FINITE STATE AUTOMATA AND MONADIC DEFINABILITY OF SINGULAR CARDINALS

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Abstract. We define a class of finite state automata acting on transfinite sequences, and use these automata to prove that no singular cardinal can be defined by a monadic second order formula over the ordinals.

A formula φ is **monadic second order** (**monadic** for short) if each of its variables is assigned a type, either the type "first order" or the type "second order." When interpreting the formula over a structure with universe A, the first order variables are taken to range over elements of A, and the second order variables are taken to range over subsets (or subclasses) of A. For more on monadic theories we refer the reader to Gurevich [4]. Let us here note that monadic formulae do not allow, at least not directly, talking about sets of *pairs* of elements of A. In particular they need not introduce Gödel sentences, and they need not allow the defining of cardinality.

Let ON be the class of all ordinals. The following are examples of statements about ordinals and sets of ordinals that can be expressed in the monadic language over $(\mathrm{ON}; <)$. We indicate how to express statements (4)–(6). The other three are obvious.

1. "α is a limit ordinal."
2. "C is unbounded in α."
3. "C is closed and unbounded in α."
4. "$\mathrm{cof}(\alpha) \geq \omega$" is expressed simply by the statement that α is a limit ordinal.
5. "$\mathrm{cof}(\alpha) \geq \omega_{n+1}$," expressed, by induction on n, using the formula formalizing the statement $(\forall C)[(C$ is closed unbounded in $\alpha) \rightarrow (\exists\beta)(\beta \in C \wedge \mathrm{cof}(\beta) \geq \omega_n)]$.
6. "$\alpha = \omega_n$," expressed by $(\mathrm{cof}(\alpha) \geq \omega_n) \wedge (\forall\beta < \alpha)(\mathrm{cof}(\beta) \ngeq \omega_n)$.

Thus, for each $n < \omega$, ω_n is definable over $(\mathrm{ON}; <)$ through a monadic formula.

It is natural to ask whether other cardinals may also be definable, and if so which ones. Magidor [6] constructs, from ω supercompact cardinals, a model in which $(\forall A \subset \omega_{\omega+1})(A$ is stationary in $\omega_{\omega+1} \rightarrow (\exists\gamma < \omega_{\omega+1})(A \cap \gamma$ is stationary in $\gamma))$. This statement can be expressed in the monadic language, and it does not hold for any ordinal, of uncountable cofinality, below $\omega_{\omega+1}$. Magidor thus obtains a model in which $\omega_{\omega+1}$ is definable over $(\mathrm{ON}; <)$ through a monadic formula.

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In this paper we tackle the definability of ω_ω. We show in ZFC that it is *not* definable. In fact, no singular cardinal is definable over $(\mathrm{ON}; <)$ through a monadic formula.

Our proof uses certain finite state automata, introduced and defined precisely in Section 2, to uniformly reduce monadic statements about α to statements in a language that allows second order quantifiers and quantifiers of the kind "for almost all $\xi < \alpha$," but does not allow standard first order quantifiers. This language is defined precisely in Section 1. The truth value of sentences in this "almost-all" language is invariant under restrictions to a club subset of the underlying domain. It follows that if a sentence of the language holds in a structure with domain τ, it holds also in a structure with domain $\mathrm{cof}(\tau)$. Hence an almost-all sentence cannot become true for the first time at a singular cardinal.

The main result connecting monadic statements and our finite state automata is Theorem 5.1, where we show that for every monadic formula φ there is an automaton $\langle \mathcal{A}, I, F\rangle$ so that $(\theta; <) \models \varphi[a_1, \dots, a_k]$ iff $\langle \mathcal{A}, I, F\rangle$ accepts the characteristic function of $\langle a_1, \dots, a_k\rangle$. Results of these kind were used by Büchi and others in proofs of decidability of the monadic theory of $(\theta; <)$, first for $\theta = \omega$, then for all countable θ in Büchi [1], for $\theta = \omega_1$ in Büchi [2], and finally for all ordinals $\theta < \omega_2$ in Büchi–Zaiontz [3]. What is new here is the scope and uniformity of the theorem—there is no limitation on θ, and the automaton $\langle \mathcal{A}, I, F\rangle$ depends on φ but *not* on θ—and the fact that our automata may consult the truth value of sentences in the almost-all language during their runs. It is through this latter feature that Theorem 5.1 reduces monadic truth to truth in the almost-all language. It should be noted that there have been earlier generalizations of the theory of automata to ordinals at ω_2 and above, specifically in Wojciechowski [9, 10]. But these generalizations, lacking the reference to the almost-all language, could not capture monadic truth.

With Theorem 5.1 at hand, a simple analysis of runs of our finite state automata, carried out in Section 6 using the fact that almost-all sentences cannot become true for the first time at a singular cardinal, shows that no singular cardinal is definable over $(\mathrm{ON}; <)$ through a monadic formula.

A similar analysis cannot be performed on regular cardinals, since they may be definable through a formula in the almost-all language (as indeed is the case for each ω_n, and for $\omega_{\omega+1}$ in Magidor's model). Let us also note that, though the conversion from φ to $\langle \mathcal{A}, I, F\rangle$ in Theorem 5.1 is effective, it does not by itself establish the decidability of the monadic theory of θ, since the almost-all theory of θ need not be decidable for $\theta \geq \omega_2$. For more on decidability, and undecidability, at the level of ω_2, see Shelah [8] and Gurevich–Magidor–Shelah [5].

§1. The "almost-all" language. Fix, for the entire section, a non-empty finite set S. We describe a language $\mathcal{L}^*_S$ suitable for talking about structures of the form $(\gamma; s, r)$ where γ is an ordinal, $s\colon \gamma \to S$, and $r\colon \gamma \to S \cup \{\uparrow\}$. (The symbol $\uparrow$ stands for "undefined." We sometimes write $r\colon \gamma \rightharpoonup S$, using the partial function symbol $\rightharpoonup$, as a shorthand for $r\colon \gamma \to S \cup \{\uparrow\}$.) The language allows second order quantifiers, and a certain cross between second and first order quantifiers stating that a property holds for a club (namely closed unbounded set) of ordinals.

Definition 1.1. For a function $t\colon \delta \to S$, where δ is an ordinal, define $\mathrm{cf}(t) = \{b \in S \mid \text{the set } \{\xi \mid t(\xi) = b\} \text{ is cofinal in } \delta\}$.

Definition 1.2. The formulae of $\mathcal{L}^*_S$ are the ones generated through the following conditions:

1. $\alpha \in A$, $s(\alpha) = b$, $r(\alpha) = b$, $b \in \mathrm{cf}(s)$, and $b \in \mathrm{cf}(s{\restriction}\alpha)$, where α is a first order variable, A a second order variable, and b an element of S, are atomic formulae in $\mathcal{L}^*_S$.
2. If φ and ψ are formulae in $\mathcal{L}^*_S$ then so are $\neg\varphi$ and $(\varphi \wedge \psi)$.
3. If φ is a formula in $\mathcal{L}^*_S$ then so is $(\exists A)\varphi$, where A is a second order variable.
4. If φ is a formula in $\mathcal{L}^*$ then so are $(\forall^*\alpha < \beta)\varphi$ and $(\forall^*\alpha)\varphi$, where α and β are first order variables.

When a formula φ in the language $\mathcal{L}^*_S$ is interpreted over the structure $(\gamma; s, r)$, its first order variables range over elements of γ, and its second order variable range over subsets of γ.

Definition 1.3. The truth value of formulae in $\mathcal{L}^*_S$ is defined subject to the conditions below. In conditions (3) and (4) we suppress the variables of φ which remain free after the quantification, for notational convenience.

1. $(\gamma; s, r) \models \alpha \in A$ just in case that $\alpha \in A$, and similarly with the other atomic formulae. (If $r(\alpha) = \uparrow$ then for all $b \in S$, $(\gamma; s, r) \not\models r(\alpha) = b$.)
2. The truth value for conjunctions and negations is defined in the obvious way.
3. $(\gamma; s, r) \models (\exists A)\varphi$ just in case that $(\gamma; s, r) \models \varphi[A]$ for some $A \subset \gamma$.
4. $(\gamma; s, r) \models (\forall^*\alpha < \beta)\varphi$ just in case that:
   (a) β is a limit ordinal of cofinality greater than ω, and
   (b) there exists a club $C \subset \beta$ so that $(\gamma; s, r) \models \varphi[\alpha]$ for all $\alpha \in C$.
   $(\gamma; s, r) \models (\forall^*\alpha)\varphi$ just in case that the same conditions hold, but with β replaced by γ.

We use "φ is true of $x_1, \ldots, x_k$ in $(\gamma; s, r)$" and "$\varphi[x_1, \ldots, x_k]$ is true in $(\gamma; s, r)$" as synonyms for $(\gamma; s, r) \models \varphi[x_1, \ldots, x_k]$.

Claim 1.4. *There are sentences $\varphi_{\mathrm{ctbl-cof}}$ and $\varphi_{\mathrm{cof}\geq\omega_1}$ in $\mathcal{L}^*_S$ so that $(\gamma; s, r) \models \varphi_{\mathrm{ctbl-cof}}$ iff $\mathrm{cof}(\gamma) \leq \omega$, and $(\gamma; s, r) \models \varphi_{\mathrm{cof}\geq\omega_1}$ iff $\mathrm{cof}(\gamma) \geq \omega_1$.*

Proof. Fix $b \in S$. Let ψ be the sentence $(\forall^*\alpha)(s(\alpha) = b \vee \neg s(\alpha) = b)$. Condition (4) in Definition 1.3 is such that $(\gamma; s, r) \models \psi$ iff $\mathrm{cof}(\gamma) \geq \omega_1$. Let $\varphi_{\mathrm{cof}\geq\omega_1} = \psi$ and let $\varphi_{\mathrm{ctbl-cof}} = \neg\psi$. ⊣

Claim 1.5. *Let $b \in S$. There is a formula $\varphi_{\mathrm{stat-}b}(x)$ in $\mathcal{L}^*_S$ so that $(\gamma; s, r) \models \varphi_{\mathrm{stat-}b}[A]$ iff $\mathrm{cof}(\gamma) \geq \omega_1$ and $\{\xi < \gamma \mid \xi \in A \wedge s(\xi) = b\}$ is stationary in γ.*

Proof. Let $\varphi_{\mathrm{stat-}b}$ be the sentence $\varphi_{\mathrm{cof}\geq\omega_1} \wedge \neg(\forall^*\alpha)\neg(\alpha \in A \wedge s(\alpha) = b)$. ⊣

Notice that $\mathcal{L}^*_S$ does not allow quantification over individual ordinals. In particular a sentence φ which is true in $(\gamma; s, r)$ is also true in $(\gamma; s^*, r^*)$ whenever s^* and r^* agree with s and r on all but finitely many ordinals. In fact this can be strengthened:

Claim 1.6. *Let φ be a sentence in $\mathcal{L}_S^*$. Then the truth value of φ in a structure $(\gamma; s, r)$ with γ of cofinality ω (or a successor) depends only on* $\mathrm{cf}(s)$.

Proof. The following stronger statement holds:

($*$) Let $\varphi(x_1, \ldots, x_k)$ be a formula with free variables of only the second order. Then the truth value of $\varphi[A_1, \ldots, A_k]$ in a structure $(\gamma; s, r)$ with γ of cofinality ω (or a successor) depends only on $\mathrm{cf}(s)$.

The proof of ($*$) is by induction on the complexity of φ. The base case consists of (1) formulae of the form $b \in \mathrm{cf}(s)$, for which ($*$) obviously holds; and (2) formulae of the form $(\forall^* \alpha)\psi(\alpha, \ldots)$, which are always false in structures $(\gamma; s, r)$ with γ a successor or a limit of cofinality ω, by condition (4) in Definition 1.3, so that again ($*$) holds. The inductive case, handling logical connectives and second order quantifiers, is straightforward. There is no need to handle formulae of the form $(\forall^* \alpha < \beta)\varphi$, since ($*$) is restricted to formulae with no free first order variables, and $(\forall^* \alpha < \beta)\varphi$ has β free. $\dashv$

Definition 1.7. Let $D \subset S$, and let φ be a sentence in $\mathcal{L}_S^*$. We write that $D \models \varphi$ to mean that $(\gamma; s, r) \models \varphi$ for some/all structures $(\gamma; s, r)$ with $\mathrm{cf}(s) = D$ and γ of cofinality ω. The terminology makes sense by the previous claim.

Definition 1.8. Two structures $(\gamma; s, r)$ and $(\gamma^*; s^*, r^*)$ are **similar**, denoted $(\gamma; s, r) \sim (\gamma^*; s^*, r^*)$, if:

1. $\mathrm{cf}(s) = \mathrm{cf}(s^*)$.
2. There are clubs C in γ and C^* in γ^*, and an order preserving bijection $f \colon C \to C^*$, so that $s^*(f(\xi)) = s(\xi)$ and $r^*(f(\xi)) = r(\xi)$ for all $\xi \in C$.

Claim 1.9. *Let φ be a sentence in $\mathcal{L}_S^*$. Let $(\gamma; s, r)$ and $(\gamma^*; s^*, r^*)$ be similar. Then $(\gamma; s, r) \models \varphi$ iff $(\gamma^*; s^*, r^*) \models \varphi$.*

Proof. The cases of successor γ and γ of cofinality ω follow from Claim 1.6, since $\mathrm{cf}(s) = \mathrm{cf}(s^*)$. So suppose that $\mathrm{cof}(\gamma) \geq \omega_1$. Shrinking C and C^* if necessary we may then assume that $\mathrm{cf}(s \restriction \xi) = \mathrm{cf}(s)$ for each $\xi \in C$, and $\mathrm{cf}(s^* \restriction \xi) = \mathrm{cf}(s^*)$ for each $\xi \in C^*$. Since $\mathrm{cf}(s) = \mathrm{cf}(s^*)$ it follows that:

(i) $\mathrm{cf}(s \restriction \xi) = \mathrm{cf}(s^* \restriction f(\xi))$ for each $\xi \in C$.

The claim follows from the following, more general statement:

($*$) Let $\varphi(x_1, \ldots, x_k)$ be a formula with k free variables. Then

$$(\gamma; s, r) \models \varphi[a_1, \ldots, a_k] \iff (\gamma^*; s^*, r^*) \models \varphi[a_1^*, \ldots, a_k^*]$$

whenever $a_1, \ldots, a_k$ and $a_1^*, \ldots, a_n^*$ are such that:

1. $a_i \in C$ and $a_i^* \in C^*$ for i so that x_i is first order.
2. $a_i^* = f(a_i)$ for i so that x_i is first order.
3. $a_i^* \cap C^* = f''(a_i \cap C)$ for i so that x_i is second order.

The proof of ($*$) is an induction on the complexity of φ. The base case consists of atomic φ, for which ($*$) follows from the conditions of Definition 1.8 and condition (i) above. The inductive cases are straightforward. Let us just note that for φ of the form $(\forall^* \alpha)\psi$ or $(\forall^* \alpha < \beta)\psi$, the clubs witnessing truth in $(\gamma; s, r)$ can be taken to be subsets of C, and similarly with $(\gamma^*; s^*, r^*)$ and C^*. $\dashv$

Definition 1.10. $r, r^*\colon \gamma \rightharpoonup S$ are **almost equal**, denoted $r \approx r^*$, if $r(\alpha) = r^*(\alpha)$ for all but finitely many $\alpha \in \gamma$.

Corollary 1.11. *Suppose that $r \approx r^*$. Then for every sentence φ in $\mathcal{L}^*_S$, $(\gamma; s, r) \models \varphi$ iff $(\gamma; s, r^*) \models \varphi$.*

Proof. Immediate from Claim 1.9. $\dashv$

Remark 1.12. Let $\varphi(x_1, \dots, x_k)$ be a formula in $\mathcal{L}^*_S$ with free variables of only the second order. The proof of Claim 1.9 shows that $(\gamma; s, r) \models \varphi[A_1, \dots, A_k]$ iff $(\gamma; s, r) \models \varphi[A'_1, \dots, A'_k]$ whenever $\mathrm{cof}(\gamma) \geq \omega_1$ and there is a club subset C of γ so that $A_i \cap C = A'_i \cap C$ for all i.

Definition 1.13. Let τ be some sentence in $\mathcal{L}^*_S$, let False be the sentence $(\tau \wedge \neg\tau)$, and let True be the sentence $\neg\mathsf{False}$. For a formula φ and a set $D \subset S$ let φ_D be the formula obtained from φ by replacing each occurrence of $b \in \mathrm{cf}(s)$ or $b \in \mathrm{cf}(s{\restriction}\alpha)$ in φ by True if $b \in D$ and by False if $b \in S - D$.

Claim 1.14. *Suppose that $\mathrm{cof}(\gamma) \geq \omega_1$. Then for every sentence φ in $\mathcal{L}^*_S$, $(\gamma; s, r) \models \varphi$ iff $(\gamma; s, r) \models \varphi_D$ where $D = \mathrm{cf}(s)$.*

Proof. Fix $(\gamma; s, r)$ with $\mathrm{cof}(\gamma) \geq \omega_1$. Let $D = \mathrm{cf}(s)$. Let $C \subset \gamma$ be a club so that $\mathrm{cf}(s{\restriction}\alpha) = D$ for all $\alpha \in C$. An induction on complexity, similar to the one used in the proof of Claim 1.9, establishes the following statement:

$(*)$ Let $\varphi(x_1, \dots, x_k)$ be a formula with k free variables. Then

$$(\gamma; s, r) \models \varphi[a_1, \dots, a_k] \iff (\gamma; s, r) \models \varphi_D[a_1, \dots, a_k]$$

whenever $a_1, \dots, a_k$ are such that $a_i \in C$ for i so that x_i is first order.

The current claim is the special case of $(*)$ with $k = 0$. $\dashv$

Definition 1.15. Given a formula $\varphi(x_1, \dots, x_k)$ in $\mathcal{L}^*_S$, let $\varphi^{\mathrm{rel}}(x_1, \dots, x_k, \delta)$ be obtained from φ be replacing each first order quantification of the form $(\forall^*\alpha)$ in φ by $(\forall^*\alpha < \delta)$, and replacing each occurrence of $\mathrm{cf}(s)$ in φ by $\mathrm{cf}(s{\restriction}\delta)$.

Claim 1.16. *Let $\delta < \gamma$. Then*

$$(\delta; s{\restriction}\delta, r{\restriction}\delta) \models \varphi[x_1, \dots, x_k] \iff (\gamma; s, r) \models \varphi^{\mathrm{rel}}[x_1, \dots, x_k, \delta].$$

Proof. Immediate by induction on the complexity of φ. $\dashv$

Definition 1.17. Let $\widehat{S}$ be a non-empty finite set. For functions $s\colon \gamma \to S$ and $\hat{s}\colon \gamma \to \widehat{S}$ define the function $s \times \hat{s}\colon \gamma \to S \times \widehat{S}$ by $(s \times \hat{s})(\alpha) = \langle s(\alpha), \hat{s}(\alpha)\rangle$. Define $r \times \hat{r}$ for partial functions $r\colon \gamma \rightharpoonup S$ and $\hat{r}\colon \gamma \rightharpoonup \widehat{S}$ similarly, with $(r \times \hat{r})(\alpha) =\uparrow$ if either $r(\alpha) =\uparrow$ or $\hat{r}(\alpha) =\uparrow$.

Lemma 1.18. *Let φ be a sentence in the language $\mathcal{L}^*_{S \times \widehat{S}}$. Then there is a sentence φ^{exist} in the language $\mathcal{L}^*_S$ so that, for limit ordinals γ,*

$$(\gamma; s, r) \models \varphi^{\mathrm{exist}} \iff (\exists \hat{s}\colon \gamma \to \widehat{S})(\exists \hat{r}\colon \gamma \rightharpoonup \widehat{S})\ (\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi.$$

Proof. We handle the cases $\mathrm{cof}(\gamma) \geq \omega_1$ and $\mathrm{cof}(\gamma) = \omega$ separately, and will compose them later. The case of $\mathrm{cof}(\gamma) = \omega$ is easily handled using Claim 1.6. The case of $\mathrm{cof}(\omega) \geq \omega_1$ is handled by converting the quantifiers $(\exists \hat{s}\colon \gamma \to \widehat{S})$ and $(\exists \hat{r}\colon \gamma \rightharpoonup \widehat{S})$ into second order quantifiers over γ.

Consider first the case that $\mathrm{cof}(\gamma) = \omega$. Let $Q = \{E \subset S \times \widehat{S} \mid E \models \varphi\}$. (We are using here the notation of Definition 1.7.) Then, for γ of cofinality ω, $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$ iff $\mathrm{cf}(s \times \hat{s}) \in Q$.

For each $E \subset S \times \widehat{S}$ let $\mathrm{proj}(E)$ be the projection of E to S, that is the set $\{b \mid (\exists \hat{b})\langle b, \hat{b}\rangle \in E\}$. Using the fact that $\widehat{S}$ is finite, it is easy to check that $(\exists \hat{s})\,\mathrm{cf}(s \times \hat{s}) = E$ is true iff $\mathrm{proj}(E) = \mathrm{cf}(s)$. So $(\exists \hat{s})\,\mathrm{cf}(s \times \hat{s}) \in Q$ iff there is $E \in Q$ with $\mathrm{proj}(E) = \mathrm{cf}(s)$.

For $D \subset S$ let $\psi_{1,D}$ be the sentence $((\bigwedge_{b \in D} b \in \mathrm{cf}(s)) \wedge (\bigwedge_{b \in S-D} \neg b \in \mathrm{cf}(s)))$ in the language $\mathcal{L}^*_S$, so that $(\gamma; s, r) \models \psi_{1,D}$ iff $\mathrm{cf}(s) = D$. Let ψ_1 be the sentence $\bigvee_{E \in Q} \psi_{1,\mathrm{proj}(E)}$. Then:

(i) For γ of cofinality ω, $(\gamma; s, r) \models \psi_1$ if and only if $(\exists \hat{s} \colon \gamma \to \widehat{S})(\exists \hat{r} \colon \gamma \rightharpoonup \widehat{S})$ $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$.

This takes care of the case of γ of cofinality ω in the proof of Lemma 1.18.

Consider next the case that $\mathrm{cof}(\gamma) \geq \omega_1$. For each $E \subset S \times \widehat{S}$ let φ_E be the formula given by Definition 1.13, so that:

(ii) For γ of cofinality $\geq \omega_1$, $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$ iff $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi_E$ where $E = \mathrm{cf}(s \times \hat{s})$.

Notice that the only references to $s \times \hat{s}$ and $r \times \hat{r}$ in φ_E come through atomic formulae of the form $(s \times \hat{s})(\alpha) = \langle b, \hat{b}\rangle$ and $(r \times \hat{r})(\alpha) = \langle b, \hat{b}\rangle$. Let $\hat{b}_1, \dots, \hat{b}_n$ enumerate $\widehat{S}$. Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be distinct second order variables which do not appear in φ. Let $\psi_{2,E}(x_1, \dots, x_n, y_1, \dots, y_n)$ be the formula obtained from φ_E by replacing every occurrence of $(s \times \hat{s})(\alpha) = \langle b, \hat{b}_i\rangle$ in φ_E by $(s(\alpha) = b \wedge \alpha \in x_i)$, and similarly replacing every occurrence of $(r \times \hat{r})(\alpha) = \langle b, \hat{b}_i\rangle$ in φ_E by $(r(\alpha) = b \wedge \alpha \in y_i\rangle$. $\psi_{2,E}$ is then a formula in $\mathcal{L}^*_S$, and:

(iii) For γ of cofinality $\geq \omega_1$,

$$(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi \iff (\gamma; s, r) \models \psi_{2,E}[A_1, \dots, A_n, B_1, \dots, B_n]$$

where $E = \mathrm{cf}(s \times \hat{s})$, $A_i = \{\xi \mid \hat{s}(\xi) = \hat{b}_i\}$, and $B_i = \{\xi \mid \hat{r}(\xi) = \hat{b}_i\}$.

Call a tuple $\langle E, A_1, \dots, A_n, B_1, \dots, B_n\rangle$, where $E \subset S \times \widehat{S}$ and $A_1, \dots, B_n \subset \gamma$, **suitable** for $s \colon \gamma \to S$ if:

(a) $\mathrm{proj}(E) = \mathrm{cf}(s)$.
(b) For each $b \in S$ and $i \leq n$, if $\{\xi \mid s(\xi) = b \wedge \xi \in A_i\}$ is stationary in γ then $\langle b, \hat{b}_i\rangle \in E$.
(c) For almost all $\alpha < \gamma$ (meaning for all α in a club subset of γ), α belongs to exactly one of $A_1, \dots, A_n$, and to at most one of $B_1, \dots, B_n$.

Note that there is a formula $\psi_{\mathrm{suit}-E}(x_1, \dots, x_n, y_1, \dots, y_n)$ in $\mathcal{L}^*_S$ so that $(\gamma; s, r) \models \psi_{\mathrm{suit}-E}[A_1, \dots, A_n, B_1, \dots, B_n]$ iff $\langle E, A_1, \dots, A_n, B_1, \dots, B_n\rangle$ is suitable for s.

CLAIM 1.19. *Let γ have cofinality $\geq \omega_1$, let $s \colon \gamma \to S$, and let $r \colon \gamma \rightharpoonup S$. Then $(\exists \hat{s} \colon \gamma \to \widehat{S})(\exists \hat{r} \colon \gamma \rightharpoonup \widehat{S})$ $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$ iff there is $E \subset S \times \hat{S}$, and there exist $A_1, \dots, A_n, B_1, \dots, B_n \subset \gamma$, so that:*

1. *$\langle E, A_1, \dots, A_n, B_1, \dots, B_n\rangle$ is suitable for s.*
2. *$(\gamma; s, r) \models \psi_{2,E}[A_1, \dots, A_n, B_1, \dots, B_n]$.*

PROOF. The direction left-to-right is clear: simply take $E = \mathrm{cf}(s \times \hat{s})$, $A_i = \{\xi \mid \hat{s}(\xi) = \hat{b}_i\}$, and $B_i = \{\xi \mid \hat{r}(\xi) = \hat{b}_i\}$.

Suppose conversely that E, $A_1, \dots, A_n$, and $B_1, \dots, B_n$ satisfy conditions (1) and (2). Using conditions (a)–(c) it is possible to find $\hat{s}\colon \gamma \to \widehat{S}$, $\hat{r}\colon \gamma \rightharpoonup \widehat{S}$, and a club $C \subset \gamma$ so that $\mathrm{cf}(s \times \hat{s}) = E$, $\{\xi \mid \hat{s}(\xi) = \hat{b}_i\} \cap C = A_i \cap C$, and $\{\xi \mid \hat{r}(\xi) = \hat{b}_i\} \cap C = B_i \cap C$ for each i. By condition (2), condition (iii) above, and Remark 1.12, $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$. ⊣

For each $E \subset S \times \widehat{S}$ let $\psi_{3,E}$ be the sentence $(\exists x_1) \cdots (\exists x_n)(\exists y_1) \cdots (\exists y_n)$ $(\psi_{\mathrm{suit}-E}(x_1, \dots, x_n, y_1, \dots, y_n) \wedge \psi_{2,E}(x_1, \dots, x_n, y_n, \dots, y_n))$ in the language $\mathcal{L}^*_S$. Let ψ_3 be the sentence $\bigvee_{E \subset S \times \widehat{S}} \psi_{3,E}$. By the last claim,

(iv) For γ of cofinality $\geq \omega_1$, $(\gamma; s, r) \models \psi_3$ iff $(\exists \hat{s}\colon \gamma \to \widehat{S})(\exists \hat{r}\colon \gamma \rightharpoonup \widehat{S})$ $(\gamma; s \times \hat{s}, r \times \hat{r}) \models \varphi$.

Now let φ^{exist} be the sentence $((\varphi_{\mathrm{ctbl-cof}} \wedge \psi_1) \vee (\varphi_{\mathrm{cof} \geq \omega_1} \wedge \psi_3))$, where ψ_1 is taken from condition (i) above, and $\varphi_{\mathrm{ctbl-cof}}$ and $\varphi_{\mathrm{cof} \geq \omega_1}$ are taken from Claim 1.4. Then φ^{exist} satisfies the requirements of Lemma 1.18. □ (Lemma 1.18)

CLAIM 1.20. *Let $\widehat{S}$ be finite non-empty. Let $\pi_1\colon \widehat{S} \to S$ and $\pi_2\colon \widehat{S} \times \widehat{S} \to S$. (We refer to π_1 and π_2 as* **projections**.*) Let φ be a sentence in $\mathcal{L}^*_S$. Then there is a sentence $\hat{\varphi}$ in $\mathcal{L}^*_{\widehat{S}}$ so that, for limit γ, $(\gamma; \hat{s}, \hat{r}) \models \hat{\varphi}$ if and only if $(\gamma; \pi_1 \circ \hat{s}, \pi_2 \circ (\hat{s} \times \hat{r})) \models \varphi$.*

PROOF. For each $b \in S$ let $P_1(b) = {\pi_1}^{-1\prime\prime}\{b\}$ and let $P_2(b) = {\pi_2}^{-1\prime\prime}\{b\}$. Let $\hat{\varphi}$ be obtained from φ by replacing every occurrence of $s(\alpha) = b$ in φ by $\bigvee_{\hat{b} \in P_1(b)} \hat{s}(\alpha) = \hat{b}$, making similar replacements to occurrences of $b \in \mathrm{cf}(s)$ and $b \in \mathrm{cf}(s \restriction \alpha)$, and replacing each occurrence of $r(\alpha) = b$ by $\bigvee_{\langle \hat{b}, \hat{c} \rangle \in P_2(b)} (\hat{s}(\alpha) = \hat{b} \wedge \hat{r}(\alpha) = \hat{c})$. (Empty disjunctions, if they occur, are taken to be the sentence False.) $\hat{\varphi}$ is then a sentence in $\mathcal{L}^*_{\widehat{S}}$, and it is easy to check that it satisfies the demands of the claim. Let us just note that the verification uses the equivalence $b \in \mathrm{cf}(\pi_1 \circ \hat{s})$ iff $(\exists \hat{b})(\pi_1(\hat{b}) = b \wedge \hat{b} \in \mathrm{cf}(\hat{s}))$. The right-to-left direction of this equivalence is immediate, and the left-to-right direction uses the fact that $\widehat{S}$ is finite. ⊣

§2. Automata.

Let Σ be a finite non-empty set. By a **Σ-automaton** we mean a tuple $\mathcal{A} = \langle S, P, T, \vec{\varphi}, \Psi, h, u \rangle$ where:

1. S and P are finite non-empty sets.
2. $T \subset S \times \Sigma \times S$.
3. $\vec{\varphi} = \langle \varphi_1, \dots, \varphi_k \rangle$ is a finite tuple of sentences in $\mathcal{L}^*_S$.
4. Ψ is a function from 2^k into S, where $k = \mathrm{lh}(\vec{\varphi})$.
5. u is a function from S into $\{U \mid U \subsetneq P\}$.
6. h is a function from S into P with the property that $h(b) \in P - u(b)$ for each $b \in S$.

$\mathcal{A}$ is called **deterministic** if T is a function from $S \times \Sigma$ into S, meaning that for each pair $\langle b, \sigma \rangle \in S \times \Sigma$ there is precisely one $b^* \in S$ so that $\langle b, \sigma, b^* \rangle \in T$.

We refer to Σ as the **alphabet**, to S as the set of **states** of $\mathcal{A}$, and to P as the set of **pebbles**. T is the **successor transition table**. $\vec{\varphi}$ and Ψ determine limit transitions in a way that we explain below. h and u determine the placement and maintenance of pebbles.

Definition 2.1. Let $\vec{\varphi}$ and Ψ be as in conditions (3) and (4) above. Given a domain $(\gamma; s, r)$ with $\gamma \in \mathrm{ON}$, $s\colon \gamma \to S$, and $r\colon \gamma \rightharpoonup S$, define $t^{\vec{\varphi}}_{(\gamma;s,r)}\colon k \to 2$ by setting $t^{\vec{\varphi}}_{(\gamma;s,r)}(i) = 1$ if $(\gamma; s, r) \models \varphi_i$ and $t^{\vec{\varphi}}_{(\gamma;s,r)}(i) = 0$ otherwise for each $i \leq k$. Define a function $\Psi \oplus \vec{\varphi}$, acting on domains $(\gamma; s, r)$ as above, by setting $(\Psi \oplus \vec{\varphi})(\gamma; s, r) = \Psi(t^{\vec{\varphi}}_{(\gamma;s,r)})$.

Remark 2.2. For $D \subset S$, set, using the terminology of Definition 1.7, $t^{\vec{\varphi}}_D(i) = 1$ if $D \models \varphi_i$ and $t^{\vec{\varphi}}_D(i) = 0$ otherwise. Set $(\Psi \oplus \vec{\varphi})(D) = \Psi(t^{\vec{\varphi}}_D)$. For γ of cofinality ω then, $(\Psi \oplus \vec{\varphi})(\gamma; s, r) = (\Psi \oplus \vec{\varphi})(\mathrm{cf}(s))$.

Let α be an ordinal and let $X\colon \alpha \to \Sigma$. A pair $\langle s, r\rangle$ where $s\colon \alpha + 1 \to S$ and $r\colon \alpha \rightharpoonup S$ is called a **run** of $\mathcal{A}$ on X just in case that it satisfies the following conditions:

(S) $\langle s(\xi), X(\xi), s(\xi + 1)\rangle \in T$ for each $\xi < \alpha$.
(L) $s(\lambda) = (\Psi \oplus \vec{\varphi})(\lambda; s\restriction\lambda, r\restriction\lambda)$ for each limit $\lambda \leq \alpha$.
(R) If there exists some $\gamma > \xi$ so that $h(s(\xi)) \notin u(s(\gamma))$ then $r(\xi) = s(\gamma)$ for the least such γ, and otherwise $r(\xi)$ is undefined.

We think of $\mathcal{A}$ is running over the input $X\colon \alpha \to \Sigma$ and producing a run $\langle s, r\rangle$ through a transfinite sequence of stages. In each stage β the automaton determines $s(\beta)$ through either condition (S) or condition (L), depending on whether β is a successor or a limit. In the case of a successor $\xi+1$, the automaton determines the state $s(\xi+1)$ based on the previous state $s(\xi)$ and the input $X(\xi)$, in line with the transition table T. Condition (S) expresses this precisely. In the case of limit λ, the automaton determines $s(\lambda)$ based on a bounded fragment of the almost-all theory of the run $(\lambda; s\restriction\lambda, r\restriction\lambda)$ produced so far. Condition (L) expresses this precisely. The fragment of the theory being consulted is the truth values of sentences in $\vec{\varphi}$. The function Ψ tells the automaton how to determine $s(\lambda)$ based on the fragment.

Having determined $s(\beta)$, the automaton places the pebble $p = h(s(\beta))$ on the ordinal β. The pebble p remains placed on β until a later stage β^* is reached with $p \notin u(s(\beta^*))$. At the first such stage β^* the automaton removes the pebble from β, and sets $r(\beta) = s(\beta^*)$. This is expressed precisely in condition (R). $r(\beta)$ remains undefined until the pebble placed on β is removed, and may indeed remain undefined throughout, if the pebble is not removed at all during the run. The use of pebbles therefore introduces a delay into part of the construction of a run. Our need for this delay will be explained later, in Remarks 4.1 and 4.14.

Notice that no pebble is ever in the uncomfortable position of having to be on two or more ordinals at the same time: when $p = h(s(\beta))$ is placed on β, condition (6) in the definition of automaton above guarantees that $p \notin u(s(\beta))$, and this results in the removal of p from any ordinal $\bar{\beta} < \beta$ on which it might have been placed before.

When reaching a limit stage λ, the automaton is commanded by condition (L) to look at the structure $(\lambda; s\restriction\lambda, r\restriction\lambda)$, check which of the sentences φ_i hold in this structure, and determine $s(\lambda)$ on the basis of this information through a finite table given by the function Ψ.

There are, conceivably, two ways to interpret this command. One would have the automaton look at the values of $r\restriction\lambda$ that are known by stage λ. The other would have the automaton look at the values reached by the end of the run. Let $(r\restriction\lambda)^{\text{local}}$ consist of the values known by stage λ, and let $(r\restriction\lambda)^{\text{global}}$ consist of the values known at the end of the run. The two functions need not be the same. There may well be ordinals $\xi < \lambda$ which still have their pebbles at stage λ, and have the pebbles removed later on. $(r\restriction\lambda)^{\text{local}}$ is not defined on these ordinals, and $r(\restriction\lambda)^{\text{global}}$ is. But there can only be finitely many such ordinals ξ, since each of these ordinals requires a separate pebble, and the set P of pebbles is finite. Thus $(r\restriction\lambda)^{\text{local}} \approx (r\restriction\lambda)^{\text{global}}$. By Corollary 1.11 then, a sentence φ of $\mathcal{L}_S^*$ is true in $(\lambda; s\restriction\lambda, (r\restriction\lambda)^{\text{local}})$ iff it is true in $(\lambda; s\restriction\lambda, (r\restriction\lambda)^{\text{global}})$. So it does not matter whether condition (L) is interpreted using $(r\restriction\lambda)^{\text{local}}$ or $(r\restriction\lambda)^{\text{global}}$. The end result of both interpretations is the same.

We generally use $(r\restriction\lambda)^{\text{local}}$ when determining $s(\lambda)$. This after all is the only practical approach, since $(r\restriction\lambda)^{\text{global}}$ is not yet known at stage λ. Condition (L) is written using what is really $(r\restriction\lambda)^{\text{global}}$ only because writing it using $(r\restriction\lambda)^{\text{local}}$ would make the notation of the definition of a run much more complicated.

At a successor stage $\xi + 1$ the automaton determines $s(\xi + 1)$ on the basis of the state $s(\xi)$ and input $X(\xi)$ at stage ξ, using a finite table T. This approach, formulated by condition (S) above, is standard for automata. If the automaton is deterministic, meaning that T is a function, then there is precisely one state b so that $\langle s(\xi), X(\xi), b\rangle \in T$, and in this case the automaton is forced to set $s(\xi + 1)$ equal to this b. But in general there may be many (or no) states b so that $\langle s(\xi), X(\xi), b\rangle \in T$, and the automaton may choose between them. Thus, in general, there may be many different runs of $\mathcal{A}$ on the same input X.

An **accepting condition** for an automaton $\mathcal{A}$ is a pair $\langle I, F\rangle$ where $I \in S$ and $F \subset S$. $\langle \mathcal{A}, I, F\rangle$ is said to **accept** $X\colon \alpha \to \Sigma$ just in case that there exists a run $\langle s, r\rangle$ of $\mathcal{A}$ on X so that $s(0) = I$ and $s(\alpha) \in F$. $\mathcal{L}(\mathcal{A}, I, F)$, the **language recognized by** $\langle \mathcal{A}, I, F\rangle$, is the class $\{X \mid X\colon \alpha \to \Sigma$ for some ordinal α, and $\langle \mathcal{A}, I, F\rangle$ accepts $X\}$.

We will show that the collection of languages recognized by automata is closed under complements, intersections, and projections.

Claim 2.3 (Closure under projections). *Let $\widehat{\Sigma}$ be a finite non-empty set. Let $\mathcal{A}$ be a $\Sigma \times \widehat{\Sigma}$–automaton, and let $\langle I, F\rangle$ be an accepting condition for $\mathcal{A}$. Then there is a Σ–automaton $\mathcal{A}^*$, and an accepting condition $\langle I^*, F^*\rangle$ for $\mathcal{A}^*$, so that $\langle \mathcal{A}^*, I^*, F^*\rangle$ accepts $X\colon \alpha \to \Sigma$ iff there exists $\widehat{X}\colon \alpha \to \widehat{\Sigma}$ so that $\langle \mathcal{A}, I, F\rangle$ accepts $X \times \widehat{X}$.*

Proof. This is a standard claim, using non-determinism to have $\mathcal{A}^*$ pick $\widehat{X}$ as part of its run, thereby absorbing the quantifier $(\exists\widehat{X})$ in the claim into the quantifier "there exists a run" in the definition of acceptance. To be slightly more precise, it is easy to design a Σ–automaton $\mathcal{A}^*$, with a set of states $S^* = S \times \widehat{\Sigma}$, so that:

1. If $\langle s \times \hat{s}, r \times \hat{r}\rangle$ is a run of $\mathcal{A}^*$ on X, then $\langle s, r\rangle$ is a run of $\mathcal{A}$ on $X \times \widehat{X}$ where $\widehat{X}$ is given by the condition $\widehat{X}(\xi) = \hat{s}(\xi + 1)$ for $\xi < \alpha$.
2. If $\langle s, r\rangle$ is a run of $\mathcal{A}$ on $X \times \widehat{X}$, then there are $\hat{s}$ and $\hat{r}$ so that:
   - $\hat{s}(\xi+1) = \widehat{X}(\xi)$ for each $\xi < \alpha$, $\hat{s}(0) = \hat{\sigma}_0$, and $\hat{s}(\lambda) = \hat{\sigma}_0$ for each limit $\lambda \leq \alpha$, where $\hat{\sigma}_0$ is some fixed element of $\widehat{\Sigma}$.
   - $\langle s \times \hat{s}, r \times \hat{r}\rangle$ is a run of $\mathcal{A}^*$ on X.

$\langle \mathcal{A}^*, \langle I, \hat{\sigma}_0\rangle, F \times \widehat{\Sigma}\rangle$ then accepts X iff there is $\widehat{X}$ so that $\langle \mathcal{A}, I, F\rangle$ accepts $X \times \widehat{X}$. ⊣

Lemma 2.4 (Closure under intersections). *Let $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ be two Σ–automata with accepting conditions $\langle I_{\mathrm{L}}, F_{\mathrm{L}}\rangle$ and $\langle I_{\mathrm{R}}, F_{\mathrm{R}}\rangle$. Then there is a Σ–automaton $\mathcal{A}_{\mathrm{C}}$, with accepting condition $\langle I_{\mathrm{C}}, F_{\mathrm{C}}\rangle$, so that $\langle \mathcal{A}_{\mathrm{C}}, I_{\mathrm{C}}, F_{\mathrm{C}}\rangle$ accepts X iff both $\langle \mathcal{A}_{\mathrm{L}}, I_{\mathrm{L}}, F_{\mathrm{L}}\rangle$ and $\langle \mathcal{A}_{\mathrm{R}}, I_{\mathrm{R}}, F_{\mathrm{R}}\rangle$ accept X. (If $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ are both deterministic, then so is $\mathcal{A}_{\mathrm{C}}$.)*

Proof. We intend to have $\mathcal{A}_{\mathrm{C}}$ produce runs that combine both the action of $\mathcal{A}_{\mathrm{L}}$ and the action of $\mathcal{A}_{\mathrm{R}}$. The only difficulty is with the pebbles, as $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ may wish to release the pebble placed on an ordinal ξ at different times. $\mathcal{A}_{\mathrm{C}}$ needs a memory cell that will hold the state causing the first release, until the time of the second release.

Let $\langle S_{\mathrm{L}}, P_{\mathrm{L}}, T_{\mathrm{L}}, \vec{\varphi}_{\mathrm{L}}, \Psi_{\mathrm{L}}, h_{\mathrm{L}}, u_{\mathrm{L}}\rangle$ be the automaton $\mathcal{A}_{\mathrm{L}}$ and let $\langle S_{\mathrm{R}}, P_{\mathrm{R}}, T_{\mathrm{R}}, \vec{\varphi}_{\mathrm{R}}, \Psi_{\mathrm{R}}, h_{\mathrm{R}}, u_{\mathrm{R}}\rangle$ be the automaton $\mathcal{A}_{\mathrm{R}}$. Without loss of generality S_{L} and S_{R} are disjoint.

Let A be the set of partial functions from $P_{\mathrm{L}} \times P_{\mathrm{R}}$ into $S_{\mathrm{L}} \cup S_{\mathrm{R}}$. Let $S_{\mathrm{C}} = S_{\mathrm{L}} \times S_{\mathrm{R}} \times A$. Let $P_{\mathrm{C}} = P_1 \times P_2$. This defines the set of states of $\mathcal{A}_{\mathrm{C}}$, and the set of pebbles. A state of $\mathcal{A}_{\mathrm{C}}$ is a triple $\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$ where b_{L} is a state of $\mathcal{A}_{\mathrm{L}}$, b_{R} is a state of $\mathcal{A}_{\mathrm{R}}$, and f is a memory function with cells $f(p_{\mathrm{L}}, p_{\mathrm{R}})$, for pebbles p_{L} and p_{R} of $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ respectively. Each cell may be empty, or it may contain a state either in S_{L} or in S_{R}.

Set $h_{\mathrm{C}} \colon S_{\mathrm{C}} \to P_{\mathrm{C}}$ to be the function defined by the condition

1. $h_{\mathrm{C}}(b_{\mathrm{L}}, b_{\mathrm{R}}, f) = \langle h_{\mathrm{L}}(b_{\mathrm{L}}), h_{\mathrm{R}}(b_{\mathrm{R}})\rangle$.

Thus the pebble placed by $\mathcal{A}_{\mathrm{C}}$ at a state $\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$ is simply the pair made of the pebble placed by $\mathcal{A}_{\mathrm{L}}$ at state b_{L} and the pebble placed by $\mathcal{A}_{\mathrm{R}}$ at state b_{R}.

Set $\langle \langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle, \sigma, \langle b^*_{\mathrm{L}}, b^*_{\mathrm{R}}, f^*\rangle\rangle \in T_{\mathrm{C}}$ just in case that:

2. $\langle b_{\mathrm{L}}, \sigma, b^*_{\mathrm{L}}\rangle \in T_{\mathrm{L}}$ and $\langle b_{\mathrm{R}}, \sigma, b^*_{\mathrm{R}}\rangle \in T_{\mathrm{R}}$.
3. $f, f^* \in A$, and f^* is defined by the conditions:
   (a) $f^*(h_{\mathrm{L}}(b_{\mathrm{L}}), h_{\mathrm{R}}(b_{\mathrm{R}})) =\uparrow$.
   (b) If $f(p_{\mathrm{L}}, p_{\mathrm{R}}) =\uparrow$, $p_{\mathrm{L}} \in u_{\mathrm{L}}(b_{\mathrm{L}})$, and $p_{\mathrm{R}} \notin u_{\mathrm{R}}(b_{\mathrm{R}})$, then $f^*(p_{\mathrm{L}}, p_{\mathrm{R}}) = b_{\mathrm{R}}$. Similarly, it $f(p_{\mathrm{L}}, p_{\mathrm{R}}) =\uparrow$, $p_{\mathrm{L}} \notin u_{\mathrm{L}}(b_{\mathrm{L}})$, and $p_{\mathrm{R}} \in u_{\mathrm{R}}(b_{\mathrm{R}})$, then $f^*(p_{\mathrm{L}}, p_{\mathrm{R}}) = b_{\mathrm{L}}$.
   (c) For $\langle p_{\mathrm{L}}, p_{\mathrm{R}}\rangle \in P_{\mathrm{L}} \times P_{\mathrm{R}}$ not covered by conditions (3a) and (3b), $f^*(p_{\mathrm{L}}, p_{\mathrm{R}}) = f(p_{\mathrm{L}}, p_{\mathrm{R}})$.

Condition (2) simply formalizes the fact that $\mathcal{A}_{\mathrm{C}}$ follows $\mathcal{A}_{\mathrm{L}}$ on the left coordinate and $\mathcal{A}_{\mathrm{R}}$ on the right coordinate. Condition (3) governs the transition of the memory function. The cell $f^*(h_{\mathrm{L}}(b_{\mathrm{L}}), h_{\mathrm{R}}(b_{\mathrm{R}}))$ corresponding to the pebble being placed at the current state is initialized to be undefined. Currently undefined cells $f(p_{\mathrm{L}}, p_{\mathrm{R}})$ so that exactly one of $p_{\mathrm{L}}, p_{\mathrm{R}}$ is released (by b_{L} or b_{R}) are updated to store the state causing the release (this is formalized in condition (3b)). In all other cases f^* continues to store the state stored by f.

Define u_{C} through the condition:

4. $\langle p_{\mathrm{L}}, p_{\mathrm{R}}\rangle \notin u_{\mathrm{C}}(\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle)$, meaning that $\langle p_{\mathrm{L}}, p_{\mathrm{R}}\rangle$ is released by the state $\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$, just in case that (at least) one of the following conditions holds:
   (a) $p_{\mathrm{L}} \notin u_{\mathrm{L}}(b_{\mathrm{L}})$ and $p_{\mathrm{R}} \notin u_{\mathrm{R}}(b_{\mathrm{R}})$.
   (b) $f(p_{\mathrm{L}}, p_{\mathrm{R}})$ is defined and belongs to S_{L}, and $p_{\mathrm{R}} \notin u_{\mathrm{R}}(b_{\mathrm{R}})$.
   (c) $f(p_{\mathrm{L}}, p_{\mathrm{R}})$ is defined and belongs to S_{R}, and $p_{\mathrm{L}} \notin u_{\mathrm{L}}(b_{\mathrm{L}})$.

Let $\pi_{1,\mathrm{L}} \colon S_{\mathrm{C}} \to S_{\mathrm{L}}$ be defined by $\pi_{1,\mathrm{L}}(b_{\mathrm{L}}, b_{\mathrm{R}}, f) = b_{\mathrm{L}}$ and let $\pi_{1,\mathrm{R}}$ be defined by $\pi_{1,\mathrm{R}}(b_{\mathrm{L}}, b_{\mathrm{R}}, f) = b_{\mathrm{R}}$.

Define $\pi_{2,\mathrm{L}} \colon S_{\mathrm{C}} \times S_{\mathrm{C}} \to S_{\mathrm{L}}$ by

$$\pi_{2,\mathrm{L}}(b_{\mathrm{C}}, \langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle) = \begin{cases} b_{\mathrm{L}} & \text{if } f(h_{\mathrm{C}}(b_{\mathrm{C}})) =\uparrow \text{ or } f(h_{\mathrm{C}}(b_{\mathrm{C}})) \in S_{\mathrm{R}}, \\ f(h_{\mathrm{C}}(b_{\mathrm{C}})) & \text{if } f(h_{\mathrm{C}}(b_{\mathrm{C}})) \in S_{\mathrm{L}}. \end{cases}$$

In the context of our use of $\pi_{2,\mathrm{L}}$ below, $b_{\mathrm{C}} \in S_{\mathrm{C}}$ is a current state, causing the placement of a pebble $h_{\mathrm{C}}(b_{\mathrm{C}})$, and $\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$ is a later state causing the release of this pebble. $\pi_{2,\mathrm{L}}(b_{\mathrm{C}}, \langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle)$ gives the state in S_{L} responsible for the release of the left coordinate of b_{C}. This is either the left coordinate of $\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$, or else it is the state stored by f.

Define $\pi_{2,\mathrm{R}} \colon S_{\mathrm{C}} \times S_{\mathrm{C}} \to S_{\mathrm{R}}$ by

$$\pi_{2,\mathrm{R}}(b_{\mathrm{C}}, \langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle) = \begin{cases} b_{\mathrm{R}} & \text{if } f(h_{\mathrm{C}}(b_{\mathrm{C}})) =\uparrow \text{ or } f(h_{\mathrm{C}}(b_{\mathrm{C}})) \in S_{\mathrm{L}}, \\ f(h_{\mathrm{C}}(b_{\mathrm{C}})) & \text{if } f(h_{\mathrm{C}}(b_{\mathrm{C}})) \in S_{\mathrm{R}}. \end{cases}$$

For a sequence $\langle f_\alpha \mid \alpha < \gamma\rangle$ of functions in A define $\lim_{\alpha \longrightarrow \gamma} f_\alpha$ to be the function $f \in A$ given by the condition: $f(p_{\mathrm{L}}, p_{\mathrm{R}})$ is equal to the eventual value of $f_\alpha(p_{\mathrm{L}}, p_{\mathrm{R}})$ as $\alpha \longrightarrow \gamma$ if $f_\alpha(p_{\mathrm{L}}, p_{\mathrm{R}})$ is eventually constant as $\alpha \longrightarrow \gamma$, and $f(p_{\mathrm{L}}, p_{\mathrm{R}})$ is undefined otherwise.

Finally, define $\vec{\varphi}_{\mathrm{C}}$ and Ψ_{C} so that:

5. $(\Psi_{\mathrm{C}} \oplus \vec{\varphi}_{\mathrm{C}})(\gamma; s_{\mathrm{C}}, r_{\mathrm{C}}) = \langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle$ just in case that:
   (a) $(\Psi_{\mathrm{L}} \oplus \vec{\varphi_{\mathrm{L}}})(\gamma; \pi_{1,\mathrm{L}} \circ s_{\mathrm{C}}, \pi_{2,\mathrm{L}} \circ (s_{\mathrm{C}} \times r_{\mathrm{C}})) = b_{\mathrm{L}}$.
   (b) $(\Psi_{\mathrm{R}} \oplus \vec{\varphi_{\mathrm{R}}})(\gamma; \pi_{1,\mathrm{R}} \circ s_{\mathrm{C}}, \pi_{2,\mathrm{R}} \circ (s_{\mathrm{C}} \times r_{\mathrm{C}})) = b_{\mathrm{R}}$.
   (c) $f = \lim_{\alpha \longrightarrow \gamma} f_\alpha$ where f_α denotes the third coordinate in $s_{\mathrm{C}}(\alpha)$ (so that $s_{\mathrm{C}}(\alpha) = \langle b_{\alpha,\mathrm{L}}, b_{\alpha,\mathrm{R}}, f_\alpha\rangle$ for some states $b_{\alpha,\mathrm{L}} \in S_{\mathrm{L}}$ and $b_{\alpha,\mathrm{R}} \in S_{\mathrm{R}}$).

Conditions (5a) and (5b) can be arranged using Claim 1.20. Condition (5c) can be arranged using references to $\mathrm{cf}(s_{\mathrm{C}})$.

Condition (5) completes the definition of $\mathcal{A}_{\mathrm{C}}$. It is not hard, using most importantly the interaction between the transition from f to f^* in condition (3) and the definitions of $\pi_{2,\mathrm{L}}$ and $\pi_{2,\mathrm{R}}$, to prove the following claims:

Claim 2.5. *Let $\langle s_{\mathrm{C}}, r_{\mathrm{C}}\rangle$ be a run of $\mathcal{A}_{\mathrm{C}}$ on $X \colon \alpha \to \Sigma$, with $s_{\mathrm{C}} = s_{\mathrm{L}} \times s_{\mathrm{R}} \times \chi$ say. Then there are $r_{\mathrm{L}}, r_{\mathrm{R}}$ so that:*

1. *$\langle s_{\mathrm{L}}, r_{\mathrm{L}}\rangle$ is a run of $\mathcal{A}_{\mathrm{L}}$ on X, and $r_{\mathrm{L}} \approx \pi_{2,\mathrm{L}} \circ (s_{\mathrm{C}} \times r_{\mathrm{C}})$.*
2. *$\langle s_{\mathrm{R}}, r_{\mathrm{R}}\rangle$ is a run of $\mathcal{A}_{\mathrm{R}}$ on X, and $r_{\mathrm{R}} \approx \pi_{2,\mathrm{R}} \circ (s_{\mathrm{C}} \times r_{\mathrm{C}})$.*

Claim 2.6. *Let $\langle s_{\mathrm{L}}, r_{\mathrm{L}}\rangle$ and $\langle s_{\mathrm{R}}, r_{\mathrm{R}}\rangle$ be runs of $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ respectively on a sequence $X\colon \alpha \to \Sigma$. Then there exists $\chi\colon \alpha+1 \to A$ and $r_{\mathrm{C}}\colon \alpha \rightharpoonup S_{\mathrm{C}}$ so that, setting $s_{\mathrm{C}} = s_{\mathrm{L}} \times r_{\mathrm{L}} \times \chi$:*

1. *$\langle s_{\mathrm{C}}, r_{\mathrm{C}}\rangle$ is a run of $\mathcal{A}_{\mathrm{C}}$ on X.*
2. *$\pi_{2,\mathrm{L}}(s_{\mathrm{C}} \times r_{\mathrm{C}}) \approx r_{\mathrm{L}}$.*
3. *$\pi_{2,\mathrm{R}}(s_{\mathrm{C}} \times r_{\mathrm{C}}) \approx r_{\mathrm{R}}$.*

Let $f_{\uparrow} \in A$ be the function which is undefined everywhere on $P_{\mathrm{L}} \times P_{\mathrm{R}}$. Let $I_{\mathrm{C}} = \langle I_{\mathrm{L}}, I_{\mathrm{R}}, f_{\uparrow}\rangle$. Let $F_{\mathrm{C}} = \{\langle b_{\mathrm{L}}, b_{\mathrm{R}}, f\rangle \mid b_{\mathrm{L}} \in F_{\mathrm{L}} \wedge b_{\mathrm{R}} \in F_{\mathrm{R}}\}$. Then $\langle \mathcal{A}_{\mathrm{C}}, I_{\mathrm{C}}, F_{\mathrm{C}}\rangle$ accepts X iff both $\langle \mathcal{A}_{\mathrm{L}}, I_{\mathrm{L}}, F_{\mathrm{L}}\rangle$ and $\langle \mathcal{A}_{\mathrm{R}}, I_{\mathrm{R}}, F_{\mathrm{R}}\rangle$ accept X. $\square$ (Lemma 2.4)

Claim 2.7. *Let $\mathcal{A}$ be a* deterministic *automaton. Let $\langle I, F\rangle$ be an accepting condition for $\mathcal{A}$. Then there is an accepting condition $\langle I^*, F^*\rangle$ for $\mathcal{A}$ so that $\langle \mathcal{A}, I, F\rangle$ accepts X iff $\langle \mathcal{A}, I^*, F^*\rangle$ does not.*

Proof. Set $I^* = I$ and $F^* = S - F$. For $X\colon \alpha \to \Sigma$, notice that $\mathcal{A}$, being deterministic, has a unique run $\langle s, r\rangle$ on X with $s(0) = I$. $\langle \mathcal{A}, I, F\rangle$ accept X iff this run ends with a state $s(\alpha)$ in F, and $\langle \mathcal{A}, I, F^*\rangle$ accepts X iff the run ends with a state $s(\alpha)$ in $S - F$. $\dashv$

To obtain closure under negations from the last claim, we have to show that every automaton is equivalent to a deterministic automaton. We do this in Section 4, after establishing some auxiliary results in Section 3.

§3. Characters. Fix, for the entire section, a finite alphabet Σ, and a Σ–automaton $\mathcal{A} = \langle S, P, T, \vec{\varphi}, \Psi, h, u\rangle$. All the definitions and results in this section are stated relative to these objects, though we suppress their mention in the notation.

Fix further an ordinal θ and an input string $X\colon \theta \to \Sigma$.

Definition 3.1. $C_X(\alpha, \beta)$, the **character** of $\mathcal{A}$ on $X{\restriction}[\alpha, \beta)$, is the set of quadruples $\langle b, D, a, b^*\rangle$ so that $b, b^* \in S$, $D \subset S$, $a \in S \cup \{\uparrow\}$, and there is a run $\langle s, r\rangle$ of $\mathcal{A}$ on $X{\restriction}[\alpha, \beta)$ with:

- $s(\alpha) = b$.
- $s(\beta) = b^*$.
- $r(\alpha) = a$ (with $a =\uparrow$ if $r(\alpha)$ is undefined).
- $\{s(\xi) \mid \alpha \leq \xi \leq \beta\} = D$.

$[\alpha, \beta)$ here is the interval of ordinals $\{\xi \mid \alpha \leq \xi < \beta\}$. By a run $\langle s, r\rangle$ of $\mathcal{A}$ on $X{\restriction}[\alpha, \beta)$ we mean a pair $\langle s, r\rangle$ so that $s\colon [\alpha, \beta] \to S$, $r\colon [\alpha, \beta) \rightharpoonup S$, and the pair $\langle s, r\rangle$ satisfies conditions (S), (L), and (R) in Section 2 for $\xi \in [\alpha, \beta)$ and limit $\lambda \in (\alpha, \beta]$.

Remark 3.2. The application of condition (L) here involves references to truth value in structures $(\lambda, s{\restriction}\lambda, r{\restriction}\lambda)$ in cases where ($\lambda > \alpha$ and) s and r are not defined on ordinals below α. There are several ways to make sense of such references but they are all equivalent since the truth value of sentences in $(\lambda; s{\restriction}\lambda, r{\restriction}\lambda)$ only depends on the restriction of s and r to tail-ends of λ, by Claim 1.9.

We read the equation $\langle b, D, a, b^*\rangle \in C_X[\alpha, \beta]$ as "$\mathcal{A}$ on $X{\restriction}[\alpha, \beta)$ can reach b^* from b, accumulating D and depebbling at a."

Remark 3.3. Notice that $\langle \mathcal{A}, I, F\rangle$ accepts $X\colon \theta \to \Sigma$ just in case that the character $C_X(0,\theta)$ has a quadruple $\langle b, D, b^*, a\rangle$ in it with $b = I$ and $b^* \in F$. Thus if we can compute the character of $\mathcal{A}$ on an input X we can tell whether X is accepted. Our plan is to construct, in the next section, a deterministic automaton that can compute the character of a given non-deterministic automaton.

Claim 3.4. *Suppose that $C_X(\alpha_1,\beta) = C_X(\alpha_2,\beta)$. Then for every $\beta^* > \beta$, $C_X(\alpha_1,\beta^*) = C_X(\alpha_2,\beta^*)$.*

Proof. Suppose that $\langle b, D, a, b^*\rangle \in C_X(\alpha_1,\beta^*)$. We show that $\langle b, D, a, b^*\rangle \in C_X(\alpha_2,\beta^*)$. A similar argument establishes the converse.

Let $\langle s, r\rangle$ witness that $\langle b, D, a, b^*\rangle \in C_X(\alpha_1,\beta^*)$. Let $D_0 = \{s(\xi) \mid \alpha_1 \leq \xi \leq \beta\}$, let $b_0 = s(\beta)$, and let $a_0 = s(\gamma)$ for $\gamma > \alpha_1$ least so that $h(s(\alpha_1)) \notin u(s(\gamma))$ if such a γ exists and is $\leq \beta$, leaving a_0 undefined otherwise. Then $\langle b, D_0, a_0, b_0\rangle \in C_X(\alpha_1,\beta)$. Since $C_X(\alpha_1,\beta) = C_X(\alpha_2,\beta)$, $\langle b, D_0, a_0, b_0\rangle$ belongs to $C_X(\alpha_2,\beta)$. A run witnessing this can be composed with $\langle s{\restriction}[\beta,\beta^*], r{\restriction}[\beta,\beta^*)\rangle$ to witness that $\langle b, D, a, b^*\rangle \in C_X(\alpha_2,\beta^*)$. ⊣

By a **character** in general we mean a set of quadruples $\langle b, D, a, b^*\rangle$ with $b, b^* \in S$, $D \subset S$, and $a \in S \cup \{\uparrow\}$. We use $\mathcal{C}$ to denote the set of all possible characters. Notice that $\mathcal{C}$ is finite, since S is finite.

Definition 3.5. For a character C and $\sigma \in \Sigma$ define $C * \sigma$ to be the set of quadruples $\langle b, D, a, b^*\rangle$ so that there exists $b' \in S$, $a' \in S \cup \{\uparrow\}$, and $D' \subset S$ with:

1. $\langle b, D', a', b'\rangle \in C$.
2. $\langle b', \sigma, b^*\rangle \in T$.
3. $D = D' \cup \{b^*\}$.
4. If $a' \in S$ then $a = a'$. If $a' =\uparrow$ and $h(b) \in u(b^*)$ then $a =\uparrow$. If $a' =\uparrow$ and $h(b) \notin u(b^*)$ then $a = b^*$.

Claim 3.6. *Suppose that $C = C_X(\alpha,\beta)$ and that $X(\beta) = \sigma$. Then $C_X(\alpha,\beta+1)$ is precisely equal to $C * \sigma$.*

Proof. Immediate from the definitions. ⊣

Definition 3.7. Let C and E be characters. Define $C * E^\omega$ to be the set of quadruples $\langle b, D, a, b^*\rangle$ so that there exists $D_0, D_1 \subset S$, $a_0, a_1 \in S \cup \{\uparrow\}$, and $b_1 \in S$, with:

1. $\langle b, D_0, a_0, b_1\rangle \in C$.
2. $\langle b_1, D_1, a_1, b_1\rangle \in E$.
3. $D = D_0 \cup D_1 \cup \{b^*\}$.
4. $b^* = (\Psi \oplus \vec{\varphi})(D_1)$.
5. One of the following conditions holds:
   (a) $a_0 \in S$ and $a = a_0$.
   (b) $a_0 =\uparrow$, $h(b) \in u(q)$ for all $q \in D_1$, $h(b) \in u(b^*)$, and $a =\uparrow$.
   (c) $a_0 =\uparrow$, $h(b) \in u(q)$ for all $q \in D_1$, $h(b) \notin u(b^*)$, and $a = b^*$.

In condition (4) we are using the notation of Remark 2.2.

Lemma 3.8. *Suppose that C, E, α, β_n $(n < \omega)$, and β are such that:*

1. *$\alpha < \beta_0 < \beta_1 < \cdots$ and $\beta = \sup_{n<\omega} \beta_n$.*

2. *For every n, $C_X(\alpha, \beta_n)$ is equal to C.*
3. *For every n and every $m > n$, $C_X(\beta_n, \beta_m) = E$.*

Then $C_X(\alpha, \beta)$ is equal to $C * E^\omega$.

PROOF. Suppose first that $\langle b, D, a, b^* \rangle \in C * E^\omega$. We aim to show that $\langle b, D, a, b^* \rangle \in C_X(\alpha, \beta)$.

Let D_0, D_1, a_0, a_1, and b_1 witness that $\langle b, D, a, b^* \rangle \in C * E^\omega$. Using the facts that $\langle b, D_0, a_0, b_1 \rangle \in C = C_X(\alpha, \beta_0)$ and $\langle b_1, D_1, a_1, b_1 \rangle \in E = C_X(\beta_n, \beta_{n+1})$ we can create functions $s\colon [\alpha, \beta) \to S$ and $r\colon [\alpha, \beta) \rightharpoonup S$ so that:

(i) $\langle s\restriction[\alpha, \beta_0], r\restriction[\alpha, \beta_0) \rangle$ is a run of $\mathcal{A}$ on $X\restriction[\alpha, \beta)$, with $s(\alpha) = b$, $s(\beta_0) = b_1$, $\{s(\xi) \mid \alpha \le \xi \le \beta_0\} = D_0$, and $r(\alpha) = a_0$.
(ii) For each n, $\langle s\restriction[\beta_n, \beta_{n+1}], r\restriction[\beta_n, \beta_{n+1}) \rangle$ is a run of $\mathcal{A}$ on $X\restriction[\beta_n, \beta_{n+1})$ with $s(\beta_n) = s(\beta_{n+1}) = b_1$, and $\{s(\xi) \mid \beta_n \le \xi \le \beta_{n+1}\} = D_1$.

Extend s to a function from $[\alpha, \beta]$ into S by setting $s(\beta) = b^*$.

For each $\xi \in [\alpha, \beta)$ define

(iii) $\tilde{r}(\xi) = s(\gamma)$ for the first $\gamma \in (\xi, \beta]$ so that $h(s(\xi)) \notin u(s(\gamma))$ if there is such an ordinal γ, and $\tilde{r}(\xi) = \uparrow$ otherwise.

$\tilde{r}$ and r need not be the same, as there may be ξ, for example in the interval $[\alpha, \beta_0)$, so that the pebble placed on ξ is removed at an ordinal $\gamma > \beta_0$. In this case $\tilde{r}(\xi)$ is defined, but $r(\xi)$ is not. The same is true in each of the intervals $[\beta_n, \beta_{n+1})$. But notice that there can be only finitely many such ordinals ξ within each interval, as there are only finitely many pebbles. Thus:

(iv) $\tilde{r}\restriction\beta_n \approx r\restriction\beta_n$ for each $n < \omega$.

Note that $\mathrm{cf}(s\restriction\beta) = D_1$. Using this and conditions (4) in Definition 3.7 it follows that:

(v) $(\Psi \oplus \vec{\varphi})(\beta, s\restriction\beta, \tilde{r}) = b^*$.

Conditions (i)–(v) taken together imply that $\langle s, \tilde{r} \rangle$ is a run of $\mathcal{A}$ on $X\restriction[\alpha, \beta)$. Using condition (5) in Definition 3.7 and conditions (i)–(iii) above it is easy to check that $a = \tilde{r}(\alpha)$. Since $s(\alpha) = b$, $s(\beta) = b^*$, and $\{s(\xi) \mid \alpha \le \xi \le \beta\} = D_0 \cup D_1 \cup \{b^*\} = D$, the run $\langle s, \tilde{r} \rangle$ witnesses that $\langle b, D, a, b^* \rangle$ belongs to $C_X(\alpha, \beta)$.

Suppose next that $\langle b, D, a, b^* \rangle$ belongs to $C_X(\alpha, \beta)$. Let $\langle s, r \rangle$, with $s\colon [\alpha, \beta] \to S$ and $r\colon [\alpha, \beta) \rightharpoonup S$, be a run witnessing this. We aim to show that $\langle b, D, a, b^* \rangle \in C * E^\omega$.

For $\delta < \eta$ in the interval $[\alpha, \beta)$ let $s_{\delta,\eta} = s\restriction[\delta, \eta]$ and define $r_{\delta,\eta}\colon [\delta, \eta) \rightharpoonup S$ by $r_{\delta,\eta}(\xi) = s(\gamma)$ where $\gamma \in (\xi, \eta]$ is least so that $h(s(\xi)) \notin u(s(\gamma))$ if there is such an ordinal γ, and $r_{\delta,\eta}(\xi) = \uparrow$ otherwise. As usual $r_{\delta,\eta}$ need not equal $r\restriction[\delta, \eta)$, as there may be pebbles placed on ordinals in $[\delta, \eta)$ which are removed at a stage later than η. But $r_{\delta,\eta} \approx r\restriction[\delta, \eta)$. From this, the definition of $r_{\delta,\eta}$, and the fact that $\langle s, r \rangle$ is a run of $\mathcal{A}$ on $X\restriction[\alpha, \beta)$, it follows that:

(vi) $\langle s_{\delta,\eta}, r_{\delta,\eta} \rangle$ is a run of $\mathcal{A}$ on $X\restriction[\delta, \eta)$.

Since the set of states S is finite, there must be a specific state $b_1 \in S$, a specific $a_1 \in S \cup \{\uparrow\}$, and an infinite set $Q \subset \{\beta_n \mid n < \omega\}$, so that:

(vii) $s(\delta) = b_1$ and $r(\delta) = a_1$ for each $\delta \in Q$.

Let $D_1 = \mathrm{cf}(s\restriction\beta)$. By throwing away an initial segment of Q if needed we may assume that $\{s(\xi) \mid \delta \leq \xi < \beta\} = D_1$ for each $\delta \in Q$. Since D_1 is finite we may, by passing to a subset of Q that is still infinite, assume that in fact:

(viii) $\{s(\xi) \mid \delta \leq \xi \leq \eta\} = D_1$ for all $\delta < \eta$ both in Q.

For each ordinal $\xi < \beta$ let $\gamma(\xi) \in (\xi, \beta]$ be the smallest ordinal so that $h(s(\xi)) \notin u(s(\gamma(\xi)))$ if there is such an ordinal. This is the ordinal where the pebble placed on ξ is removed in the run $\langle s, r\rangle$. Note that $\gamma(\delta)$ is defined and strictly smaller than β for all but finitely many $\delta \in Q$. Throwing away an initial segment of Q if necessary we may therefore assume that $\gamma(\delta) < \beta$ for all $\delta \in Q$. Shrinking Q further, but still keeping it cofinal in β, we may assume that $\gamma(\delta)$ is smaller than the next element of Q above δ, and from this it follows that:

(ix) $r_{\delta,\eta}(\delta) = r(\delta)$ for $\delta < \eta$ both in Q.

From conditions (vi)–(ix) it follows that $\langle b_1, D_1, a_1, b_1\rangle \in C_X(\delta, \eta)$ for all $\delta < \eta$ both in Q. Since $Q \subset \{\beta_n \mid n < \omega\}$, and $C_X(\beta_n, \beta_m) = E$ for all $n < m < \omega$, it certainly follows that $\langle b_1, D_1, a_1, b_1\rangle \in E$.

Let $\nu \in Q$ be large enough that $\gamma(\alpha)$, if defined, is either smaller than ν or equal to β. Let $a_0 = r_{\alpha,\nu}(\alpha)$ and let $D_0 = \{s(\xi) \mid \alpha \leq \xi \leq \nu\}$. Then $\langle s_{\alpha,\nu}, r_{\alpha,\nu}\rangle$ witnesses that $\langle b, D_0, a_0, b_1\rangle \in C_X(\alpha, \nu)$. Since $\nu \in Q = \{\beta_n \mid n < \omega\}$, and since $C_X(\alpha, \beta_n) = C$ for all n, it follows that $\langle b, D_0, a_0, b_1\rangle \in C$.

It is now easy to check that D_0, D_1, a_0, a_1, and b_1 witness that $\langle b, D, a, b^*\rangle$ belongs to $C * E^\omega$. ⊣

§4. Determinism. Fix throughout the section a finite alphabet Σ and a Σ–automaton $\mathcal{A} = \langle S, P, T, \vec{\varphi}, \Psi, h, u\rangle$.

Fix an ordinal θ and an input string $X\colon \theta \to \Sigma$. We describe a process that computes $C_X(0, \theta)$. We will later check that this process can be carried out by a deterministic automaton, thereby showing that any language that is recognizable by a non-deterministic automaton is also recognizable by a deterministic automaton.

Let $\mathcal{C}$ be the set of characters corresponding to the automaton $\mathcal{A}$. Let $\#\mathcal{C}$ denote the number of elements of $\mathcal{C}$. Let $H = \{0, \ldots, \#\mathcal{C}\}$.

Call an ordinal α **essential** at β if $\alpha < \beta$, so that $C_X(\alpha, \beta)$ makes sense, and there is no $\bar{\alpha} < \alpha$ so that $C_X(\bar{\alpha}, \beta) = C_X(\alpha, \beta)$. Notice that there are at most $\#\mathcal{C}$ ordinals which are essential at any given β.

By induction we define a sequence of sets $K_\gamma \subset H$ for $\gamma \leq \theta$, and ordinals α_i^γ for $i \in K_\gamma$ (and also for some $i \notin K_\gamma$), so that $\{\alpha_i^\gamma \mid i \in K_\gamma\}$ is precisely the set of all ordinals which are essential at γ. Notice that K_γ then has at most $\#\mathcal{C}$ elements, and is therefore a *proper* subset of H. Set h_γ to be the least element of $H - K_\gamma$, and set $\alpha_{h_\gamma}^\gamma = \gamma$. These assignments will be used during the induction.

Set $K_0 = \emptyset$ to begin with.

Assuming that K_β and $\langle \alpha_i^\beta \mid i \in K_\beta\rangle$ are known for all $\beta < \gamma$, define $K_{<\gamma} = \{i \mid i \in K_\beta \cup \{h_\beta\}$ for a tail-end of $\beta < \gamma$, and α_i^β is eventually constant as $\beta \longrightarrow \gamma\}$. For $i \in K_{<\gamma}$ set α_i^γ equal to the eventual value of α_i^β as $\beta \longrightarrow \gamma$. (If γ is a successor then $K_{<\gamma} = K_{\gamma-1} \cup \{h_{\gamma-1}\}$ and $\alpha_i^\gamma = \alpha_i^{\gamma-1}$.)

Let $K_\gamma = \{i \in K_{<\gamma} \mid \alpha_i^\gamma$ is essential at $\gamma\}$.

It is easy to verify by induction on γ that $\{\alpha_i^\gamma \mid i \in K_\gamma\}$ is precisely the set of all ordinals which are essential at γ. The proof uses Claim 3.4, which implies that ordinals which are not essential at some $\beta < \gamma$ are also not essential at γ.

We refer to ordinals which belong to $\{\alpha_i^\gamma \mid i \in K_{<\gamma}\}$ but not to $\{\alpha_i^\gamma \mid i \in K_\gamma\}$ as **discarded** at stage γ. Numbers i which belong to $K_{<\gamma}$ but not to K_γ are **released** at stage γ. Thus an ordinal $\alpha = \alpha_i^\gamma$ is discarded at stage γ if in stage γ the character from α "merges" with the character from a smaller ordinal, precisely, if there is $\bar{\alpha} < \alpha$ so that $C_X(\alpha, \gamma) = C_X(\bar{\alpha}, \gamma)$. All but finitely many ordinals must be discarded eventually, since the set of possible characters is finite.

Remark 4.1. We do not know at stage α whether α will be discarded, and if so, with which of the characters from smaller ordinals will the character from α merge. But using pebbles we will be able to design an automaton $\widehat{\mathcal{A}}$ with run $\langle \hat{s}, \hat{r} \rangle$ so that $\hat{r}(\alpha)$ has this information. The use of pebbles in $\widehat{\mathcal{A}}$ lets us delay the definition of $\hat{r}(\alpha)$ until (if ever) reaching a stage where α is discarded.

Let R_γ be the order on $K_{<\gamma}$ defined by $i \; R_\gamma \; j$ iff $\alpha_i^\gamma < \alpha_j^\gamma$. For $i \; R_\gamma \; j$ define $C_{i,j}^\gamma$ to be $C_X(\alpha_i^\gamma, \alpha_j^\gamma)$. For $i \in K_{<\gamma}$ Let C_i^γ denote $C_X(\alpha_i^\gamma, \gamma)$. Notice that with these definitions, j is released at γ iff there exists $i \; R_\gamma \; j$ so that $C_i^\gamma = C_j^\gamma$. Let $f_\gamma(j) = \langle i, C_{i,j}^\gamma \rangle$ for the R_γ least such i. f_γ is then a function from the set of released $j \in H$ into $H \times \mathcal{C}$.

Definition 4.2. $\hat{b}_\gamma$ is the tuple

$$\langle K_{<\gamma}, R_\gamma, (i \mapsto C_i^\gamma), (i, j \mapsto C_{i,j}^\gamma), f_\gamma, K_\gamma \rangle,$$

where $i \mapsto C_i^\gamma$ and $i, j \mapsto C_{i,j}^\gamma$ denote the obvious functions, the former defined on all $i \in K_{<\gamma}$ and the latter on all pairs $i, j \in K_{<\gamma}$ with $i \; R_\gamma \; j$.

Definition 4.3. Define $\widehat{S}$ to be the set

$$\mathcal{P}(H) \times \mathcal{P}(H \times H) \times (H \rightharpoonup \mathcal{C}) \times ((H \times H) \rightharpoonup \mathcal{C}) \times (H \rightharpoonup (H \times \mathcal{C})) \times \mathcal{P}(H),$$

where $\mathcal{P}(A)$ denotes the powerset of A and $(A \rightharpoonup B)$ denotes the set of partial functions from A to B.

The tuple $\hat{b}_\gamma$ belongs to $\widehat{S}$ for each γ. Notice that $\widehat{S}$ is finite, as both H and $\mathcal{C}$ are finite.

Definition 4.4. Let $\widehat{I}$ denote the tuple $\langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle$ in $\widehat{S}$. Define a function $z \colon \widehat{S} \rightharpoonup \mathcal{C}$ by setting $z(\hat{b}) = C_0$ for each tuple $\hat{b} = \langle K_<, R, (i \mapsto C_i), (i, j \mapsto C_{i,j}), f, K \rangle$ in $\widehat{S}$ with $0 \in K_{<\gamma}$, and leaving $z(\hat{b})$ undefined on the other tuples.

Notice that $\hat{b}_0$ is precisely equal to $\widehat{I}$, and that for each $\gamma > 0$, $z(\hat{b}_\gamma)$ is defined and equal to $C_X(0, \gamma)$, as $0 \in K_{<\gamma}$ and $\alpha_0^\gamma = 0$ for all $\gamma > 0$.

Remark 4.5. f_γ and K_γ can be determined from knowledge of $K_{<\gamma}$, R_γ, C_i^γ for each $i \in K_{<\gamma}$, and $C_{i,j}^\gamma$ for each pair $i \; R_\gamma \; j$. Thus the entire state $\hat{b}_\gamma$ can be determined (independently of X and γ) from knowledge of $K_{<\gamma}$, R_γ, C_i^γ for each $i \in K_{<\gamma}$, and $C_{i,j}^\gamma$ for each pair $i \; R_\gamma \; j$.

Claim 4.6. *$\hat{b}_{\gamma+1}$ can be determined (independently of X and γ) from knowledge of $\hat{b}_\gamma$ and $X(\gamma)$.*

PROOF. Using the last remark it is enough to determine $K_{<\gamma+1}$, $R_{\gamma+1}$, $C_i^{\gamma+1}$ for each $i \in K_{<\gamma+1}$, and $C_{i,j}^{\gamma+1}$ for $i\ R_{\gamma+1}\ j$.

$K_{<\gamma+1}$ is equal to $K_\gamma \cup \{h_\gamma\}$ where $h_\gamma = \min(H - K_\gamma)$. $R_{\gamma+1} = (R_\gamma \restriction K_\gamma) \cup \{\langle i, h_\gamma\rangle \mid i \in K_\gamma\}$. $C_i^{\gamma+1}$ for $i \in K_\gamma$ is equal to $C_i^\gamma * X(\gamma)$ by Claim 3.6, and $C_{h_\gamma}^{\gamma+1}$ is equal to $C_\emptyset * X(\gamma)$ where $C_\emptyset = \{\langle b, \{b\}, \uparrow, b\rangle \mid b \in S\}$ is the character $C_X(\gamma,\gamma)$ (this character is the same regardless of γ and X). Finally, $C_{i,j}^{\gamma+1} = C_{i,j}^\gamma$ for $i,j \in K_\gamma$, and $C_{i,h_\gamma}^{\gamma+1} = C_i^\gamma$ for $i \in K_\gamma$. ⊣

Let $\hat{s}\colon \theta{+}1 \to \widehat{S}$ be the function $(\gamma \mapsto \hat{b}_\gamma)$. By the last claim, there is a function $\widehat{T}\colon \widehat{S}\times\Sigma \to \widehat{S}$, independent of X, so that $\hat{b}(\gamma{+}1) = \widehat{T}(\hat{b}(\gamma), X(\gamma))$. Our intention is to show that $\hat{s}$ is produced as a run of a deterministic automaton on X (and that the automaton of course is defined independently of X). This function $\widehat{T}$ provides the successor transition table for the automaton. We continue now to work on the limits.

CLAIM 4.7. *For limit γ, each of $K_{<\gamma}$, R_γ, and $C_{i,j}^\gamma$ for $i\ R_\gamma\ j$, can be determined (independently of X and γ) from knowledge of* $\mathrm{cf}(\hat{s}\restriction\gamma)$.

PROOF. It is easy to check that $i \in K_{<\gamma}$ iff $i \in K_\beta$ for a tail-end of $\beta < \gamma$, and since K_β is coded as part of $\hat{s}(\beta)$ the truth value of the right-hand-side condition can be determined from knowledge of $\mathrm{cf}(\hat{s}\restriction\gamma)$. Similarly, $i\ R_\gamma\ j$ iff $i\ R_\beta\ j$ for a tail-end of $\beta < \gamma$, and $C_{i,j}^\gamma$ is equal to the eventual value of $C_{i,j}^\beta$ as $\beta \longrightarrow \gamma$. ⊣

CLAIM 4.8. *Let γ be a limit of cofinality ω and let $k \in K_{<\gamma}$. Then there exists $i \in H$, $j \in H$, and $C, E \in \mathcal{C}$ so that:*

1. *i belongs to $K_{<\gamma}$.*
2. *The set $\{\beta < \gamma \mid j$ is released at β, $f_\beta(j) = \langle i, E\rangle$, and $C_{k,j}^\beta$ is defined and equal to $C\}$ is cofinal in γ.*

PROOF. Let $\bar{\gamma} = \max\{\alpha_l^\gamma \mid l \in K_{<\gamma}\}$. $\bar{\gamma}$ is smaller than γ since γ is a limit. Notice that the set $Q = \{\beta \in (\bar{\gamma}, \gamma) \mid \min\{\alpha_l^\beta \mid l \in K_{<\beta} \wedge \alpha_l^\beta > \bar{\gamma}\}$ is discarded at $\beta\}$ is cofinal in γ, since otherwise the eventual value of $\min\{\alpha_l^\beta \mid l \in K_{<\beta} \wedge \alpha_l^\beta > \bar{\gamma}\}$ as $\beta \longrightarrow \gamma$ would belong to $\{\alpha_l^\gamma \mid l \in K_{<\gamma}\}$. For each $\beta \in Q$ let $j(\beta)$ be such that $\alpha_{j(\beta)}^\beta = \min\{\alpha_l^\beta \mid l \in K_{<\beta} \wedge \alpha_l^\beta > \bar{\gamma}\}$. Notice that $k\ R_\beta\ j(\beta)$ since $\alpha_{j(\beta)}^\beta > \bar{\gamma}$ and $\alpha_k^\beta = \alpha_k^\gamma \leq \bar{\gamma}$. Thus $C_{k,j(\beta)}^\beta$ is defined.

For each $\beta \in Q$ let $g(\beta) = \langle i, j, C, E\rangle$ where $j = j(\beta)$, $\langle i, E\rangle = f_\beta(j)$, and $C = C_{k,j}^\beta$. g takes values in the finite set $H \times H \times \mathcal{C} \times \mathcal{C}$. Thus there is a fixed tuple $\langle i, j, C, E\rangle$ so that $g(\beta) = \langle i, j, C, E\rangle$ for cofinally many $\beta \in Q$. Every such β belongs to the set in condition (2) of the claim, so this set is cofinal in γ. As for condition (1): $\alpha_i^\beta < \alpha_j^\beta$ by the definition of f_β, and from the definition of $j(\beta)$ it follows that $\alpha_i^\beta \leq \bar{\gamma}$. Applying this with $\beta < \gamma$ large enough that all ordinals $\leq \bar{\gamma}$ which do not belong to $\{\alpha_l^\gamma \mid l \in K_{<\gamma}\}$ have been discarded by stage β, it follows that $i \in K_{<\gamma}$. ⊣

LEMMA 4.9. *Let γ be a limit of cofinality ω and let $k \in K_{<\gamma}$. Let i, j, C, and E satisfy the conditions of the previous claim. Then $C_k^\gamma = C * E^\omega$.*

PROOF. Using condition (2) of the previous claim and the fact that $\mathrm{cof}(\gamma) = \omega$, we may fix an increasing sequence of ordinals β_n ($n < \omega$), cofinal in γ, so that for each n, j is released at β_n, $f_{\beta_n}(j) = \langle i, E \rangle$, and $C^{\beta_n}_{k,j}$ is defined and equal to C.

Let $\alpha_n = \alpha^{\beta_n}_j$, so that α_n is discarded at stage β_n. The set $\{\alpha_n \mid n < \omega\}$ cannot be bounded in γ; if it were then its ordinals would all be discarded by some stage strictly below γ, contradicting the fact that $\{\beta_n \mid n < \omega\}$ is cofinal in γ.

By thinning the set $\{\beta_n \mid n < \omega\}$ if needed we may therefore assume that:

(i) $\langle \alpha_n \mid n < \omega \rangle$ is increasing, and $\alpha_{n+1} > \beta_n$ for each n.

Let α denote α^γ_i. The assignment makes sense as $i \in K_{<\gamma}$ by condition (1) of Claim 4.8. α^γ_i is equal to α^β_i for all sufficiently large $\beta < \gamma$, and without loss of generality we may assume that this is the case for all $\beta \in \{\beta_n \mid n < \omega\}$. So $\alpha^{\beta_n}_i = \alpha$ for each n. Recall that $\alpha_n = \alpha^{\beta_n}_j$ is discarded at stage β_n and $f_{\beta_n}(j) = \langle i, E \rangle$. Using the definition of f_{β_n} it follows that:

(ii) $C_X(\alpha, \alpha_n) = E$.

(iii) $C_X(\alpha, \beta_n) = C_X(\alpha_n, \beta_n)$.

By condition (iii) and Claim 3.4, $C_X(\alpha, \beta^*) = C_X(\alpha_n, \beta^*)$ for every $\beta^* > \beta_n$. By condition (i) then:

(iv) $C_X(\alpha, \alpha_m) = C_X(\alpha_n, \alpha_m)$ for all $m > n$.

Note that $C_X(\alpha, \alpha_m) = E$ by condition (ii). Thus from condition (iv) it follows that in fact:

(v) $C_X(\alpha_n, \alpha_m) = E$ for all n and all $m > n$.

α^β_k is equal to α^γ_k for all sufficiently large $\beta < \gamma$, and without loss of generality we may assume that this is the case for all $\beta \in \{\beta_n \mid n < \omega\}$. We have $C^{\beta_n}_{k,j} = C$ for each n. Since $\alpha^{\beta_n}_k = \alpha^\gamma_k$ and $\alpha^{\beta_n}_j = \alpha_n$ this means that:

(vi) $C_X(\alpha^\gamma_k, \alpha_n) = C$ for each n.

With conditions (v) and (vi) we may apply Lemma 3.8 and conclude finally that $C_X(\alpha^\gamma_k, \gamma) = C * E^\omega$. $\dashv$

COROLLARY 4.10. *Let γ be a limit of cofinality ω. Then for each $k \in K^{<\gamma}_k$, C^γ_k can be determined (independently of X and γ) from knowledge of* $\mathrm{cf}(\hat{s} \restriction \gamma)$.

PROOF. Looking at $\mathrm{cf}(\hat{s} \restriction \gamma)$ one can tell which tuples $\langle i, j, C, E \rangle$ satisfy the conditions of Claim 4.8, and then use Lemma 4.9 to determine C^γ_k. $\dashv$

COROLLARY 4.11. *Let γ be a limit of cofinality ω. Then $\hat{b}_\gamma$ can be determined (independently of X and γ) from knowledge of* $\mathrm{cf}(\hat{s} \restriction \gamma)$.

PROOF. Immediate putting Remark 4.5, Claim 4.7 and Corollary 4.10 together. $\dashv$

REMARK 4.12. Our handling of limits of cofinality ω here is very similar to the handling of countable limits in Büchi [2]. Both our handling of this issue and Büchi's use ideas which trace back to the subset construction of McNaughton [7].

We pass now to limits of cofinality greater than ω.

DEFINITION 4.13. Set $\widehat{P} = H$. For a state $\hat{b} = \langle K_<, R, (i \mapsto C_i), (i, j \mapsto C_{i,j}), f, K\rangle$ set $\hat{h}(\hat{b}) = \min(H - K)$ and $\hat{u}(\hat{b}) = K$. $\hat{h}$ is then a function from $\widehat{S}$ into $\widehat{P}$, $\hat{u}$ is a function from $\widehat{S}$ into $\{U \mid U \subsetneq \widehat{P}\}$, and $\hat{h}(\hat{b}) \notin \hat{u}(\hat{b})$ for each state $\hat{b}$.

For each $\xi < \theta$ let $\hat{\rho}(\xi) > \xi$ be the ordinal at which ξ is discarded, if there is such an ordinal, and let $\hat{r}(\xi) = \hat{b}_{\hat{\rho}(\xi)}$. If ξ is not discarded then leave $\hat{r}(\xi)$ undefined.

We intend to show that $\hat{s} = \langle \hat{b}_\gamma \mid \gamma \leq \theta\rangle$ and $\hat{r}$ form a run of a deterministic automaton $\widehat{A}$ on X. Definition 4.13 determines the handling of pebbles in runs of $\widehat{A}$. The definition is such that the pebble placed on ξ during the run $\langle \hat{s}, \hat{r}\rangle$ is precisely equal to h_ξ, and the pebble is released precisely when ξ is discarded. From this and the definition of $\hat{r}$ it follows that condition (R) in Section 2 holds for $\langle \hat{s}, \hat{r}\rangle$.

REMARK 4.14. We are using the availability of pebbles in automata to delay the definition of $\hat{r}(\xi)$ in runs of $\widehat{A}$, so that it is made not at stage ξ but later on at the stage $\hat{\rho}(\xi)$ in which ξ is discarded. The run is constructed so that $\hat{r}(\xi) = s(\hat{\rho}(\xi))$, and this is essential to the proof of Claim 4.15 below.

We continue now to define the deterministic Σ–automaton $\widehat{\mathcal{A}}$. We already defined the set of states $\widehat{S}$, the successor transition function $\widehat{T}$, the set of pebbles $\widehat{P}$, and the functions $\hat{h}$ and $\hat{u}$. We also saw that $\hat{s}(\gamma)$ can be determined from knowledge of $\mathrm{cf}(\hat{s}\restriction\gamma)$ for limit γ of cofinality ω. It remains to see that for limit γ of cofinality greater than ω, $\hat{s}(\gamma)$ can be determined from knowledge of the truth value of finitely many fixed sentences in $(\gamma; \hat{s}\restriction\gamma, \hat{r}\restriction\gamma)$.

CLAIM 4.15. *Let γ be a limit of cofinality greater than ω_1. Then there exists a club $Z \subset \gamma$ so that for every $\xi < \beta$ both in Z with $\mathrm{cof}(\beta) = \omega$, $C_X(\xi, \beta)$ can be determined (independently of X, ξ, β, Z, and γ) from knowledge of $\hat{s}(\xi)$, $\hat{r}(\xi)$, and $\mathrm{cf}(\hat{s}\restriction\gamma)$.*

PROOF. Let $Z \subset \gamma$ be a club so that for each $\xi \in Z$:

(i) $\mathrm{cf}(s\restriction\xi) = \mathrm{cf}(s\restriction\gamma)$.
(ii) $(\forall\zeta < \xi)(\hat{\rho}(\zeta) < \gamma \to \hat{\rho}(\zeta) < \xi)$.
(iii) $\hat{\rho}(\xi)$ is defined and smaller than the next element of Z above ξ.

Conditions (i) and (ii) can be obtained through a closure argument using the fact that $\mathrm{cof}(\gamma) \geq \omega_1$. Condition (iii) is obtained by a closure argument using the fact that $\hat{\rho}(\xi)$ is defined and smaller than γ for all but finitely many $\xi < \gamma$.

Fix $\xi < \beta$ both in Z with $\mathrm{cof}(\beta) = \omega$. We describe how to determine $C_X(\xi, \beta)$, using only knowledge of $\hat{s}(\xi)$, $\hat{r}(\xi)$, and $\mathrm{cf}(\hat{s}\restriction\gamma)$.

By condition (iii), ξ is discarded at stage $\hat{\rho}(\xi) < \beta$. Let $\langle k, D\rangle$ be such that $f_{\hat{\rho}(\xi)}(h_\xi) = \langle k, D\rangle$. h_ξ can be determined from knowledge of $\hat{s}(\xi)$, and $f_{\hat{\rho}(\xi)}$ can be determined from knowledge of $\hat{s}(\hat{\rho}(\xi)) = \hat{r}(\xi)$. Hence k can be determined from knowledge of $\hat{s}(\xi)$ and $\hat{r}(\xi)$.

The definition of $f_{\hat{\rho}(\xi)}$ is such that $C_X(\alpha_k^{\hat{\rho}(\xi)}, \hat{\rho}(\xi)) = C_X(\xi, \hat{\rho}(\xi))$. (We are using here the fact that $\alpha_{h_\xi}^{\hat{\rho}(\xi)} = \xi$.) Since $\beta > \hat{\rho}(\xi)$ it follows by Claim 3.4 that $C_X(\alpha_k^{\hat{\rho}(\xi)}, \beta) = C_X(\xi, \beta)$.

Now $\alpha_k^{\hat{\rho}(\xi)}$ is an ordinal below ξ which had not been discarded by stage $\hat{\rho}(\xi)$. From condition (ii) it follows that the ordinal is not discarded by stage β (in fact not even by stage γ), and therefore $\alpha_k^{\hat{\rho}(\xi)} = \alpha_k^\beta$.

Thus $C_X(\xi, \beta) = C_X(\alpha_k^\beta, \beta) = C_k^\beta$. C_k^β can be determined from knowledge of $\mathrm{cf}(\hat{s}\restriction\beta)$ by Corollary 4.10, and $\mathrm{cf}(\hat{s}\restriction\beta)$ can be determined from $\mathrm{cf}(\hat{s}\restriction\gamma)$ by condition (i). $\dashv$

Using the last claim, fix a function $\Lambda\colon \widehat{S} \times \widehat{S} \times \mathcal{P}(\widehat{S}) \to \mathcal{C}$ (independently of X), so that for every γ of cofinality greater than ω:

($*$) there is a club $Z \subset \gamma$, so that $C_X(\xi, \beta) = \Lambda(\hat{s}(\xi), \hat{r}(\xi), \mathrm{cf}(\hat{s}\restriction\gamma))$ for all $\xi < \beta$ both in Z with $\mathrm{cof}(\beta) = \omega$.

Lemma 4.16. *Let γ be a limit of cofinality greater than ω, and let $k \in K_{<\gamma}$. Then $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha_k^\gamma, \gamma)$ iff there is $a_0 \in S \cup \{\uparrow\}$, $D_0 \subset S$, $D^* \subset S$, $s\colon \gamma \to S$, and $r\colon \gamma \rightharpoonup S$ so that:*

1. $(\forall^*\beta < \gamma)\langle b, D_0, a_0, s(\beta)\rangle \in C_k^\beta$.
2. $(\forall^*\xi < \gamma)(\forall^*\beta < \gamma)$ *$r(\xi)$ and $\hat{r}(\xi)$ are both defined, and if* $\mathrm{cof}(\beta) = \omega$ *then* $\langle s(\xi), D^*, r(\xi), s(\beta)\rangle$ *belongs to* $\Lambda(\hat{s}(\xi), \hat{r}(\xi), \mathrm{cf}(\hat{s}\restriction\gamma))$.
3. $(\forall^*\beta < \gamma)$ *if* $\mathrm{cof}(\beta) = \omega$ *then* $s(\beta) = (\Psi \oplus \vec{\varphi})(D^*)$.
4. $(\forall^*\beta < \gamma)$ *if* $\mathrm{cof}(\beta) > \omega$ *then* $s(\beta) = (\Psi \oplus \vec{\varphi})(\beta; s\restriction\beta, r\restriction\beta)$.
5. (a) $b^* = (\Psi \oplus \vec{\varphi})(\gamma; s, r)$.
   (b) $D = D_0 \cup D^* \cup \{b^*\}$.
   (c) *One of the following conditions holds:*
      - $a_0 \in S$ *and* $a = a_0$.
      - $a_0 = \uparrow$, $h(b) \in u(q)$ *for every* $q \in D^*$, $h(b) \in u(b^*)$, *and* $a = \uparrow$.
      - $a_0 = \uparrow$, $h(b) \in u(q)$ *for every* $q \in D^*$, $h(b) \notin u(b^*)$, *and* $a = b^*$.

(In conditions (3), (4), and (5a) we are using the terminology of Definition 2.1 and Remark 2.2.)

Remark 4.17. One should think of a pair $\langle s, r\rangle$ witnessing the conditions of Lemma 4.16 as a skeleton for a run of $\mathcal{A}$ on $X\restriction[\alpha_k^\kappa, \gamma)$ witnessing that $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha_k^\gamma, \gamma)$. For example conditions (3) and (4) say that, on a club, the skeleton behaves like a run of $\mathcal{A}$. In the proof of the lemma we shall see that if there is a run witnessing $\langle b, D, a, b^*\rangle \in C_X(\alpha_k^\gamma, \gamma)$, then this run satisfies conditions (1)–(5), and in the other direction, any skeleton satisfying the conditions can be completed to a run witnessing $\langle b, D, a, b^*\rangle \in C_X(\alpha_k^\gamma, \gamma)$.

Proof of Lemma 4.16. Suppose first that $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha_k^\gamma, \gamma)$, and let $\langle s, r\rangle$ be a run of $\mathcal{A}$ on $X\restriction[\alpha_k^\gamma, \gamma)$ witnessing this. Let $D^* = \mathrm{cf}(s)$, let $D_0 = \{s(\xi) \mid \alpha_k^\gamma \leq \xi < \gamma\}$, let $a_0 = r(\alpha_k^\gamma)$ if the pebble $h(s(\alpha_k^\gamma))$ placed on α_k^γ is released before stage γ, and let $a_0 = \uparrow$ if the pebble is released at γ or not released at all. It is easy to check that conditions (1)–(5) hold for a_0, D_0, D^*, s, and r. The runs witnessing condition (1) are $\langle s\restriction[\alpha_k^\gamma, \beta], r\restriction[\alpha_k^\gamma, \beta)\rangle$ for sufficiently large β. Using ($*$) above we may assume that $\Lambda(\hat{s}(\xi), \hat{r}(\xi), \mathrm{cf}(\hat{s}\restriction\gamma))$ in condition (2) is equal to $C_X(\xi, \beta)$, and for almost all ξ and β, the run $\langle s\restriction[\xi, \beta], r\restriction[\xi, \beta)\rangle$ witnesses that $\langle s(\xi), D^*, r(\xi), s(\beta)\rangle$ belongs to this character. Conditions (3) and (4) hold because $\langle s, r\rangle$ is a run of $\mathcal{A}$. Condition (5) holds because $\langle s, r\rangle$ witnesses that $\langle b, D, a, b^*\rangle \in C_X(\alpha_k^\gamma, \gamma)$.

Suppose next that $\langle b, D, a, b^*\rangle$, a_0, D_0, D^*, s, and r satisfy conditions (1)–(5). We work to show that $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha^\gamma_k, \gamma)$.

Let $Z_1 \subset \gamma$ be a club witnessing $(*)$. Let Z_2 be the intersection of the clubs witnessing the truth of the "for almost all" statements in conditions (1)–(4). Let $Z \subset Z_1 \cap Z_2$ be a club so that every $\xi \in Z$ which is not a limit point of Z has cofinality ω.

Let ξ_0 be the first element of Z, and for each $\xi \in Z$ let $\beta(\xi)$ be the first element of Z above ξ. (Notice that $\beta(\xi)$ has cofinality ω.) Fix a run $\langle s_0, r_0\rangle$ of $\mathcal{A}$ on $X\restriction[\alpha^\gamma_k, \xi_0)$ witnessing that $\langle b, a_0, D_0, s(\xi_0)\rangle$ belongs to $C_X(\alpha^\gamma_k, \xi_0)$. This is possible using condition (1). For each $\xi \in Z$ fix a run $\langle s_\xi, r_\xi\rangle$ of $\mathcal{A}$ on $X\restriction[\xi, \beta(\xi))$ witnessing that $\langle s(\xi), D^*, r(\xi), s(\beta(\xi))\rangle$ belongs to $C_X(\xi, \beta(\xi))$. This is possible using condition (2) and $(*)$ above.

Define $s^* \colon [\alpha^\gamma_k, \gamma] \to S$ through the conditions: $s^*\restriction[\alpha^\gamma_k, \xi_0] = s_0$, $s^*\restriction[\xi, \beta(\xi)] = s_\xi$ for each $\xi \in Z$, and $s^*(\gamma) = b^*$. Notice then that:

(i) $s^*(\alpha^\gamma_k) = b$.
(ii) $s^*(\xi) = s(\xi)$ for each $\xi \in Z$.
(iii) $\{s^*(\zeta) \mid \alpha^\gamma_k \le \zeta \le \xi_0\} = D_0$.
(iv) $\{s^*(\zeta) \mid \xi \le \zeta \le \beta(\xi)\} = D^*$ for each $\xi \in Z$.
(v) If there is an ordinal $\rho \in (\alpha^\gamma_k, \xi_0]$ so that $h(s^*(\alpha^\gamma_k)) \notin u(s^*(\rho))$ then $a_0 = s^*(\rho)$ for the least such ρ, and otherwise $a_0 = \uparrow$.
(vi) There is an ordinal $\rho \in (\xi, \beta(\xi)]$ so that $h(s^*(\xi)) \notin u(s^*(\rho))$, and $s^*(\rho)$ for the least such ρ is equal to $r(\xi)$.

These conditions follow from the facts that $\langle s_0, r_0\rangle$ witnesses the membership of $\langle b, D_0, a_0, s(\xi_0)\rangle$ in $C_X(\alpha^\gamma_k, \xi_0)$, $r(\xi) \ne \uparrow$, and $\langle s_\xi, r_\xi\rangle$ witnesses the membership of $\langle s(\xi), D^*, r(\xi), s(\beta(\xi))\rangle$ in $C_X(\xi, \beta(\xi))$.

For $\xi \in [\alpha^\gamma_k, \gamma)$ let $\rho^*(\xi)$ be the first ordinal $\rho^* > \xi$ so that $h(s^*(\xi)) \notin u(s^*(\rho^*))$ if there is such an ordinal, and undefined otherwise. Let $r^*(\xi) = s^*(\rho^*(\xi))$. Notice then that:

(vii) $r^*(\alpha^\gamma_k) = a$.

This follows from conditions (iv) and (v) above, the fact that $s^*(\gamma) = b^*$, and condition (5c) in Lemma 4.16. Notice further that for $\xi \in Z$:

(viii) $r^*(\xi) = r(\xi)$.

This follows from condition (vi) above.

Claim 4.18. *For every $\beta \in Z \cup \{\gamma\}$ which is a limit point of Z, $s^*(\beta) = (\Psi \oplus \vec{\varphi})(\beta; s^*\restriction\beta, r^*\restriction\beta)$.*

Proof. Suppose first that β has cofinality ω. By condition (iv) and since β is a limit point of Z, $\mathrm{cf}(s^*\restriction\beta)$ is equal to D^*. So $(\Psi \oplus \vec{\varphi})(\beta; s^*\restriction\beta, r^*\restriction\beta) = (\Psi \oplus \vec{\varphi})(D^*) = s^*(\beta)$ where the last equality uses condition (3) in Lemma 4.16 and condition (ii) above.

Suppose next that β has cofinality ω_1 or greater. Notice that by conditions (ii) and (viii), s^* and r^* agree with s and r on the set $Z \cap \beta$, which is closed unbounded in β. By Claim 1.9 it follows that a sentence of $\mathcal{L}^*_S$ is true in $(\beta; s^*\restriction\beta, r^*\restriction\beta)$ iff it is true in $(\beta; s\restriction\beta, r\restriction\beta)$. From this in turn it follows that $(\Psi \oplus \vec{\varphi})(\beta; s^*\restriction\beta, r^*\restriction\beta)$ is equal to $(\Psi \oplus \vec{\varphi})(\beta; s\restriction\beta, r\restriction\beta)$. Finally $(\Psi \oplus \vec{\varphi})(\beta; s\restriction\beta, r\restriction\beta)$ is equal to $s^*(\beta)$

using condition (4) in Lemma 4.16 and condition (ii) above if $\beta \in Z$, and using condition (5a) if $\beta = \gamma$. $\dashv$

Claim 4.19. *For every ζ in the interval $[\alpha^\gamma_k, \xi_0)$, $\langle s^*(\zeta), X(\zeta), s^*(\zeta+1)\rangle \in T$. If ζ is a limit ordinal in the interval $(\alpha^\gamma_k, \xi_0]$ then $s^*(\zeta) = (\Psi \oplus \vec{\varphi})(\zeta; s^*\restriction\zeta, r^*\restriction\zeta)$.*

Proof. Both statements follow from the fact that $s^*\restriction[\alpha^\gamma_k, \xi_0] = s_0$, the definition of r^*, and the fact that $\langle s_0, r_0\rangle$ is a run of $\mathcal{A}$ on $X\restriction[\alpha^\gamma_k, \xi_0)$. $\dashv$

Claim 4.20. *Let ξ belong to Z. Then for every ζ in the interval $[\xi, \beta(\xi))$, $\langle s^*(\zeta), X(\zeta), s^*(\zeta+1)\rangle \in T$. If ζ is a limit ordinal in the interval $(\xi, \beta(\xi)]$ then $s^*(\zeta) = (\Psi \oplus \vec{\varphi})(\zeta; s^*\restriction\zeta, r^*\restriction\zeta)$.*

Proof. Both statements follow from the fact that $s^*\restriction[\xi, \beta(\xi)) = s_\xi$, the definition of r^*, and the fact that $\langle s_\xi, r_\xi\rangle$ is a run of $\mathcal{A}$ on $X\restriction[\xi, \beta(\xi))$. $\dashv$

The last three claims combine to show that $\langle s^*, r^*\rangle$ is a run of $\mathcal{A}$ on $X\restriction[\alpha^\gamma_k, \gamma)$. By conditions (i) and (vii) above, $s^*(\alpha^\gamma_k) = b$ and $r^*(\alpha^\gamma_k) = a$. $s^*(\gamma) = b^*$ by definition. By conditions (iii), (iv), and (5b), $\{s^*(\zeta) \mid \alpha^\gamma_k \le \zeta \le \gamma\} = D$. The run $\langle s^*, r^*\rangle$ therefore witnesses that $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha^\gamma_k, \gamma)$.
$\square$ (Lemma 4.16)

Corollary 4.21. *There is a sentence $\hat{\varphi}_{b,D,a,b^*}$ in the language $\mathcal{L}^*_{\widehat{S}}$ so that for every limit γ of cofinality greater than ω, and every $k \in K_{<\gamma}$, $\langle b, D, a, b^*\rangle$ belongs to $C_X(\alpha^\gamma_k, \gamma)$ iff $(\gamma; \hat{s}\restriction\gamma, \hat{r}\restriction\gamma) \models \hat{\varphi}_{b,D,a,b^*}$.*

Proof. Each of the five conditions in Lemma 4.16 can be written as a sentence in the langauge $\mathcal{L}^*_{\widehat{S}\times S}$ over the structure $(\gamma; \hat{s}\times s, \hat{r}\times r)$. The statement that there are $s\colon \gamma \to S$ and $r\colon \gamma \rightharpoonup S$ so that the conditions hold can be written as a sentence in the language $\mathcal{L}^*_{\widehat{S}}$ over the structure $(\gamma; \hat{s}, \hat{r})$, using Lemma 1.18. $\dashv$

Let $\widehat{\Phi}$ be the collection of the following sentences in $\mathcal{L}^*_{\widehat{S}}$:

- The sentences $\hat{\varphi}_{b,D,a,b^*}$ of the previous corollary, for all $b, b^* \in S$, $a \in S \cup \{\uparrow\}$, and $D \subset S$.
- The sentence $\hat{\varphi}_{\text{ctbl}-\text{cof}}$ which is true in a structure $(\gamma; \hat{s}, \hat{r})$ iff γ has countable cofinality.
- The sentences $\hat{b} \in \text{cf}(\hat{s})$, for all $\hat{b} \in \widehat{S}$.

Corollary 4.22. *For each limit ordinal γ, $\hat{s}(\gamma)$ can be determined (independently of X and γ) from knowledge of the truth value of the sentences of $\widehat{\Phi}$ in $(\gamma; \hat{s}\restriction\gamma, \hat{r}\restriction\gamma)$.*

Proof. Knowledge of the truth values of the sentences of $\widehat{\Phi}$ in $(\gamma; \hat{s}\restriction\gamma, \hat{r}\restriction\gamma)$ allows determining:

(i) $\text{cf}(\hat{s}\restriction\gamma)$.
(ii) Whether $\text{cof}(\gamma) = \omega$.
(iii) The truth value of each of the sentences $\hat{\varphi}_{b,D,a,b^*}$ in $(\gamma; \hat{s}\restriction\gamma, \hat{r}\restriction\gamma)$.

From the knowledge of $\text{cf}(\hat{s}\restriction\gamma)$ one can determine $K_{<\gamma}$, R_γ, and $C^\gamma_{i,j}$ for $i\ R_\gamma\ j$, see Claim 4.7. If $\text{cof}(\gamma) = \omega$ then one can also determine C^γ_k for each $k \in K_{<\gamma}$ from the knowledge of $\text{cf}(\hat{s}\restriction\gamma)$, see Corollary 4.10. If $\text{cof}(\gamma) > \omega$ then one can determine C^γ_k, namely $C_X(\alpha^\gamma_k, \gamma)$, from the truth value of the sentences $\hat{\varphi}_{b,D,a,b^*}$, using Corollary 4.21.

Altogether then one can determine each of $K_{<\gamma}$, R_γ, C^γ_k for $k \in K_{<\gamma}$, and $C^\gamma_{i,j}$ for all pairs $i \; R_\gamma \; j$. By Remark 4.5 one can therefore determine $\hat{s}(\gamma)$. ⊣

Let $\vec{\phi}$ list all the sentences in the set $\widehat{\Phi}$. Recall that $t^{\vec{\phi}}_{(\gamma;\hat{s},\hat{r})}$ is the function, from $\mathrm{lh}(\vec{\phi})$ into 2, defined by the condition $t^{\vec{\phi}}_{(\gamma;\hat{s};\hat{r})}(i) = 1$ if $(\gamma;\hat{s},\hat{r}) \models \phi_i$ and $t^{\vec{\phi}}_{(\gamma;\hat{s};\hat{r})}(i) = 0$ otherwise. Thus $t^{\vec{\phi}}_{(\gamma;\hat{s};\hat{r})}$ codes the truth value of each of the sentences of $\widehat{\Phi}$ in $(\gamma;\hat{s},\hat{r})$. By the last corollary there is a function $\widehat{\Psi}\colon 2^{\mathrm{lh}(\vec{\phi})} \to \widehat{S}$ (independent of X) so that $\hat{s}(\gamma) = \widehat{\Psi}(t^{\vec{\phi}}_{(\gamma;\hat{s},\hat{r})})$ for every limit γ.

We have now completed the definition of a deterministic Σ–automaton $\widehat{\mathcal{A}} = \langle \widehat{S}, \widehat{P}, \widehat{T}, \widehat{\Psi}, \vec{\phi}, \hat{h}, \hat{u}\rangle$ so that $\langle \hat{s}, \hat{r}\rangle$ as defined above is a run of $\mathcal{A}$ on X.

We also defined, in Definition 4.4, a state $\widehat{I} \in \widehat{S}$ and a function $z\colon \widehat{S} \rightharpoonup \mathcal{C}$ so that $\hat{s}(0) = \widehat{I}$ and $z(\hat{s}(\gamma)) = C_X(0,\gamma)$ for each $\gamma > 0$.

Our definitions of $\widehat{\mathcal{A}}$, $\widehat{I}$, and z were independent of the input string X. We have therefore proved the following theorem:

Theorem 4.23. *Let Σ be a finite alphabet and let $\mathcal{A}$ be a Σ–automaton. Then there is a* deterministic *Σ–automaton $\widehat{\mathcal{A}}$, with a set of states $\widehat{S}$ say, a particular state $\widehat{I} \in \widehat{S}$, and a function $z\colon \widehat{S} \rightharpoonup \mathcal{C}$ (where $\mathcal{C}$ is the set of characters for the original automaton $\mathcal{A}$), so that: for every ordinal θ and every input string $X\colon \theta \to \Sigma$, if $\langle \hat{s}, \hat{r}\rangle$ is the unique run of $\widehat{\mathcal{A}}$ on X with $\hat{s}(0) = \widehat{I}$, then $z(\hat{s}(\gamma)) = C_X(0,\gamma)$ for each $\gamma \in (0,\theta]$.* ⊣

Corollary 4.24. *Let Σ be a finite alphabet and let $\mathcal{A}$ be a Σ–automaton. Let $\langle I, F\rangle$ be an accepting condition for $\mathcal{A}$. Then there is a* deterministic *Σ–automaton $\widehat{\mathcal{A}}$, with an accepting condition $\langle \widehat{I}, \widehat{F}\rangle$, so that for every ordinal θ and every input string $X : \theta \to \Sigma$, $\langle \widehat{\mathcal{A}}, \widehat{I}, \widehat{F}\rangle$ accepts X iff $\langle \mathcal{A}, I, F\rangle$ accepts X.*

Proof. Let Q be the set of characters C so that

$$(\exists D)(\exists a)(\exists b^*)(\langle I, D, a, b^*\rangle \in C \wedge b^* \in F).$$

Then $\langle \mathcal{A}, I, F\rangle$ accepts an input string X of length θ iff $C_X(0,\theta)$ belongs to Q.

Let $\widehat{A}$, $\widehat{I}$, and z be as in the previous theorem.

If $\langle \mathcal{A}, I, F\rangle$ accepts the unique input string of length 0 then let $\widehat{F} = z^{-1}{}''Q \cup \{\widehat{I}\}$, and otherwise let $\widehat{F} = z^{-1}{}''Q$.

Then $\langle \widehat{\mathcal{A}}, \widehat{I}, \widehat{F}\rangle$ accepts $X : \theta \to \Sigma \iff \theta = 0$ and $\langle \mathcal{A}, I, F\rangle$ accepts X, or $\theta > 0$ and $C_X(0,\theta) \in Q \iff \langle \mathcal{A}, I, F\rangle$ accepts X. ⊣

§5. Formulae to Automata. For $a \in \mathrm{ON}$ define $\chi_{\mathsf{f}}(a)\colon \mathrm{ON} \to 2$ through the condition $\chi_{\mathsf{f}}(\gamma) = 1$ if $\gamma = a$ and $\chi_{\mathsf{f}}(\gamma) = 0$ otherwise. For $a \subset \mathrm{ON}$ define $\chi_{\mathsf{s}}(a)\colon \mathrm{ON} \to 2$ through the condition $\chi_{\mathsf{s}}(\gamma) = 1$ if $\gamma \in a$ and $\chi_{\mathsf{s}}(\gamma) = 0$ otherwise. (f and s here stand for "first order" and "second order.")

Given a monadic second order formula φ with free variables $x_1, \ldots, x_k$ let $\mathrm{sig}(\varphi)\colon k \to \{\mathsf{s}, \mathsf{f}\}$ be the function defined by the condition $\mathrm{sig}(\varphi)(i) = \mathsf{s}$ if x_i is a second order variable, and $\mathrm{sig}(\varphi)(i) = \mathsf{f}$ if x_i is a first order variable.

A sequence $\langle a_1, \ldots, a_k\rangle$ is said to **fit the signature** of φ if a_i is an ordinal for i such that $\mathrm{sig}(\varphi)(i) = \mathsf{f}$, and a set of ordinals for i such that $\mathrm{sig}(\varphi)(i) = \mathsf{s}$. Given a sequence $\langle a_1,\dots,a_k\rangle$ which fits the signature of φ define $\chi(a_1,\dots,a_k)\colon \mathrm{ON}\to 2^k$ to be the function $\chi_{\mathrm{sig}(\varphi)(1)}(a_1)\times\cdots\times\chi_{\mathrm{sig}(\varphi)(k)}(a_k)$.

The work in the previous sections leads to the following theorem, which provides the crucial link between monadic second order formulae over the ordinals and our finite state automata. The theorem relies heavily on the determinism proved in Section 4.

Theorem 5.1. *Let φ be a monadic second order formula in the language of order, with k free variables say. Then there is a deterministic finite state automaton $\mathcal{A}$, with accepting condition $\langle I,F\rangle$, so that: for every ordinal θ, and for every sequence $a_1,\dots,a_k$ of elements and subsets of θ which fits the signature of φ, $(\theta;<)\models\varphi[a_1,\dots,a_k]$ iff $\langle\mathcal{A},I,F\rangle$ accepts $\chi(a_1,\dots,a_k)\restriction\theta$.*

Proof. The proof is by induction on the complexity of φ. The case that φ is atomic is a simple exercise. The inductive case of conjunction is a direct application of Lemma 2.4. The inductive case of negation is a direct application of Claim 2.7. Finally the inductive case of existential quantification (either first or second order) is an application of Claim 2.3 followed by an application of Corollary 4.24 to obtain a deterministic automaton equivalent to the non-deterministic automaton produced by Claim 2.3. $\dashv$

Remark 5.2. Theorem 5.1 holds also in the case that θ is the class of all ordinals, with both the quantifiers of φ and the quantifiers appearing in the "almost-all" sentences used in $\mathcal{A}$ interpreted as ranging over classes.

Remark 5.3. The construction of $\mathcal{A}$ from φ is effective. Thus, there is in fact a recursive function which assigns to each formula φ an automaton $\mathcal{A}$ and accepting condition $\langle I,F\rangle$ witnessing Theorem 5.1.

Theorem 5.1 has a converse:

Claim 5.4. *Given an automaton $\mathcal{A}$ on a finite alphabet $\Sigma=\{\sigma_1,\dots,\sigma_n\}$ and an accepting condition $\langle I,F\rangle$, there is a monadic second order formula φ so that $\langle\mathcal{A},I,F\rangle$ accepts $X\colon\theta\to\Sigma$ iff $(\theta;<)\models\varphi[a_1,\dots,a_n]$ where $a_i=\{\xi\mid X(\xi)=\sigma_i\}$.*

Proof. This can be seen through coding strings of states of $\mathcal{A}$ by sets of ordinals, and having φ express the existence of an accepting run: Let $S=\{e_1,\dots,e_k\}$ be the set of states of $\mathcal{A}$. For $s\colon\theta\to S$ and $r\colon\theta\rightharpoonup S$ define $b_i=\{\xi<\theta\mid s(\xi)=e_i\}$ and $c_i=\{\xi<\theta\mid r(\xi)=e_i\}$. It is enough to show that there are formulae ψ_i so that $(\theta;<)\models\psi_i[a_1,\dots,a_n,b_1,c_1,\dots,b_k,c_k]$ iff $\langle s^\frown e_i,r\rangle$ is a run of $\mathcal{A}$ on X. Using second order existential quantification one can then obtain φ from the formulae ψ_i for i such that $e_i\in F$. Now ψ_i must simply express conditions (S), (L), and (R) in Section 2. It is clear that each of the conditions can be expressed in the monadic language, and in fact only first order quantifiers are needed for conditions (S) and (R). Let us just note that T, Ψ, h, and u are finite objects, and the references to these objects in conditions (S), (L), and (R) can be removed, replacing the conditions by long, but finite, disjunctions of cases, ranging over all possible configurations which satisfy the requirements involving T, Ψ, h, and u. The almost-all formulae from $\vec{\varphi}$ appear as subformulae in the disjunction in the case of condition (L). In the case of conditions (S) and (R) the disjunction is first order. $\dashv$

§6. Second order definability. One of the important aspects of Theorem 5.1 is the independence of the conversion from φ to $\langle \mathcal{A}, I, F\rangle$ from the length θ of the input string. We now use this aspect to show that singular cardinals cannot be defined by second order sentences over $(\mathrm{ON}; <)$.

Let Σ be the singleton alphabet $\{\emptyset\}$. Let $\mathcal{A}$ be a deterministic Σ–automaton. Let θ be closed under ordinal multiplication by ω, meaning that $\alpha < \theta \rightarrow \alpha \cdot \omega < \theta$. Note that θ is then closed also under ordinal addition. Let $X\colon \theta \rightarrow \Sigma$ be the constant function $X(\alpha) = \emptyset$ (the only possible input in the case of the singleton alphabet $\{\emptyset\}$). Let $\langle s, r\rangle$ be a run of $\mathcal{A}$ on X. The claims below are formulated with reference to these objects.

Claim 6.1. *Let $\alpha < \theta$ and define $\bar{s}$ and $\bar{r}$ setting $\bar{s}(\xi) = s(\alpha + \xi)$ for $\xi \leq \theta$ and $\bar{r}(\xi) = r(\alpha + \xi)$ for $\xi < \theta$. Then $\langle \bar{s}, \bar{r}\rangle$ is a run of $\mathcal{A}$ on X.*

Proof. This is a standard claim for automata running on a constant input string. The proof is a simple induction showing that $\langle \bar{s}, \bar{r}\rangle$ satisfies conditions (S), (L), and (R) in Section 2, using the same conditions for $\langle s, r\rangle$ and the fact that $X(\alpha+\xi) = X(\xi)$ for each ξ. Let us just comment that, for each limit γ, the structures $(\alpha + \gamma; s, r)$ and $(\gamma; \bar{s}, \bar{r})$ are similar (see Definition 1.8) and therefore by Claim 1.9 satisfy the same $\mathcal{L}^*_S$ sentences. This is important for condition (L) in Section 2. ⊣

Claim 6.2. *Let $\alpha_1, \alpha_2 < \theta$. Suppose that $s(\alpha_1) = s(\alpha_2)$. Then for every $\xi \leq \theta$, $s(\alpha_1 + \xi) = s(\alpha_2 + \xi)$, and $r(\alpha_1) = r(\alpha_2)$.*

Proof. Define $\langle \bar{s}_1, \bar{r}_1\rangle$ through the conditions $\bar{s}_1(\xi) = s(\alpha_1 + \xi)$ and $\bar{r}_1(\xi) = r(\alpha_1) + \xi$. Define $\bar{s}_2$ and $\bar{r}_2$ similarly using α_2. By the previous claim then, both $\langle \bar{s}_1, \bar{r}_1\rangle$ and $\langle \bar{s}_2, \bar{r}_2\rangle$ are runs of $\mathcal{A}$ on X. Both have the same first state: $\bar{s}_1(0) = s(\alpha_1) = s(\alpha_2) = \bar{s}_2(0)$. Since $\mathcal{A}$ is deterministic, $\bar{s}_1$ must equal $\bar{s}_2$ and $\bar{r}_1$ must equal $\bar{r}_2$. ⊣

Claim 6.3. *Let $D = \mathrm{cf}(s{\restriction}\theta)$. Let $\delta < \theta$ be large enough that $\{s(\xi) \mid \delta \leq \xi < \theta\} = D$. Let $\eta \in (\delta, \theta)$ be large enough that $\{s(\xi) \mid \delta \leq \xi < \eta\} = D$, and picked so that $s(\eta) = s(\delta)$. (This is possible since $s(\delta) \in \mathrm{cf}(s{\restriction}\theta)$ and therefore there are cofinally many $\zeta < \theta$ so that $s(\zeta) = s(\delta)$.) Let γ be such that $\eta = \delta + \gamma$.*

Then for every $\alpha \in (\delta, \theta]$ which is closed under ordinal addition of γ (meaning that $\nu < \alpha \rightarrow \nu + \gamma < \alpha$), $\mathrm{cf}(s{\restriction}\alpha)$ is precisely equal to D.

Proof. Using the previous claim and the fact that $s(\delta) = s(\delta + \gamma)$ we see that $\{s(\xi) \mid \delta + \gamma \leq \xi \leq \delta + \gamma \cdot 2\}$ is equal to $\{s(\xi) \mid \delta \leq \xi \leq \delta + \gamma\}$, which we know is equal to D. From this and the fact that $s(\xi) \in D$ for all $\xi > \delta$ it follows that for every $\beta \in [\delta, \eta]$, $\{s(\xi) \mid \beta \leq \xi \leq \beta + \gamma \cdot 2\}$ is equal to D.

Fix now some $\alpha \in [\delta, \theta]$ which is closed under addition of γ. Since $\alpha > \delta$, certainly $\mathrm{cf}(s{\restriction}\alpha) \subset D$. Thus it is enough to show that $\{s(\xi) \mid \nu \leq \xi < \alpha\} \supset D$ for cofinally many $\nu < \alpha$.

Let $\nu < \alpha$ be given. Increasing ν if needed we may assume that $\nu > \delta$, so that $s(\nu) \in D$. There is therefore some $\beta \in [\delta, \eta]$ with $s(\beta) = s(\nu)$. By the previous claim, $\{s(\xi) \mid \nu \leq \xi \leq \nu + \gamma \cdot 2\}$ is equal to $\{s(\xi) \mid \beta \leq \xi \leq \beta + \gamma \cdot 2\}$, which we know is equal to D. Since α is closed under addition of γ, $\nu + \gamma \cdot 2 < \alpha$. So $\{s(\xi) \mid \nu \leq \xi < \alpha\} \supset \{s(\xi) \mid \nu \leq \xi < \nu + \gamma \cdot 2\} = D$. ⊣

Claim 6.4. *Let δ and γ be as in the previous claim. Let $C = \{\alpha \leq \theta \mid \alpha > \delta$ and α is closed under addition of $\gamma\}$. Then for every $\alpha, \beta \in C$, if $\operatorname{cof}(\alpha) = \operatorname{cof}(\beta)$ then $s(\alpha) = s(\beta)$.*

Proof. Suppose for contradiction that $\alpha, \beta \in C$ have the same cofinality, yet $s(\alpha) \neq s(\beta)$. Let $\tau = \operatorname{cof}(\alpha) = \operatorname{cof}(\beta)$. Picking α and β of minimal cofinality we may assume that:

(i) If $\bar{\alpha}, \bar{\beta} \in C$ are such that $\operatorname{cof}(\bar{\alpha}) = \operatorname{cof}(\bar{\beta}) < \tau$, then $s(\bar{\alpha}) = s(\bar{\beta})$.

If τ is equal to ω, then condition (L) in Section 2 is such that $s(\alpha) = (\Psi \oplus \vec{\varphi})(\operatorname{cf}(s \restriction \alpha))$ and $s(\beta) = (\Psi \oplus \vec{\varphi})(\operatorname{cf}(s \restriction \beta))$ (see Definition 2.1 and Remark 2.2). By the previous claim both $\operatorname{cf}(s \restriction \alpha)$ and $\operatorname{cf}(s \restriction \beta)$ are equal to D. So $s(\alpha) = (\Psi \oplus \vec{\varphi})(D) = s(\beta)$.

Suppose then that $\operatorname{cof}(\tau) > \omega$. Let $Y \subset C$ be a closed unbounded subset of α of order type τ. (Y can be picked a subset of C since $C \cap \alpha$ is a closed unbounded subset of α.) Similarly let $Z \subset C$ be a closed unbounded subset of β of order type τ. Let $f \colon Y \to Z$ be the unique order preserving bijection.

Notice that $\operatorname{cof}(f(\xi)) = \operatorname{cof}(\xi)$ for each ξ which is a limit point of Y. Using condition (i) it follows that $s(f(\xi)) = s(\xi)$ for each ξ which is a limit point of C, and hence by Claim 6.2, also $r(f(\xi)) = r(\xi)$. The structures $(\alpha; s \restriction \alpha, r \restriction \alpha)$ and $(\beta; s \restriction \beta, r \restriction \beta)$ are therefore similar. By Claim 1.9 they satisfy the same sentences of $\mathcal{L}^*_S$, and therefore $(\Psi \oplus \vec{\varphi})(\alpha; s \restriction \alpha, r \restriction \alpha)$ is equal to $(\Psi \oplus \vec{\varphi})(\beta; s \restriction \beta, r \restriction \beta)$. Using condition (L) in Section 2 it follows that $s(\alpha) = s(\beta)$. $\dashv$

Corollary 6.5. *Let Σ be the singleton alphabet $\{\emptyset\}$, let $\mathcal{A}$ be a deterministic Σ–automaton, and let $X \colon \theta \to \Sigma$ be the constant input $X(\alpha) = \emptyset$ for $\alpha < \theta$. Let $\langle s, r \rangle$ be a run of $\mathcal{A}$ on X.*

Suppose that θ is a limit ordinal and that θ is closed under multiplication by its cofinality. (In particular $\operatorname{cof}(\theta) < \theta$.) Then there is $\bar{\theta}$ strictly smaller than θ so that $s(\bar{\theta}) = s(\theta)$.

Proof. Let δ, γ, and C be as in the previous claim. Let $\tau = \operatorname{cof}(\theta)$.

Notice that C is equal to $\{\delta + \gamma \cdot \omega \cdot \xi \mid \xi \geq 1 \wedge \delta + \gamma \cdot \omega \cdot \xi < \theta\} \cup \{\theta\}$. Since θ is closed under multiplication by τ, the ordinal $\bar{\theta} = \delta + \gamma \cdot \tau$ is smaller than θ. This ordinal belongs to C, and has cofinality τ. So both $\bar{\theta}$ and θ are ordinals of cofinality τ in C. By the last claim $s(\bar{\theta}) = s(\theta)$. $\dashv$

A monadic sentence φ **pinpoints** an ordinal θ if θ is least so that $(\theta; <) \models \varphi$. θ can be **pinpointed** if there is a monadic sentence φ which pinpoints it.

Theorem 6.6. *Let θ be a limit ordinal closed under multiplication by its cofinality. Then θ cannot be pinpointed.*

Proof. Suppose for contradiction that φ pinpoints θ. Using Theorem 5.1 fix a deterministic automaton $\mathcal{A}$ with accepting condition $\langle I, F \rangle$ so that $(\gamma; <) \models \varphi$ iff $\langle \mathcal{A}, I, F \rangle$ accepts $\chi() \restriction \gamma$.

φ has no free variables, and $\chi() \restriction \gamma$ is simply the constant input $X \colon \gamma \to 2^0 = \{\emptyset\}$ defined by $X(\alpha) = \emptyset$ for all $\alpha < \gamma$.

Since φ holds in $(\theta; <)$, $\langle \mathcal{A}, I, F \rangle$ accepts $\chi() \restriction \theta$. Let $\langle s, r \rangle$ be the accepting run, so that $s(0) = I$ and $s(\theta) \in F$. By the last claim there is $\bar{\theta} < \theta$ so that $s(\bar{\theta}) = s(\theta)$, hence $s(\bar{\theta}) \in F$. Thus $\langle \mathcal{A}, I, F \rangle$ accepts $\chi() \restriction \bar{\theta}$ (the witnessing run is

$\langle s\restriction\bar\theta+1, r\restriction\theta\rangle$), and therefore $(\bar\theta,<)\models\varphi$. But this contradicts the assumption that φ pinpoints θ. ⊣

An ordinal θ is **definable** by a monadic formula over $(\mathrm{ON};<)$ if there is a monadic formula φ with one free variable so that $(\mathrm{ON};<)\models\varphi[\alpha]$ iff $\alpha=\theta$.

Lemma 6.7. *Suppose that θ is definable by a monadic formula over* $(\mathrm{ON};<)$. *Then θ can be pinpointed.*

Proof. Let φ be a formula defining θ. Using Theorem 5.1 fix a deterministic automaton $\mathcal{A}$ with an accepting condition $\langle I,F\rangle$ so that $(\mathrm{ON};<)\models\varphi[\alpha]$ iff $\langle\mathcal{A},I,F\rangle$ accepts X_α, where $X_\alpha\colon\mathrm{ON}\to 2$ is the function determined by the condition $X_\alpha(\xi)=1$ if $\xi=\alpha$ and $X_\alpha(\xi)=0$ otherwise.

Since $(\mathrm{ON};<)\models\varphi[\theta]$, $\langle\mathcal{A},I,F\rangle$ accepts X_θ. Let $\langle s,r\rangle$ be the accepting run, so that $s(0)=I$ and $s(\mathrm{ON})\in F$. (By $s(\mathrm{ON})$ we mean the final state reached by $\mathcal{A}$ running on the class-length input X_α.) Let $b^*=s(\theta)$.

Claim 6.8. *There is no $\bar\theta<\theta$ so that $s(\bar\theta)=b^*$.*

Proof. Suppose for contradiction $\bar\theta<\theta$ and $s(\bar\theta)=s(\theta)$. Define $s^*\colon\mathrm{ON}{+}1\to S$ through the conditions:

- $s^*(\xi)=s(\xi)$ for $\xi\le\bar\theta$.
- $s^*(\bar\theta+\xi)=s(\theta+\xi)$.

(Notice that there is no conflict between the two conditions, as $s(\bar\theta)=s(\theta)$.) Define $r^*\colon\mathrm{ON}\rightharpoonup S$ setting $r^*(\xi)=s^*(\zeta)$ for the least $\zeta>\xi$ so that $h(s^*(\xi))\notin u(s^*(\zeta))$ if there is such a ζ, and $r^*(\xi)={\uparrow}$ otherwise.

It is easy to check that $\langle s^*,r^*\rangle$ is a run of $\mathcal{A}$ on $X_{\bar\theta}$. Since $s^*(0)=I$ and $s^*(\mathrm{ON})=s(\mathrm{ON})\in F$, $\langle s^*,r^*\rangle$ witnesses that $\langle\mathcal{A},I,F\rangle$ accepts $X_{\bar\theta}$. It follows that $(\mathrm{ON};<)\models\varphi[\bar\theta]$, and this is a contradiction since φ is only true of θ. ⊣

For each ordinal α let $Y_\alpha\colon\alpha\to 2$ be the constant function 0, that is the function $Y(\xi)=0$ for all $\xi<\alpha$. Notice that $X_\theta\restriction\theta$ is simply Y_θ.

Let $F^*=\{b^*\}$. Then $\langle\mathcal{A},I,F^*\rangle$ accepts $Y_\theta=X_\theta\restriction\theta$ (the run witnessing this is $\langle s\restriction\theta+1, r\restriction\theta\rangle$). By the last claim and since $\mathcal{A}$ is deterministic, $\langle\mathcal{A},I,F^*\rangle$ does not accept $Y_{\bar\theta}$ for any $\bar\theta<\theta$.

Using Claim 5.4, that is using the fact that the existence of accepting runs can be expressed in the monadic langauge, fix a sentence ψ so that $(\alpha;<)\models\psi$ iff $\langle\mathcal{A},I,F^*\rangle$ accepts Y_α. Then $(\theta;<)\models\psi$, and for all $\bar\theta<\theta$, $(\bar\theta;<)\not\models\psi$. Hence ψ pinpoints θ. □ (Lemma 6.7)

Theorem 6.9. *Let θ be a limit ordinal closed under ordinal multiplication by its cofinality. Then θ can*not *be defined by a monadic formula over* $(\mathrm{ON};<)$.

Proof. If θ can be defined then by Lemma 6.7 it can be pinpointed, contradicting Theorem 6.6. ⊣

All *cardinals* are closed under ordinal multiplication, so every cardinal κ with $\mathrm{cof}(\kappa)<\kappa$ is closed under ordinal multiplication by its cofinality. Using Theorem 6.9 it follows finally that no singular cardinal can be defined by a monadic formula over $(\mathrm{ON};<)$.

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