Inner models and ultrafilters in $L(\mathbb{R})$

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Part 3:

1. The finite intersection property.
2. Correct iterations.
3. Uniqueness of the s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$, $\lambda < \boldsymbol{\delta}^2_1$.
4. Uniqueness for $\lambda \geq \boldsymbol{\delta}^2_1$.
5. Some questions.

Return to definitions from last time. Prove the finite intersection property.

M fine structural over a real u, satisfying L.C. assumptions.

τ least so that $\mathsf{L}(M\|\,\tau) \models$ "τ is Woodin."

Defined $a(M) = \pi_{M,\infty}{}''\tau$. Then defined $C_M = \{a(P) \mid P \text{ an iterate of } M\}$.

Want to show the C_Ms have the finite intersection property.

Claim: Let M and N be fine structural over reals u and v, satisfying L.C. assumption. Then there are iterations $M \to M^*$ and $N \to N^*$ so that $a(M^*) = a(N^*)$.

Proof: A back-and-forth argument. Create iterations $M \to M_1 \to M_2 \cdots \to M_\omega$ and $N \to N_1 \cdots \to N_\omega$ so that $a(N_{k+1}) \supset a(M_k)$ and $a(M_{k+1}) \supset a(N_k)$. Then take $M^* = M_\omega$, $N^* = N_\omega$. $\square$

Suppose now that $v \geq_T M$. The statement

there is an iteration $M \to M^*$ so that $a(M^*) = \pi_{N^*,\infty}{}''\tau^{N^*}$

is true in $\mathsf{L}(\mathbb{R})$, hence true in the symmetric collapse of N^*.

By elementarity, the statement

there is an iteration $M \to M^*$ so that $a(M^*) = \pi_{N,\infty}{}''\tau^{N}$

is true in the symmetric collapse of N, hence true in $\mathsf{L}(\mathbb{R})$.

We proved:

Claim: Let M and N be fine structural over reals u and v, satisfying L.C. assumption. Suppose $v \geq_T M$. Then $a(N) \in C_M$.

The assumption of the claim holds with N replaced by any iterate P of N. So:

Claim: Let M and N be fine structural over reals u and v, satisfying L.C. assumption. Suppose $v \geq_T M$. Then $C_N \subset C_M$.

From this get the finite intersection property.

Discussion so far suppressed correctness of iterations.

Recall: iteration trees involve choices at limits. M is iterable if the choices can be made in a way that secures wellfoundedness. An iteration of M is **correct** if it sticks to these choices.

M fine structural over u; has ω Woodin cardinals δ_0, δ_1, ..., with sup δ; $\mathcal{P}(\delta)^M$ is ctbl in V; and M is iterable.

Let $\kappa < \delta_0$ be least cardinal strong to δ_0.

Let τ be least so that $\mathsf{L}(M\|\tau) \models$ "τ is Woodin." ($\tau < \kappa$ then.)

Theorem (Woodin): $\pi_{M,\infty}(\tau) = \aleph_\omega$.

Theorem (Steel): $\pi_{M,\infty}(\kappa) = \boldsymbol{\delta}^2_1$.

Theorem (Woodin): $\pi_{M,\infty}(\delta_0) = \Theta$.

Correctness for trees using extenders below τ is roughly $\mathbf{\Pi}^1_2$.

Correctness gets more complicated as we allow extenders higher in M. Stays in $\mathsf{L}(\mathbb{R})$ up to κ (meaning that $\mathsf{L}(\mathbb{R})$ can identify correct iterations for trees using extenders below the image of κ).

Arguments so far therefore work for $\lambda < \boldsymbol{\delta}^2_1$, recovering the supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ (Solovay), on $\mathcal{P}_{\omega_2}(\lambda)$ (Becker), on $\mathcal{P}_{\boldsymbol{\delta}^1_n}(\lambda)$ (Becker-Jackson), and producing ultrafilters on $[\mathcal{P}_{\omega_1}(\lambda)]^{<\omega_1}$.

Theorem (Woodin, using inner models): ω_1 is Θ–supercompact in $\mathsf{L}(\mathbb{R})$. (Have a s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$ for each $\lambda < \Theta$, and a sequence $\langle \mu_\lambda \mid \lambda < \Theta \rangle$ of such measures in $\mathsf{L}(\mathbb{R})$.)

Theorem (Woodin): For $\lambda < \boldsymbol{\delta}^2_1$, the s.c measure on $\mathcal{P}_{\omega_1}(\lambda)$ is unique.

Woodin ($\approx$ 1980) defined a filter $\mathcal{F}$ on $\mathcal{P}_{\omega_1}(\lambda)$ and showed that $\mathcal{F} \subset \mu$ for every s.c. measure μ on $\mathcal{P}_{\omega_1}(\lambda)$.

Using Kechris–Harrington determinacy for games on λ, $\mathcal{F}$ is an ultrafilter. From this and $\mathcal{F} \subset \mu$ get $\mathcal{F} = \mu$.

Kechris–Harrington determinacy holds for games on $\lambda < \boldsymbol{\delta}^2_1$.

Get uniqueness of the s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$, $\lambda < \boldsymbol{\delta}^2_1$.

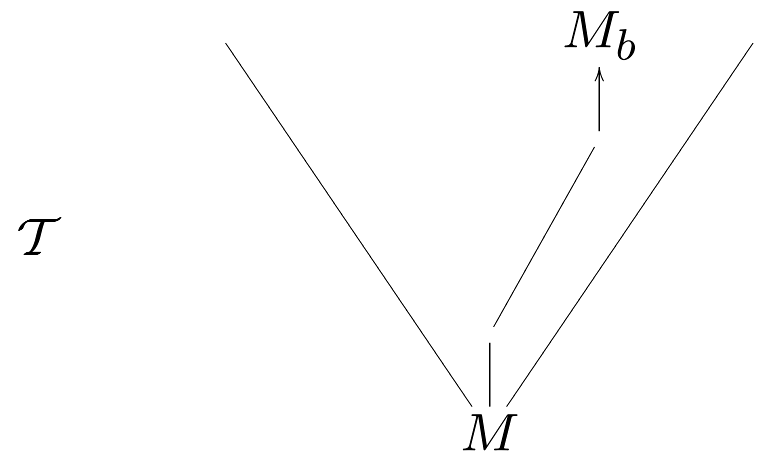
Arguments of previous talks work for $\lambda < \boldsymbol{\delta}^2_1$. Adapt them now to work for $\lambda < \Theta$. (Main issue is correctness.) Then adapt Woodin's uniqueness argument to work for the filter $\mathcal{F}$ generated by the C_Ms.

To reach Θ, must allow trees with extenders reaching to δ_0.

An iteration tree is *normal* if it uses extenders of increasing lengths. An iteration tree is *full* if it is normal, and if the extenders used by the tree have lengths cofinal in the image of δ_0.

Let $\mathcal{T}$ be a full iteration tree. Let b and c be cofinal branches through $\mathcal{T}$, with direct limit models M_b and M_c, and direct limit maps j_b and j_c. Then $j_b(\delta_0) = j_c(\delta_0)$, and $M_b \| j_b(\delta_0) = M_c \| j_c(\delta_0)$.

Refer to $M_b \| j_b(\delta_0)$ as $\Delta(\mathcal{T})$. Does not depend on last branch of $\mathcal{T}$.



Correctness for all branches of a full tree *except* the last one can be identified in $\mathsf{L}(\mathbb{R})$.

Inside $\mathsf{L}(\mathbb{R})$ can*not* identify the correct final branch b, the final model M_b, or the final embedding j_b. (Can identify $M_b \| j_b(\delta_0) = \Delta(\mathcal{T})$.)

Call a full tree k–*correct* if it is correct up to the final branch, and the embedding of its last branch moves the type of k indiscernibles for $\mathsf{L}(\mathbb{R})$ correctly.

M is k–*iterable* if choice of branches at limits can be made so that all models on the tree are k–iterable.

For a fixed k, k–correctness and k–iterability can be identified in $\mathsf{L}(\mathbb{R})$.

Woodin defined a directed system of k–iterable models. Showed that it agrees with the true directed system up to an ordinal λ_k, with $\langle \lambda_k \mid k < \omega \rangle$ cofinal in Θ.

We are interested in a s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$ for some fixed $\lambda < \Theta$.

Fix k so that $\lambda_k > \lambda$.

Can now replace the true directed system with the directed system of k–iterable models (which can be identified inside $\mathsf{L}(\mathbb{R})$).

By a *nice sequence* over M we mean a sequence $\langle \mathcal{T}_k, \bar{M}_{k+1} \mid k < \omega \rangle$ which can be expanded to an iteration $\langle M_k, \mathcal{T}_k, b_k \mid k < \omega \rangle$ with $M_0 = M$, each $\mathcal{T}_k$ a full iteration tree on M_k, and $\bar{M}_{k+1} = \Delta(\mathcal{T})_k$.

Define $a(\bar{M}_k \mid k < \omega)$ to be

$$\bigcup_{k<\omega} \pi_{\bar{M}_k,\infty}{}''(\bar{\lambda}_k)$$

where now $\pi_{\bar{M}_k,\infty}$ comes from the k–correct directed system.

$a(\bar{M}_k \mid k < \omega)$ represents what previously was $a(M_\omega)$.

Set now $C_M = \{a(\bar{M}_k \mid k < \omega) \mid \langle \bar{M}_k \mid k < \omega \rangle$ a nice sequence over $M\}$.

Let $\mathcal{F}$ be the filter generated by the sets C_M.

Previous argument adapts to show $\mathcal{F}$ is a s.c. measure.

Woodin's argument for uniqueness (From Cabal 79-81, adapted to current definitions):

Let μ be a s.c. measure on λ. Suppose $\mu \neq \mathcal{F}$. Have a set A assigned different measures by $\mathcal{F}$ and μ. Switching to the complement of A if needed we may assume that $\mu(A) = 1$, and $A \notin \mathcal{F}$. Have then some M so that $A \cap C_M = \emptyset$.

For each $x \in \mathcal{P}_{\omega_1}(\lambda)$ consider the following game G_x:

I	α_1		α_2		$\dots$	
II		$\bar{M}_1$		$\bar{M}_2$		$\dots$

Rule for I: $\alpha_k \in x$.

Rule for II: $\alpha_k \in (\pi_{\bar{M}_k,\infty}{}''\bar{\lambda}_k) \subset x$ for each k; and $\bar{M}_{k+1} = \Delta(\mathcal{T}_k)$ with $\mathcal{T}_k$ a full tree (or finite comp. of full trees) on $\bar{M}_k$ (on M if $k = 0$).

Infinite runs won by player II.

Using definition of C_M, can check that:

(1) II has a w.q.s. $\Rightarrow x \in C_M$.

(2) $x \in C_M \Rightarrow$ II has a w.q.s.

Recall, have a set A of μ measure 1, with $A \cap C_M = \emptyset$.

I wins G_x for each $x \in A$. Let σ_x be a w.s.

For each x, $\sigma_x(\emptyset) \in x$.

By normality of μ, can find $A_1 \subset A$ of measure 1, and an ordinal α_1, so that $\sigma_x(\emptyset) = \alpha_1$ for all $x \in A_1$.

Now fix $\bar{M}_1$ so that $\alpha_1 \in \pi_{\bar{M}_1,\infty}{}''\bar{\lambda}_1$.

Repeat for $\sigma_x(\langle \alpha_1, \bar{M}_1 \rangle)$.

Continue this way. Get A_k, α_k, $\bar{M}_k$ so that

$$\alpha_1, \bar{M}_1, \ldots, \alpha_k, \bar{M}_k$$

is according to σ_x for each $x \in A_k$.

μ is ctbly additive. So $\cap_{k<\omega} A_k$ has measure 1.

Let $y = \bigcup_{k<\omega} \pi_{\bar{M}_k,\infty}{}''\bar{\lambda}_k$.

μ is fine, so within each measure 1 set can find some $x \supset y$.

Fix $x \in \bigcap_{k<\omega} A_k$ with $x \supset y$.

Then $\langle \alpha_1, \bar{M}_1, \ldots \rangle$ is an infinite play according to σ_x, and won by player II, contradiction.

Theorem: (In L($\mathbb{R}$), assuming L.C.) For each $\lambda < \Theta$, there is a unique s.c. measure on $\mathcal{P}_{\omega_1}(\lambda)$.

Harrington–Kechris determinacy:

Let λ be an ordinal. Let $\rho\colon \mathbb{R} \to \lambda$ be a norm. Let $A \subset \lambda^\omega$.

Let $G(A)$ be the game where players I and II alternate playing *reals* x_n. Player I wins if $\langle \rho(x_n) \mid n < \omega \rangle \in A$. Otherwise player II wins.

Theorem (H-K): For $\lambda < \boldsymbol{\delta}^2_1$, $G(A)$ is determined.

There is a simple proof of this theorem using the directed system and proofs of determinacy from large cardinals.

Works for $\lambda \leq \boldsymbol{\delta}^2_1$.

Question: Is H-K determinacy true above $\boldsymbol{\delta}^2_1$?

Perhaps more interesting:

Got ultrafilters on $[\cdots]^{<\omega_1}$.

Is it possible to get ultrafilters on sets of longer sequences? An u.f. on $[\mathcal{P}_{\omega_2}(\aleph_\omega)]^{<\omega_2}$ for example? (Not in L($\mathbb{R}$), which doesn't satisfy ω_1-DC, but in L($\mathbb{R}$)$[G]$ where G is generic for col$(\omega, {<}\omega_1)$.)

Could have interesting applications to forcing over L($\mathbb{R}$).

Got an u.f. on $[\omega_1]^{<\omega_1}$. Is there a similar large cardinal construction of an u.f. on $[\boldsymbol{\delta}^1_3]^{<\boldsymbol{\delta}^1_3}$?

Does it subsume the weak partition property for $\boldsymbol{\delta}^1_3$?

Does it lead to a L.C. proof of the strong partition property for $\boldsymbol{\delta}^1_3$?