathbb{I} \restriction \kappa, \mathbb{P}^*, D^*) \to (H^V_\theta, \mathbb{I}, \mathbb{P}, D)$$

with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ and $\pi(\kappa) = \lambda$. Because

$$\pi[G \restriction \kappa] = G \restriction \kappa \subset G,$$

π lifts to an elementary embedding

$$\widetilde{\pi} : (H^V_\eta[G \restriction \kappa], \mathbb{P}^*_{G\restriction\kappa}, D^*_{G\restriction\kappa}) \to (H^V_\theta[G], \mathbb{P}_G, D_G).$$

Let

$$g = \{p \in P_G \mid \text{there is } p^* \in g^* \text{ such that } \widetilde{\pi}(p^*) \leq_{P_G} p\}.$$

Then g is a $\widetilde{\pi}[D^*_{G\restriction\kappa}]$-generic filter on $\mathbb{P}_G$. But $\widetilde{\pi}[D^*_{G\restriction\kappa}] = D_G$ since $D^*_{G\restriction\kappa}$ and D_G are ω_1-sequences and $\omega_1 < \kappa$. There is no such filter because $u \in G$ and by clause (5) in our choice of u. This contradiction completes the proof of Theorem 12.

Proof of Corollary 18

Assume that there is a proper class of cardinals θ such that $\theta^{<\lambda} = \theta$, $2^\theta = \theta^+$ and λ is (θ, Σ^2_1)-subcompact. By Lemma 20, for each such θ, we may pick a function $f_\theta : \lambda \to \lambda$ that is (θ, Σ^2_1)-fast. Since ${}^\lambda\lambda$ is a set, there is a single function f and a proper class of θ such that $f_\theta = f$. Use f to define a semi-proper iteration $\mathbb{I}$ as in the proof of Theorem 12. Clearly this works.

Proof of Theorem 15

Theorem 15 is a slight refinement of Shelah's theorem that SPFA implies MM. We follow the line of reasoning, due to Todorcevic (see [2]) that SPFA implies SRP, SRP implies WRP, and WRP implies that stationary preserving partial orderings are semi-proper. But when it comes time to apply SPFA, we use a slightly more complicated poset that is θ-linked. What follows is a self-contained account for the convenience of the reader. We stress that, except for one modest aspect, the argument is known and not due to the authors.

Let η be a regular cardinal and $\mathbb{P}$ be a stationary preserving poset. Suppose that $\mathbb{P} \subseteq H_\eta$ and $\mathbb{P}$ is η-cc. Let $\theta = |H_\eta|^{\aleph_0}$. Assume that SPFA($\theta$-linked) holds. We must show that $\mathbb{P}$ is semi-proper.

Towards a contradiction, suppose that $\mathbb{P}$ is not semi-proper. Let

$$S_0 = \{N \in [H_\eta]^{\aleph_0} \mid \exists\, p \in N \cap P\, \forall\, q <_{\mathbb{P}} p \;\; q \text{ is not } (N, \mathbb{P})\text{-semi-generic}\}.$$

Then S_0 is stationary in $[H_\eta]^{\aleph_0}$. Apply Fodor's lemma to find $p \in P$ so that if we let

$$S = \{N \in [H_\eta]^{\aleph_0} \mid p \in N \text{ and } \forall\, q <_{\mathbb{P}} p \;\; q \text{ is not } (N, \mathbb{P})\text{-semi-generic}\},$$

then S is stationary in $[H_\eta]^{\aleph_0}$. The following lemma says that if WRP holds for S, then we have the contradiction we seek.

Lemma 34. *There is no* $X \subseteq H_\eta$ *such that*

$$\omega_1 \subseteq X,$$

$$|X| = \aleph_1$$

and

$$S \cap [X]^{\aleph_0} \text{ is stationary in } [X]^{\aleph_0}.$$

Proof. Suppose otherwise. Let G be a V-generic filter on $\mathbb{P}$ with $p \in G$. Since $\mathbb{P}$ preserves stationary subsets of ω_1,

$$(S \cap [X]^{\aleph_0} \text{ is stationary in } [X]^{\aleph_0})^{V[G]}.$$

In $V[G]$, define a partial function $f : X \to \omega_1$ by $f(\tau) = \tau_G$ whenever τ is a $\mathbb{P}$-name in X for a countable ordinal. Pick $N \in S$ such that N is closed under f. Let $q \in G$ be such that $q <_{\mathbb{P}} p$ and $q \Vdash N$ is closed under f. Then q is $(N, \mathbb{P})$-semi-generic. Contradiction! □

We define a poset $\mathbb{Q} = (Q, <_{\mathbb{Q}})$ that is designed to add a set X as in Lemma 34. Here is where the complication arises in the proof because we need $\mathbb{Q}$ to be θ-linked.

The elements of Q are sequences of the form $\langle (M_\alpha, F_\alpha) \mid \alpha \leq \delta \rangle$ with the following properties.

- $\delta < \omega_1$.
- Each M_α is a countable elementary substructure of H_η with

$$M_\alpha \cap \omega_1 \geq \alpha.$$

- $\langle M_\alpha \mid \alpha \leq \delta \rangle$ is an $\in$-increasing (hence $\subseteq$-increasing) and continuous.
- Each F_α is a countable family of functions from H_η to H_η.
- $\langle F_\alpha \mid \alpha \leq \delta \rangle$ is a $\subseteq$-increasing and continuous.
- Each M_α is closed under the functions in F_α.
- For each $\alpha \leq \delta$, if there exists N so that
  - $M_\alpha \subseteq N$,
  - $M_\alpha \cap \omega_1 = N \cap \omega_1$,
  - N is closed under the functions in F_α and
  - $N \in S$,

  then $M_\alpha \in S$.

We define the ordering $\leq_{\mathbb{Q}}$ by

$$\langle (M'_\alpha, F'_\alpha) \mid \alpha \leq \delta' \rangle \leq_{\mathbb{Q}} \langle (M_\alpha, F_\alpha) \mid \alpha \leq \delta \rangle$$

iff $\delta' \geq \delta$ and, for every $\alpha < \delta$, $M'_\alpha = M_\alpha$ and $F'_\alpha \supseteq F_\alpha$.

Define the *stem* of a condition $\langle (M_\alpha, F_\alpha) \mid \alpha \leq \delta \rangle$ to be the sequence $\langle M_\alpha \mid \alpha \leq \delta \rangle$. Any two conditions with the same stem are compatible. The number of stems is at most θ, so $\mathbb{Q}$ is θ-linked. We will see that $\mathbb{Q}$ is semi-proper but first let us show that this gives the desired contradiction. The next lemma is essentially the fact that SRP implies WRP.

Lemma 35. *For each $\delta < \omega_1$, let D_δ be the set of conditions of length $\delta + 1$. Then there is no filter G on $\mathbb{Q}$ such that $G \cap D_\delta \neq \emptyset$ for all $\delta < \omega_1$.*

Proof. Assume otherwise. Let $\langle (M_\alpha, F_\alpha) \mid \alpha \leq \omega_1 \rangle$ be the union of G and

$$X = \bigcup \{M_\alpha \mid \alpha < \omega_1\}.$$

By Lemma 34, it is enough to see that $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$. Assume otherwise. As always, we endow H_η with predicates for membership and a fixed wellordering of H_η. Note that

$$\langle (M_\alpha, F_\alpha) \mid \alpha < \omega_1 \rangle \subseteq H_\eta.$$

Let $\mathcal{A}$ be the expansion of H_η by this additional predicate. Since S is stationary in $[H_\eta]^{\aleph_0}$ but not in $[X]^{\aleph_0}$, there is a countable $N \prec \mathcal{A}$ such that $N \in S$ but $N \cap X \notin S$. Let $\delta = N \cap \omega_1$. Then

- $M_\delta = N \cap X \subseteq N$,
- $M_\delta \cap \omega_1 = \delta = N \cap \omega_1$,
- N is closed under the functions in $F_\delta = \bigcup \{F_\alpha \mid \alpha < \delta\}$ and
- $N \in S$.

Hence $M_\delta \in S$ by the definition of $\mathbb{Q}$. This contradicts the fact that $M_\delta = N \cap X \notin S$. $\square$

Of course, for each $\delta < \omega_1$, D_δ is dense in $\mathbb{Q}$. The next lemma is the last step in the proof of Theorem 15. It is essentially the fact that SPFA implies SRP.

Lemma 36. *$\mathbb{Q}$ is semi-proper.*

Proof. Let $\kappa >> \theta$ be a regular cardinal. Consider a countable $N \prec H_\kappa$ with everything relevant in N. Let $q \in N \cap Q$. We seek a condition $r^* <_{\mathbb{Q}} q$ such that r^* is $(N, \mathbb{Q})$-semi-generic.

Let $\delta = N \cap \omega_1$ and G be an N-generic filter on $\mathbb{Q}$ with $q \in G$. Define

- $\langle (M_\alpha, F_\alpha) \mid \alpha < \delta \rangle = \bigcup G$,
- $M = N \cap H_\eta$,
- $F = \bigcup \{F_\alpha \mid \alpha < \delta\}$ and
- $r = \langle (M_\alpha, F_\alpha) \mid \alpha < \delta \rangle \cup \{(\delta, (M, F))\}$.

Observe the following.

- F consists of all functions from H_η to itself that are elements of N. This is by N-genericity.
- If $r \in Q$, then $r <_{\mathbb{Q}} q$ and r is $(N, \mathbb{Q})$-generic.
- If $M \in S$, then $r \in Q$.

By the definition of $\mathbb{Q}$, the only way that r could fail to be a condition is if $M \notin S$ but there exists $M^* \prec H_\eta$ such that

- $M \subseteq M^*$,
- $M \cap \omega_1 = \delta = M^* \cap \omega_1$,
- M^* is closed under the functions in F and
- $M^* \in S$.

Assume that this is the case. Let N^* be the Skolem hull of $N \cup M^*$ in H_κ. We claim that

$$N^* \cap H_\eta = M^*.$$

For suppose that $a \in N^* \cap H_\eta$. Then there is a Skolem term τ and there are $b \in N$ and $c \in M^*$ such that

$$a = \tau^{H_\kappa}[b, c].$$

Let f be the partial function from H_η to itself given by

$$f : z \mapsto \tau^{H_\kappa}[b, z].$$

Then $f \in N$. Hence $a = f(c) \in M^*$, which proves the claim.

We have found $N^* \prec H_\kappa$ such that

- $N \subseteq N^*$,
- $N \cap \omega_1 = \delta = N^* \cap \omega_1$,
- $M^* = N^* \cap H_\eta \in S$.

Let G^* be an N^*-generic filter on $\mathbb{Q}$ with $q \in G^*$. Define

- $\langle (M^*_\alpha, F^*_\alpha) \mid \alpha < \delta \rangle = \bigcup G^*$,
- $F^* = \bigcup \{F^*_\alpha \mid \alpha < \delta\}$ and
- $r^* = \langle (M^*_\alpha, F^*_\alpha) \mid \alpha < \delta \rangle \cup \{(\delta, (M^*, F^*))\}$.

Then $r^* \in Q$. Hence $r^* <_{\mathbb{Q}} q$ and r^* is $(N^*, \mathbb{Q})$-generic. Finally, observe that r^* is $(N, \mathbb{Q})$-semi-generic because of the agreement between N and N^*. $\square$

3. The chain condition hierarchy

We conclude the paper with an upper bound on the large cardinal consistency strength of SPFA(θ^+-cc). This involves a strengthening the hypothesis of Theorem 12.

Definition 37. Let $\lambda \leq \theta$ be cardinals. Then λ is (θ, Σ^2_2)-*subcompact* iff for all $Q \subseteq H_\theta$ and first order formulas $\varphi(x)$, if there exists $B \subseteq H_{\theta^+}$ such that for all $B' \subseteq H_{\theta^+}$

$$(H_{\theta^+}, B, B') \models \varphi(Q),$$

then there exist $\kappa \leq \eta < \lambda$, $P \subseteq H_\eta$ and $A \subseteq H_{\eta^+}$ such that for all $A' \subseteq H_{\eta^+}$,

$$(H_{\eta^+}, A, A') \models \varphi(P)$$

and there exists an elementary embedding

$$\pi : (H_\eta, P) \to (H_\theta, Q)$$

with $\pi \restriction \kappa = \text{identity} \restriction \kappa$ and $\pi(\kappa) = \lambda$. We also say that $[\lambda, \theta]$ is a Σ^2_2-*indescribable* interval of cardinals in this case.

We remark that, by now, it should be clear to the reader how to define (θ, Γ)-subcompact cardinals and what it means for $[\lambda, \theta]$ to be a Γ-indescribable interval of cardinals whenever Γ is a level of the typed Levy hierarchy over H_θ.

Theorem 38. *Let $\lambda \leq \theta$ be cardinals such that $\theta^{<\lambda} = \theta$ and $2^\theta = \theta^+$. Suppose that λ is (θ, Σ^2_2)-subcompact. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \text{SPFA}(\theta^+\text{-cc}).$$

Moreover,

$$V^{\mathbb{I}} \models \lambda = \mathfrak{c} = \aleph_2.$$

The main difference between the proofs of Theorems 38 and 12 is that if $\mathbb{P} \subseteq H_\theta$, then it is a Σ^2_2 statement about $\mathbb{P}$ that there exists $\mathbb{Q} \subseteq H_{\theta^+}$ such that

- $\mathbb{Q}$ is θ^+-cc (for all $A \subseteq \mathbb{Q}$, if A is an antichain, then $A \in H_{\theta^+}$) and
- there exists $F : H^{<\omega}_{\theta^+} \to H_{\theta^+}$ that witnesses $\mathbb{Q}$ is semi-proper.

The rest of the proof of Theorem 38 is a routine modification to the proof of Theorem 12, including Lemmas 20 and 21, replacing Σ^2_1 by Σ^2_2 and θ-linked by θ^+-cc. Here are some special cases of Theorem 38.

Corollary 39. *Let λ be a Σ^2_2-indescribable cardinal and assume that $2^\lambda = \lambda^+$. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \text{SPFA}(\mathfrak{c}^+\text{-cc}).$$

By Corollaries 6 and 39, the consistency strengths of PFA($\mathfrak{c}^+$-cc) and SPFA($\mathfrak{c}^+$-cc) are somewhere between a Σ^2_1 indescribable cardinal and a Σ^2_2 indescribable cardinal. It would be interesting and seemingly within the scope of current techniques to know these consistency strengths exactly.

Corollary 40. *Suppose that λ is a (λ^+, Σ^2_2)-subcompact cardinal and $2^{\lambda^+} = \lambda^{++}$. Then there is a length λ semi-proper iteration $\mathbb{I}$ that is amenable to H_λ such that*

$$V^{\mathbb{I}} \models \mathrm{SPFA}(\mathfrak{c}^{++}\text{-cc}).$$

The least cardinal κ that is (κ^+, Σ^2_2)-subcompact is strictly less than the least cardinal λ that is λ^+-supercompact. This is because Lemma 10 holds with Σ^2_2 substituted for Σ^2_1.

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