THE INEFFABLE TREE PROPERTY AND FAILURE OF THE SINGULAR CARDINALS HYPOTHESIS

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1. Introduction

A long standing project in set theory is to analyze how much compactness can be obtained in the universe. Compactness is the phenomenon where if a certain property holds for all small substructures of an object, then it holds for the entire object. Compactness properties of particular interest are combinatorial principles that follow from large cardinals, but can be forced to hold at successors. Key examples include (in order of increasing strength) failure of squares, the tree property, and the ineffable tree property (ITP). These principles "capture" the combinatorial essence of certain large cardinals. At an inaccessible cardinal, the tree property is equivalent to weak compactness; ITP is equivalent to supercompactness. Forcing these principles at successors tells us to what extent small cardinals can behave like large cardinals.

An old question of Magidor addressing these issues is: can we get principles like the tree property or ITP simultaneously for every regular cardinal greater than ω_1? A positive answer would require many failures of SCH. In this paper we focus on ITP, the strongest of our key examples, and its relation to singular cardinal combinatorics. This is of particular interest because failure of SCH is an example of anticompactness, and so it is difficult to combine it with principles like ITP.

Definition 1.1. *Let μ be a regular uncountable cardinal.*

- *A* μ-list *is a sequence of functions $(d_\alpha)_{\alpha<\mu}$ such that $d_\alpha : \alpha \to 2$ for all $\alpha < \mu$. Such a list is* thin *if for every $\alpha < \mu$, $|\{d_\beta \restriction \alpha \mid \alpha \leq \beta < \mu\}| < \mu$.*
- *If $(d_\alpha)_{\alpha<\mu}$ is a thin μ-list, an* ineffable branch *of the list is a function $b : \mu \to 2$ such that the set $\{\alpha < \mu : b \restriction \alpha = d_\alpha\}$ is stationary in μ.*
- *The cardinal μ has the* ineffable tree property *if and only if every thin μ-list has an ineffable branch. We abbreviate this assertion by $ITP(\mu)$.*

It is clear that if μ has the ineffable tree property then it has the tree property. When μ is inaccessible, then by a classical result μ has the ineffable tree property if and only if μ is ineffable: we note that in this context all μ-lists are thin. Weiss [12] showed that if ω_2 has the ineffable tree property then ω_2 is ineffable in L, and that conversely if μ is ineffable then Mitchell forcing at μ produces an extension where $2^\omega = \omega_2 = \mu$ and the ineffable tree property of μ is preserved.

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Definition 1.2. *Let μ and λ be regular uncountable cardinals with $\mu \leq \lambda$.*

- *A* $P_\mu(\lambda)$-list *is a sequence of functions $(d_x)_{x \in P_\mu(\lambda)}$ such that $d_x : x \to 2$ for all $x \in P_\mu(\lambda)$. Such a list is* thin *if for every $x \in P_\mu(\lambda)$, $|\{d_y \upharpoonright x \mid x \subseteq y \in P_\mu(\lambda)\}| < \mu$.*
- *If $(d_x)_{x \in P_\mu(\lambda)}$ is a thin $P_\mu(\lambda)$-list, an* ineffable branch *of the list is a function $b : \lambda \to 2$ such that the set $\{x \in P_\mu(\lambda) \mid b \upharpoonright x = d_x\}$ is stationary in $P_\mu(\lambda)$.*
- *The pair (μ, λ) has the* ineffable tree property *if and only if every thin $P_\mu(\lambda)$-list has an ineffable branch. We abbreviate this assertion by $ITP(\mu, \lambda)$.*

Definition 1.3. *Let μ be a regular uncountable cardinal, then μ has the* super tree property *if and only if $ITP(\mu, \lambda)$ holds for all regular $\lambda \geq \mu$. We abbreviate this assertion by ITP_μ.*

Since μ is club in $P_\mu(\mu)$, it is not hard to see that $ITP(\mu)$ is equivalent to $ITP(\mu, \mu)$. The more general property $ITP(\mu, \lambda)$ is closely related to the property of supercompactness: in particular a classical result by Magidor [5] shows that for μ inaccessible, μ is supercompact if and only if ITP_μ holds. Weiss [12] showed that if μ is supercompact then Mitchell forcing at μ produces a model where ITP_{ω_2} holds, and Viale and Weiss [11] showed that this conclusion follows from PFA.

In some recent work, Hachtman and Sinapova [4] showed that if μ is the successor of a singular limit of supercompact cardinals then ITP_μ holds, and that this situation is also consistent when $\mu = \aleph_{\omega+1}$. In their construction, however, SCH holds.

This raises the following natural questions:

Question. *Is it possible for $ITP(\mu)$ (or ITP_μ) to hold when μ is the successor of a singular cardinal ν, and the Singular Cardinals Hypothesis fails at ν? Can this hold for a small value of μ?*

Our main results are:

- In Theorem 2.1, we show it is consistent that there exists ν a strong limit cardinals of cofinality ω, such that $2^\nu > \nu^+$ and $ITP(\nu^+)$ holds.
- In Theorem 3.1, we show it is consistent that there exists ν a strong limit cardinal of cofinality ω, such that $2^\nu > \nu^+$ and ITP_{ν^+} holds.
- In Theorem 4.3, we show it is consistent that $\aleph_{\omega^2}$ is strong limit, $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+2}$ and $ITP(\aleph_{\omega^2+1}, \lambda)$ holds for all regular $\lambda \geq \aleph_{\omega^2+1}$.

Of course each of these results entails the previous one, but for expository reasons we will work up to the proof of Theorem 4.3 in steps.

2. The one-cardinal ITP

Neeman [7] constructed a model where ν is a singular strong limit cardinal of cofinality ω, $2^\nu > \nu^+$, and ν^+ has the tree property. We will show that in fact the ineffable tree property holds at ν^+ in this model.

Theorem 2.1. *In the model of* [7], ITP(ν^+) *holds.*

Proof. We begin by recalling Neeman's construction. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\kappa = \kappa_0$, and assume that the supercompactness of κ is indestructible under κ-directed closed forcing. Let $\nu =$

$\sup_{n<\omega} \kappa_n$ and $\mu = \nu^+$. Let ρ be regular with $\rho > \mu$, and let E be $\mathrm{Add}(\kappa, \rho)$ generic over V.

In $V[E]$ the cardinal κ is supercompact, in particular there is a supercompactness measure U^* on $\mathcal{P}_\kappa(\mu)$. For each $n < \omega$ let U_n be the projection of U^* to $\mathcal{P}_\kappa(\kappa_n)$.[1] In $V[E]$ define the diagonal supercompact Prikry forcing $\mathbb{P}$ from the sequence of measures U_n. Let G be $\mathbb{P}$-generic over $V[E]$: we will show that $ITP(\mu)$ holds in $V[E][G]$.

We work in $V[E]$ unless otherwise noted. Let $\langle \dot{d}_\alpha \mid \alpha < \mu \rangle$ be a $\mathbb{P}$-name for a thin μ-list. We recall that $\mu = \kappa^+$ in $V[E][G]$, and for each $\alpha < \mu$ we let $\{\dot{\sigma}^\alpha_\xi \mid \xi < \kappa\}$ be a $\mathbb{P}$-name for an enumeration of $\{\dot{d}_\beta \restriction \alpha \mid \beta \geq \alpha\}$.

We recall that every condition in $\mathbb{P}$ has a *stem* h and a *top part* A, where h is a finite sequence $(x_0, \ldots x_{n-1})$ with $x_i \in P_\kappa \kappa_i$ (subject to some technical conditions) and A is an infinite sequence $(A_n, A_{n+1}, \ldots)$ with $A_i \in U_i$. For our purposes the main points are that that are κ_{n-1} stems of length n, and that each such stem lies in V_{κ_n}. Let h be a stem and ϕ a sentence of the forcing language: then we define $h \Vdash^* \phi$ to abbreviate "there is an appropriate top part A such that $h^\frown A \Vdash \phi$".

Lemma 2.2. *There exist an unbounded set $I \subseteq \mu$, a natural number n^*, and a function $x \mapsto h_x$ with domain $A_0 \in U^*$ such that for all $x \in A_0$: h_x is a stem of length n^*, and for all $\alpha \in I \cap x$ there is $\xi < \kappa$ such that*

$$h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^\alpha_\xi.$$

Proof. Let $j : V[E] \longrightarrow M$ be the ultrapower by U^*. Let G^* be generic for $j(\mathbb{P})$ over M, and work for the moment in the model $M[G^*]$. We note that $j(\kappa) > \mu$ and G^* adds no bounded subsets of $j(\kappa)$, in particular μ is regular and uncountable in $M[G^*]$.

For each $\alpha < \mu$, let $p_\alpha \in G^*$ decide the value of $\xi < j(\kappa)$ for which $j(\dot{d})_{\sup j``\mu} \restriction j(\alpha) = j(\dot{\sigma})^{j(\alpha)}_\xi$. The stem of p_α is a finite initial segment of the generic ω-sequence added by G^*, so there are just countably many possibilities for this stem. We may therefore find a stem h^* for $j(\mathbb{P})$, such that in $M[G^*]$ there exists an unbounded set $I^* \subseteq \mu$ with $\mathrm{stem}(p_\alpha) = h^*$ for all $\alpha \in I^*$.

Working in $V[E]$, define

$$I = \{\alpha < \mu \mid \exists \xi < j(\kappa)\ h^* \Vdash^* j(\dot{d})_{\sup j``\mu} \restriction j(\alpha) = j(\dot{\sigma})^{j(\alpha)}_\xi\}$$

Clearly $I^* \subseteq I$ and hence I is unbounded.[2]

Let h^* have length n^* and let $h^* = [x \mapsto h_x]_{U^*}$ where h_x is a stem of length n^* for all x. For every $\alpha \in I$ let

$$A_\alpha = \{x \mid \exists \xi < \kappa\ h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^\alpha_\xi\}.$$

Then $A_\alpha \in U^*$ for all $\alpha \in I$. Let $A_0 = \Delta_{\alpha \in I} A_\alpha$, that is $\{x \mid \forall \alpha \in I \cap x\ x \in A_\alpha\}$. Then $A_0 \in U^*$, and we may assume that the domain of $x \mapsto h_x$ is exactly A_0. Then I, n^*, $x \mapsto h_x$ and A_0 are as required. $\square$

[1] For our purposes in this section we could just choose U_n as any supercompactness measure on $P_\kappa \kappa_n$, but the more uniform choice is important in Section 4.3 and would also be useful if we were aiming at a fine analysis of the PCF structure of the model.

[2] In fact $V[E]$ and M agree on the power set of μ, and M and $M[G^*]$ agree to rank $j(\kappa)$, so $I^* \in V[E]$. We prefer the form of the argument we gave here since it also works in more general situations.

Lemma 2.3. *There are a stem $\bar{h}$ of length n^* and a stationary set $T \subseteq \mu$ such that for all $\gamma_1 < \gamma_2$ from T, $\bar{h} \Vdash^* \dot{d}_{\gamma_1} = \dot{d}_{\gamma_2} \restriction \gamma_1$.*

Proof. Let $i : V \to N$ witness that κ_{n^*+1} is μ-supercompact in V. We will construct a generic embedding $i : V[E] \to N[F]$ extending $i : V \to N$, defined in a generic extension $V[F]$ of $V[E]$.

In $V[E]$, we can factorise $i(\mathrm{Add}(\kappa, \rho))$ as $\mathbb{Q}_0 \times \mathbb{Q}_1$, where conditions in $\mathbb{Q}_0$ have supports contained in $\kappa \times i``\rho$ and conditions in $\mathbb{Q}_1$ have supports contained in $\kappa \times (i(\rho) \setminus i``\rho)$. Clearly $i \restriction \mathrm{Add}(\kappa, \rho)$ is an isomorphism between $\mathrm{Add}(\kappa, \rho)$ and $\mathbb{Q}_0$, so working over V we may view E as generic for $\mathbb{Q}_0$. Forcing over $V[E]$ with $\mathbb{Q}_1$ we may obtain a generic object F such that $V[E] \subseteq V[F]$, $i``E \subseteq F$ and i lifts in $V[F]$ to an embedding $i : V[E] \to N[F]$.[3]

Let $\gamma \in i(I) \setminus \sup(i``\mu)$. For each $\delta < \kappa$ and stem h of length n^*, we work in $V[F]$ to define

$$b_{\delta,h} = \{(\alpha, \xi) \in I \times \kappa \mid h \Vdash^*_{i(\mathbb{P})} i(\dot{\sigma})^\gamma_\delta \restriction i(\alpha) = i(\dot{\sigma})^{i(\alpha)}_\xi\}$$

Immediately from the definition, keeping in mind that $i(h) = h$ and $i \restriction \kappa + 1 = id$:

- $b_{\delta,h}$ is a partial function from I to κ, with $b_{\delta,h} \in V[F]$.
- If $b_{\delta,h}(\alpha) = \xi$, then working in $V[E]$ we may compute $b_{\delta,h} \restriction \alpha$ as follows: for $\alpha' \in \alpha \cap I$, $\alpha' \in \mathrm{dom}(b_{\delta,h})$ iff $h \Vdash^*_{\mathbb{P}} \dot{\sigma}^\alpha_\xi \restriction \alpha' = \dot{\sigma}^{\alpha'}_{\xi'}$ for some $\xi' < \kappa$, and $b_{\delta,h}(\alpha') = \xi'$ for the unique ξ' with this property. We note that this computation involved the stem h but not the ordinal δ.
- By the previous remark, for all $\alpha \in \mathrm{dom}(b_{\delta,h})$ we have $b_{\delta,h} \restriction \alpha \in V[E]$.

Recall that F was added by forcing over $V[E]$ with $\mathbb{Q}_1$. The poset $\mathbb{Q}_1 \times \mathbb{Q}_1$ has κ^+-cc in $V[E]$, so $\mathbb{Q}_1$ has the κ^+-approximation property. It follows that if $\mathrm{dom}(b_{\delta,h})$ is unbounded in I, then $b_{\delta,h} \in V[E]$.

In general whether or not $\mathrm{dom}(b_{\delta,h})$ is unbounded depends on the choice of F, and *a priori* the best we can do in $V[E]$ is to collect the possible values of $b_{\delta,h}$ with unbounded domains, and some information about those values with bounded domains.

Working in $V[E]$, to each pair (δ, h) we associate:

- The set $\mathcal{C}_{\delta,h}$ of possible values of $b_{\delta,h}$ with $\mathrm{dom}(b_{\delta,h})$ unbounded.
- The supremum $\gamma_{\delta,h}$ of the possible values of $\sup(\mathrm{dom}(b_{\delta,h}))$ with $\mathrm{dom}(b_{\delta,h})$ bounded.

Since $\mathbb{Q}_1$ is κ^+-cc, $|\mathcal{C}_{\delta,h}| \leq \kappa$ and $\gamma_{\delta,h} < \mu$.

Now let $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h}$. Since c is a possible value of $b_{\delta,h}$ it has the corresponding coherence property:

($\dagger_1$) If $\alpha \in \mathrm{dom}(c)$ with $c(\alpha) = \xi$, then for $\alpha' \in I \cap \alpha$, we have that $\alpha' \in \mathrm{dom}(c)$ with $c(\alpha') = \xi'$ if and only if $h \Vdash^* \dot{\sigma}^\alpha_\xi \restriction \alpha' = \dot{\sigma}^{\alpha'}_{\xi'}$.

In particular, if $c, c' \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h}$ and $\alpha \in \mathrm{dom}(c) \cap \mathrm{dom}(c')$ with $c(\alpha) = c'(\alpha)$, then $c \restriction \alpha = c' \restriction \alpha$. Since there are fewer than μ possibilities for h, we may choose $\bar{\alpha} < \mu$ such that:

- For all h, if c and c' are distinct elements of $\bigcup_\delta \mathcal{C}_{\delta,h}$ then there is no $\alpha \geq \bar{\alpha}$ such that $c(\alpha) = c'(\alpha)$.
- For all δ and h, it is forced by $\mathbb{Q}_1$ that if $\mathrm{dom}(b_{\delta,h})$ is bounded then $\mathrm{dom}(b_{\delta,h}) \subseteq \bar{\alpha}$.

[3] If we assume that i witnesses ρ-supercompactness, then the factorisation of $i(\mathrm{Add}(\kappa, \rho))$ happens in N and the construction is slightly simpler.

Claim 2.4. *Let $\alpha \in I \setminus \bar{\alpha}$. Then for U^*-many x there exists $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$ such that*

$$h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$$

Proof. For a fixed α, we will prove the statement which is i applied to the claim. Let $A' = \{x \in i(A_0) \mid \gamma, i(\alpha) \in x\}$, then by fineness of supercompactness measures $A' \in i(U^*)$. Suppose $x \in A'$. Applying i to the conclusion of Lemma 2.2, there are δ and ξ less than κ such that:

- $i(h)_x \Vdash^* i(\dot{d})_{\sup(x)} \restriction \gamma = i(\dot{\sigma})^{\gamma}_{\delta}$
- $i(h)_x \Vdash^* i(\dot{d})_{\sup(x)} \restriction i(\alpha) = i(\dot{\sigma})^{i(\alpha)}_{\xi}$.

It follows that

$$i(h)_x \Vdash^* i(\dot{d})_{\sup(x)} \restriction i(\alpha) = i(\dot{\sigma})^{\gamma}_{\delta} \restriction i(\alpha) = i(\dot{\sigma})^{i(\alpha)}_{\xi} = i(\dot{\sigma}^{\alpha}_{\xi})$$

Hence setting $h' = i(h)_x$, by definition we have that $b_{\delta,h'}(\alpha) = \xi$. Note that h' is below the critical point and so $h' = i(h')$. Since $\alpha \geq \bar{\alpha}$, $\mathrm{dom}(b_{\delta,h'})$ is unbounded. Let $c = b_{\delta,h'}$, then $c \in V[E]$ and $c \in \mathcal{C}_{\delta,h'}$.

Let $c' = i(c)$, so that $c' \in i(\mathcal{C})_{\delta,h'}$ and $c'(i(\alpha)) = \xi$. We just showed that in $N[F]$ there is a set $A' \in i(U^*)$ with the following property: for all $x \in A'$, there are $\delta < \kappa$ and $c' \in i(\mathcal{C})_{\delta,i(h)_x}$ such that $i(h)_x \Vdash^* i(\dot{d})_{\sup(x)} \restriction i(\alpha) = i(\dot{\sigma})^{i(\alpha)}_{c'(i(\alpha))}$.

By elementarity of $i : V[E] \to N[F]$, in $V[E]$ there is a measure one set $B_\alpha \in U^*$ with the following property: for all $x \in B_\alpha$, there are $\delta < \kappa$ and $c \in \mathcal{C}_{\delta,h_x}$ such that $h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$.

□

Let A_1 be the set of $x \in P_\kappa(\mu)$ with the following properties, each of which holds on a U^*-large set:

1. $x \in A_0$.
2. For all $\alpha \in I \cap x$, $x \in B_\alpha$.
3. $I \cap x$ is unbounded in x.
4. $\bar{\alpha} \in x$.

Fix $x \in A_1$. It follows from Claim 2.4 that for all $\alpha \in I \cap x \setminus \bar{\alpha}$, there is $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$, such that $h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$. The key point is that there is a unique such c which works for every α.

To see this let $\alpha < \beta$ both lie in $I \cap x \setminus \bar{\alpha}$, and let $c, d \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$ be such that $h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$ and $h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \beta = \dot{\sigma}^{\beta}_{d(\beta)}$. Then $h_x \Vdash^* \dot{\sigma}^{\beta}_{d(\beta)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$, so $d(\alpha) = c(\alpha)$ by property $(\dagger_1)$ of d. Since $\alpha \geq \bar{\alpha}$, $c = d$.

We have shown that there is a unique $c \in \bigcup_\delta \mathcal{C}_{\delta,h_x}$ such that $h_x \Vdash^* \dot{d}_{\sup(x)} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)}$ for all $\alpha \in I \cap x \setminus \bar{\alpha}$. As we already argued, if $\alpha < \alpha'$ and both lie in $I \cap x \setminus \bar{\alpha}$, then $h_x \Vdash^* \dot{\sigma}^{\alpha}_{c(\alpha)} = \dot{\sigma}^{\alpha'}_{c(\alpha')} \restriction \alpha$. Since $|x| < \kappa$, $I \cap x$ is unbounded in x and the measures appearing in the definition of $\mathbb{P}$ are κ-complete,

$$(\dagger_2) \qquad h_x \Vdash^* \dot{d}_{\sup(x)} = \bigcup_{\alpha \in I \cap x \setminus \bar{\alpha}} \dot{\sigma}^{\alpha}_{c(\alpha)}$$

Let $S = \{\sup(x) : x \in A_1\}$, and note that S is stationary. For each $\beta \in S$ we may choose $x_\beta \in A_1$ such that $\sup(x_\beta) = \beta$, and then let $c_\beta \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_{x_\beta}}$ be the unique witness to $(\dagger_2)$ for x_β. Appealing to Fodor's lemma, we may find a stationary set $T \subseteq S$ and a fixed $\bar{h}$ and $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,\bar{h}}$ such that $c_\beta = c$ and $h_{x_\beta} = \bar{h}$ for all $\beta \in T$.

Now we can finish the proof of the lemma by collecting some measure one sets. Let $\gamma_1 < \gamma_2$ from T, let $x_1 = x_{\gamma_1}$ and $x_2 = x_{\gamma_2}$. If $\alpha < \alpha'$ with $\alpha \in I \cap x_1 \setminus \bar{\alpha}$ and $\alpha' \in I \cap x_2 \setminus \bar{\alpha}$, then by the coherence property from ($\dagger_1$)

$$\bar{h} \Vdash^* \dot{\sigma}^{\alpha'}_{c(\alpha')} \restriction \alpha = \dot{\sigma}^{\alpha}_{c(\alpha)} \tag{$\dagger_3$}$$

Collect the measure one sets witnessing ($\dagger_2$) for x_1 and x_2, and the measure one sets witnessing all instances of ($\dagger_3$) for relevant α and α'. Intersecting this family of fewer than κ many sets, we see that

$$\bar{h} \Vdash^* \dot{d}_{\gamma_2} \restriction \gamma_1 = \dot{d}_{\gamma_1}$$

□

The remainder of the argument follows Neeman's argument very closely.

Lemma 2.5. *Suppose that h is a stem extending $\bar{h}$ and T_h is a stationary subset of T such that for all $\gamma_1 < \gamma_2$ from T_h, $h \Vdash^* \dot{d}_{\gamma_1} = \dot{d}_{\gamma_2} \cap \gamma_1$. Then there are $\rho_h < \mu$ and measure one sets A^h_γ for $\gamma \in T_h \setminus \rho$ such that for all $\beta < \gamma$ from $T_h \setminus \rho$ and all $x \in A^h_\beta \cap A^h_\gamma$, $h \frown x \Vdash^* \dot{d}_\beta = \dot{d}_\gamma \cap \beta$.*

Proof. The proof is exactly parallel to the proof of [7, Lemma 3.5]. □

Lemma 2.6. *There are $\rho < \mu$ and conditions p_γ for $\gamma \in T \setminus \rho$ with stem $\bar{h}$ such that for all $\beta < \gamma$ from $T \setminus \rho$, $p_\beta \wedge p_\gamma \Vdash \dot{d}_\beta = \dot{d}_\gamma \cap \beta$.*

Proof. The proof is exactly parallel to the proof of [7, Lemma 3.14]. □

To finish the proof, we need a minor variation on a well-known fact about λ-cc forcing.

Lemma 2.7. *Let λ be a regular uncountable cardinal, let $\mathbb{Q}$ be λ-cc and let U be stationary in λ. Then for any sequence $\langle q_i : i \in U \rangle$ of conditions in $\mathbb{Q}$, there is $i \in U$ such that q_i forces that $\{j \in U : q_j \in G\}$ is stationary in $V[G]$.*

Proof. Suppose not. By λ-cc, for every $i \in U$ there is a club set C_i such that q_i forces $\{j \in U : q_j \in G\}$ is disjoint from C_i. If C is the diagonal intersection of the club sets C_i, then $q_i \Vdash q_j \notin G$ for $i, j \in C \cap U$ with $i < j$. So $\{q_i : i \in C \cap U\}$ is an antichain, contradiction. □

To finish the proof, we apply this lemma to the sequence $\langle p_\gamma : \gamma \in T \setminus \rho \rangle$.

□

3. The two-cardinal ITP

We will now show that the two-cardinal tree property holds in Neeman's model. More precisely:

Theorem 3.1. *If G is $\mathbb{P}$-generic, then ITP_μ holds in $V[E][G]$.*

Proof. Fix a regular $\lambda > \mu$. We start by collecting some information:

- By a classical theorem of Solovay, $\lambda^{<\kappa_n} = \lambda$ for all n. So $\lambda^{\kappa_n} = \lambda$ for all n, and hence $\lambda^\nu = \lambda^{\sum_n \kappa_n} = \prod_n \lambda^{\kappa_n} = \lambda^\omega = \lambda$. So $|P_\mu \lambda| = \lambda$.

- If G is $\mathbb{P}$-generic over $V[E]$, then the μ-chain condition of $\mathbb{P}$ implies that stationary subsets of $(\mathcal{P}_\mu(\lambda))^{V[E]}$ remain stationary in $(\mathcal{P}_\mu(\lambda))^{V[E][G]}$. So it is enough to show $ITP(\mu,\lambda)$ for lists indexed by $(\mathcal{P}_\mu(\lambda))^{V[E]}$.
- Let $U^* \in V[E]$ be a normal measure on $\mathcal{P}_\kappa(\lambda)$ and let $j : V[E] \to M$ be the induced embedding. Since $|P_\mu\lambda| = \lambda$, we can use some coding to form diagonal intersections of $P_\mu\lambda$-indexed sequences of elements of U^*.

  To be explicit: let $e : \mathcal{P}_\mu(\lambda) \to \lambda$ be a bijection, let $\langle A_z : z \in P_\mu\lambda\rangle$ with $A_z \in U^*$, and define $\Delta_z A_z = \{x \in P_\kappa\lambda : \forall z\ (e(z) \in x \implies x \in A_z)\}$. Clearly $\Delta_z A_z \in U^*$. For use later we note that $j(e) \restriction j``P_\mu(\lambda)$ sets up a bijection between $j``P_\mu(\lambda)$ and $j``\lambda$.

Let $\dot{d} = \langle \dot{d}_z \mid z \in \mathcal{P}_\mu(\lambda)\rangle$ be a $\mathbb{P}$-name for a $\mathcal{P}_\mu(\lambda)$-thin list. Suppose that for each $z \in \mathcal{P}_\mu(\lambda)$, the z-th level $\{\dot{d}_y \restriction z \mid z \subseteq y \in \mathcal{P}_\mu(\lambda)\}$ is forced to be enumerated by $\{\dot{\sigma}^z_\xi \mid \xi < \kappa\}$.

Since $|P_\mu(\lambda)| = \lambda$, $j``(P_\mu(\lambda)) \in M$ and so $z^* \in M$ where $z^* = \bigcup j"\mathcal{P}_\mu(\lambda)$. Let $h(x) = \{z \in P_\mu(\lambda) : e(z) \in x\}$, then $[h]_{U^*} = j(h)(j``\lambda) = j``P_\mu(\lambda)$. We let $g(x) = \bigcup h(x)$, so that $g : P_\kappa(\lambda) \to P_\mu(\lambda)$ and $[g]_{U^*} = \bigcup j``P_\mu(\lambda) = z^*$.

We remark that here z^* and g are the analogues of $\sup(j"\mu)$ and $x \mapsto \sup(x)$, that we used in the one cardinal version.

Lemma 3.2. *If $A \in U^*$, then $g``A$ is stationary in $\mathcal{P}_\mu(\lambda)$*

Proof. Let $C \subseteq \mathcal{P}_\mu(\lambda)$ be club. Then $j``C$ is an upwards-directed subset of $j(C)$ and $|j``C| = \lambda < j(\kappa) < j(\mu)$, so that $z^* = \bigcup j``C \in j(C)$. So $\{x : g(x) \in C\} \in U^*$, and there is $x \in A$ with $g(x) \in C$. □

Lemma 3.3. *There exist a cofinal set $I \subseteq \mathcal{P}_\mu(\lambda)$, a natural number n^*, and a function $x \mapsto h_x$ with domain $A_0 \in U^*$ such that for all $x \in A_0$: h_x is a stem of length n^*, and for all $z \in I$ with $e(z) \in x$ there is $\xi < \kappa$ such that*

$$h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_\xi$$

Proof. Let G^* be $j(\mathbb{P})$-generic and work in $M[G^*]$. For all $z \in \mathcal{P}_\mu(\lambda)$ $j(z) \subseteq z^*$, so there are $\xi < j(\kappa)$ and $p_z \in G^*$ such that

$$p_z \Vdash j(\dot{d})_{z^*} \restriction j(z) = j(\dot{\sigma})^{j(z)}_\xi.$$

Since μ and λ remain regular in $M[G^*]$, there exist a stationary (hence cofinal) $I^* \subseteq \mathcal{P}_\mu(\lambda)$ in $M[G^*]$, a natural number n^* and a stem h^* of length n^* with the following property: for all $z \in I^*$ there is some $\xi < j(\kappa)$ such that

$$h^* \Vdash^* j(\dot{d})_{z^*} \restriction j(z) = j(\dot{\sigma})^{j(z)}_\xi$$

Working in $V[E]$, define

$$I = \{z \in P_\mu(\lambda) \mid \exists \xi < j(\kappa)\ h^* \Vdash^* j(\dot{d})_{z^*} \restriction j(z) = j(\dot{\sigma})^{j(z)}_\xi\}$$

Clearly $I^* \subseteq I$ and hence I is cofinal.

Let $h^* = [x \mapsto h_x]_{U^*}$, so that for all $z \in I$ there is $A_z \in U^*$ with the following property: for all $x \in A_z$ there exists $\xi < \kappa$ such that

$$h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_\xi.$$

Take $A_0 = \Delta_{z \in I} A_z$, then everything is as required.

□

Lemma 3.4. *There are a stem $\bar{h}$ of length n^* and a stationary set $T \subseteq \mathcal{P}_\mu(\lambda)$ such that for all $z_1 \subseteq z_2$ both from T, $\bar{h} \Vdash^* \dot{d}_{z_1} = \dot{d}_{z_2} \restriction z_1$.*

Proof. As in the proof of Lemma 2.3, let $i : V \to N$ witness that κ_{n^*+1} is λ-supercompact in V and construct a generic embedding $i : V[E] \to N[F]$ extending $i : V \to N$ defined in a generic extension $V[F]$ of $V[E]$. Let $u^* \in i(I)$ be such that $\bigcup i``\mathcal{P}_\mu(\lambda) \subseteq u^*$.

For $\delta < \kappa$ and h of length n^*, define

$$b_{\delta,h} = \{(z,\xi) \in I \times \kappa \mid h \Vdash^*_{i(\mathbb{P})} i(\dot{\sigma})^{u^*}_\delta \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi\}.$$

- $b_{\delta,h}$ is a partial function from I to κ, with $b_{\delta,h} \in V[F]$.
- If $b_{\delta,h}(z) = \xi$, then working in $V[E]$ we may compute $b_{\delta,h} \restriction P(z)$ as follows: for $z' \subseteq z$ with $z' \in I$, $z' \in \mathrm{dom}(b_{\delta,h})$ iff $h \Vdash^*_{\mathbb{P}} \dot{\sigma}^z_\xi \restriction z' = \dot{\sigma}^{z'}_{\xi'}$ for some $\xi' < \kappa$, and $b_{\delta,h}(z') = \xi'$ for the unique ξ' with this property. We note that this computation involved the stem h but not the ordinal δ.
- By the previous remark, for all $z \in \mathrm{dom}(b_{\delta,h})$ we have $b_{\delta,h} \restriction P(z) \in V[E]$.

Claim 3.5. *For each pair (δ, h), if $\mathrm{dom}(b_{\delta,h})$ is cofinal in $\mathcal{P}_\mu(\lambda)$, then $b_{\delta,h} \in V[E]$.*

Proof. Let $d \subseteq \mathrm{dom}(b_{\delta,h})$ with $d \in V[E]$ and $|d| < \mu$. As the domain is cofinal, there is z in the domain with $\bigcup d \subseteq z$, and so $b_{\delta,h} \restriction d \in V[E]$. Since $V[F]$ is an extension of $V[E]$ with the μ-approximation property, $b_{\delta,h} \in V[E]$. □

As in the proof of Lemma 2.3, let $\mathcal{C}_{\delta,h}$ be the set of possible values for $b_{\delta,h}$ with $\mathrm{dom}(b_{\delta,h})$ cofinal, where $|\mathcal{C}_{\delta,h}| \leq \kappa$. As before, the elements of $\bigcup_\delta \mathcal{C}_{\delta,h}$ enjoy the coherence properties of $b_{\delta,h}$.

Arguing exactly as before, we find $\bar{z} \in P_\mu(\lambda)$ such that:

- For all h, if c and c' are distinct elements of $\bigcup_\delta \mathcal{C}_{\delta,h}$ then there is no $z \supseteq \bar{z}$ such that $c(z) = c'(z)$.
- For all δ and h, it is forced by $\mathbb{Q}_1$ that if $\mathrm{dom}(b_{\delta,h})$ is not cofinal then $\mathrm{dom}(b_{\delta,h})$ contains no z with $z \supseteq \bar{z}$.

Claim 3.6. *Let $z \in I$ with $\bar{z} \subseteq z$. Then for U^*-many $x \in A_z$, $e(z) \in x$ and there exists $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$ such that*

$$h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_{c(z)}$$

Proof. Let $A' = \{x \in i(A_0) \mid i(e)(u^*), i(e(z)) \in x\}$. Suppose $x \in A'$. Then by applying elementarity to the conclusion of lemma 3.3, there are δ and ξ less than κ such that:

- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction u^* = i(\dot{\sigma})^{u^*}_\delta$.
- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi$

Combining these it follows that

$$i(h)_x \Vdash^* i(\dot{\sigma})^{u^*}_\delta \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi.$$

Let $h' := i(h)_x$, so that by definition $b_{\delta,h'}(z) = \xi$. Since $\bar{z} \subseteq z$, $\mathrm{dom}(b_{\delta,h'})$ is unbounded. Let $c = b_{\delta,h'}$, then $c \in V[E]$ and $c \in \mathcal{C}_{\delta,h'}$.

Let $c' = i(c)$, so that $c' \in i(\mathcal{C})_{\delta,h'}$ and $c'(i(z)) = \xi$. We just showed that in $N[F]$ there is a set $A' \in i(U^*)$ with the following property: for all $x \in A'$, there are $\delta < \kappa$ and $c' \in i(\mathcal{C})_{\delta,i(h)_x}$ such that $c'(i(z)) = \xi$ and $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi$.

By elementarity there is a measure one set $B_z \in U^*$ with the following property: for all $x \in B_z$, there are $\delta < \kappa$ and $c \in C_{\delta,h_x}$ such that $h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_{c(z)}$. □

Let A_1 be the set of x with the following properties, each of which holds on a U^*-large set:

(1) $x \in A_0$.
(2) For all $z \in I$ such that $e(z) \in x$, $x \in B_z$.
(3) The set $\{z \in I : e(z) \in x\}$ is cofinal in $\{z : e(z) \in x\}$.
(4) $e(\bar{z}) \in x$.

Fix $x \in A_1$. It follows from Claim 3.6 that for all $z \in I$ such that $e(z) \in x$ and $\bar{z} \subseteq z$, there is $c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$ such that $h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_{c(z)}$. We claim that there is a unique such c which works uniformly for every relevant z.

To see this let $z_0, z_1 \in I$ be such that $\bar{z} \subseteq z_0 \cap z_1$ and $e(z_0), e(z_1) \in x$. Find $z' \in I$ such that $e(z') \in x$ and $z_0 \cup z_1 \subseteq z'$. Let $c_0, c_1, c' \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x}$ be such that:

- $h_x \Vdash^* \dot{d}_{g(x)} \restriction z_0 = \dot{\sigma}^{z_0}_{c_0(z_0)}$.
- $h_x \Vdash^* \dot{d}_{g(x)} \restriction z_1 = \dot{\sigma}^{z_1}_{c_1(z_1)}$.
- $h_x \Vdash^* \dot{d}_{g(x)} \restriction z' = \dot{\sigma}^{z'}_{c'(z')}$.

By the coherence properties of the various branches, $z_0, z_1 \in \mathrm{dom}(c')$ and $c'(z_0) = c_0(z_0)$, $c'(z_1) = c_1(z_1)$. Since $\bar{z} \subseteq z_0 \cap z_1$, $c' = c_0$ and $c' = c_1$, so $c_0 = c_1$ as required.

We have shown that there is a unique $c \in \bigcup_\delta \mathcal{C}_{\delta,h_x}$ such that $h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_{c(z)}$ for all $z \in I$ with $e(z) \in x$. Since $|x| < \kappa$ and $\{z \in I : e(z) \in x\}$ is cofinal in $\{z : e(z) \in x\}$, we see that

$$(\dagger_4) \qquad h_x \Vdash^* \dot{d}_{g(x)} = \bigcup \{\dot{\sigma}^z_{c(z)} \mid z \in I, \bar{z} \subseteq z, e(z) \in x\}.$$

Let $S = \{g(x) : x \in A_1\}$, and note that S is stationary in $P_\mu(\lambda)$ by Lemma 3.2. For each $w \in S$ we choose $x_w \in A_1$ such that $g(x_w) = w$, and then c_w witnessing $(\dagger_4)$. for x_w. By Fodor's Lemma we find a stationary set $T \subseteq S$, a stem $\bar{h}$ and a function $\bar{c}$ such that $c_w = \bar{c}$ and $h_{x_w} = \bar{h}$ for all $w \in T$. If $z_1 \subseteq z_2$ with $z_1, z_2 \in T$ then exactly as in the proof of Lemma 2.3 we may intersect appropriate measure one sets to see that

$$\bar{h} \Vdash^* \dot{d}_{z_1} = \dot{d}_{z_2} \restriction z_1$$

So the set T is as required. □

Let $\bar{h}$ be the stem of length n and T be the stationary set satisfying the conclusion of lemma 3.4. We finish the argument as in the proof of Theorem 2.1.

Lemma 3.7. *Suppose that h is a stem extending $\bar{h}$ and T_h is a stationary subset of T such that for all $z_1 \subseteq z_2$ from T_h, $h \Vdash \dot{d}_{z_1} = \dot{d}_{z_2} \cap z_1$. Then there are $z_h \in \mathcal{P}_\mu(\lambda)$ and measure one sets A^h_z for $z \in T_h \cap \{z \mid z_h \subseteq z\}$ such that for all $z_h \subseteq y \subseteq z$ with $y, z \in T_h$ and all $x \in A^h_y \cap A^h_z$, $h \frown x \Vdash^* \dot{d}_y = \dot{d}_z \cap y$.*

Lemma 3.8. *There are $z^* \in \mathcal{P}_\mu(\lambda)$ and conditions $\langle p_z \mid z \in T \cap \{z \mid z^* \subseteq z\}\rangle$ with stem $\bar{h}$, such that if $z^* \subseteq y \subseteq z$ with $y, z \in T$, then $p_y \wedge p_z \Vdash \dot{d}_y = \dot{d}_z \cap y$.*

This finishes the proof, since by a lemma analogous to Lemma 2.7 there is a condition which forces that the set $\{z \mid p_z \in G\}$ is stationary in $\mathcal{P}_\mu(\lambda)$. □

4. The two-cardinal ITP at a small cardinal

In this section we use a different model for the tree property at the successor of a singular cardinal where SCH fails, namely Sinapova's model [8] where $\aleph_{\omega^2}$ is singular strong limit, $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+2}$ and $\aleph_{\omega^2+1}$ has the tree property. We will show that in a suitable version of this model, $ITP(\aleph_{\omega^2+1}, \lambda)$ holds for all regular $\lambda \geq \aleph_{\omega^2+1}$.

The initial hypothesis is the same as in the preceding sections, namely we have an increasing ω-sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals and we let $\kappa = \kappa_0$. By doing some preparatory forcing we may assume in addition that GCH holds above κ, and κ is indestructible under κ-directed closed forcing.

Let $\nu = \sup_n \kappa_n$, $\mu = \nu^+$, $\rho = \nu^{++}$. Our intention is that in the final model $\kappa = \aleph_{\omega^2}$, $\mu = \aleph_{\omega^2+1}$, $\rho = \aleph_{\omega^2+2} = 2^{\aleph_\omega}$.

We force over V with a full support iteration $\mathbb{C}$ of length ω, forcing at stage n with $Coll(\kappa_n, < \kappa_{n+1})$. Let H be $\mathbb{C}$-generic, so that in $V[H]$ we have $\kappa_n = \kappa^{+n}$ for all $n < \omega$. We then force over $V[H]$ with $\mathbb{A} = Add(\kappa, \rho)^V = Add(\kappa, \rho)^{V[H]}$, obtaining a generic extension $V[H][E]$. Since $H \times E$ is generic for κ-directed closed forcing, κ is still indestructibly supercompact in $V[H][E]$. In $V[H][E]$ we have $\kappa_n = \kappa^{+n}$, $\nu = \kappa^{+\omega}$, $\mu = \kappa^{+\omega+1}$, $\rho = \kappa^{+\omega+2}$, $2^\kappa = 2^\mu = \rho$, and $\sigma^{<\kappa} = \sigma$ for all regular $\sigma > \kappa$.

Next we want to force with a diagonal style supercompact Prikry forcing with interleaved collapses to make $\kappa = \aleph_{\omega^2}$. However, we have to be very careful in how we select the normal measures with which to define this forcing. The reason is that when proving $ITP(\mu, \lambda)$, at the stage when we fix the length of the stem, we need a λ-supercompact elementary embedding j with critical point κ, so that $j(\mathbb{P})$ preserves μ and λ. This was automatic when the Prikry forcing had no interleaved collapses. But now, we need μ and λ to be among the (few) cardinals below $j(\kappa)$ that are preserved by $j(\mathbb{P})$. In the next subsection, we will prove that such measures exist, uniformly for all λ.

4.1. **Measures and filters.**

Using techniques of Gitik and Sharon [3], we will construct in $V[H][E]$ sequences of supercompactness measures $\langle U_n : n < \omega \rangle$ and filters $\langle F_n : n < \omega \rangle$ such that:

- U_n is a supercompactness measure on $P_\kappa \kappa_n$.
- If $j_n : V[H][E] \to M_n = Ult(V[H][E])$ is the ultrapower map, then F_n is $Coll(\kappa^{+\omega+5}, < j_n(\kappa))^{M_n}$-generic over M_n.
- For unboundedly many regular $\lambda > \mu$, there is a $Coll(\mu^{++}, < \lambda)^{V[H]}$-name $\dot{U}_\lambda$ such that $\dot{U}_\lambda$ is forced to be a supercompactness measure on $P_\kappa(\lambda)$ whose projection to each $P_\kappa(\kappa_n)$ is U_n.

The first two bullet points are the hypotheses needed to build the forcing poset of [8], the third one will be used to argue for ITP in the generic extension.

To construct the measures U_n and filters F_n, suppose towards contradiction that for all possible choices of $\langle U_n : n < \omega \rangle$ and $\langle F_n : n < \omega \rangle$ satisfying the first two bullet points there is only a bounded set of λ satisfying the third bullet point. Choose λ so large that the third bullet point fails for all choices of U_n and F_n. Let K be $Coll(\mu^{++}, < \lambda)^{V[H]}$-generic over $V[H][E]$. By the indestructibility of κ in $V[H][E]$, let

$$j : V[H][K][E] \to M[H^*][K^*][E^*]$$

be the ultrapower map formed from a supercompactness measure on $P_\kappa(\lambda)$ in $V[H][K][E]$. Let $\bar{\jmath} : V \to M$ be the restriction to V. Let

$$j_\mu : V[H][K][E] \to M_\mu[H^*_\mu][K^*_\mu][E^*_\mu]$$

be the μ-supercompactness embedding derived from j, that is to say the ultrapower of $V[H][K][E]$ by the supercompactness measure $\{A : j``\mu \in j(A)\}$, and let $\bar{\jmath_\mu} : V \to M_\mu$ be its restriction to V. Let

$$k : M_\mu[H^*_\mu][K^*_\mu][E^*_\mu] \to M[H^*][K^*][E^*]$$

be the usual map given by $k : [f] \mapsto j(f)(j``\mu)$, so that $j = k \circ j_\mu$. As usual $\mu + 1 \subseteq \mathrm{ran}(k)$, so that in particular $\mathrm{crit}(k) > \mu$. Let $\bar{k} = k \restriction M_\mu$.

Since j_μ is an ultrapower map,

$$M_\mu[H^*_\mu][K^*_\mu][E^*_\mu] = \{j_\mu(f)(j_\mu``\mu) : f \in V[H][E][K], \mathrm{dom}(f) = P_\kappa(\mu)\}.$$

M_μ is the class of elements of the form $j_\mu(f)(j``\mu)$ where $f \in V[H][E][K]$ and $f : P_\kappa(\mu) \to V$. Similarly $M[H^*][K^*][E^*] = \{j(f)(j``\lambda) : f \in V[H][E][K], \mathrm{dom}(f) = P_\kappa(\lambda)\}$ and M is the class of elements of the form $j(f)(j``\lambda)$ where $f \in V[H][E][K]$ and $f : P_\kappa(\lambda) \to V$. It follows that $\bar{k} : M_\mu \to M$ is elementary and $\bar{\jmath} = \bar{k} \circ \bar{\jmath}_\mu$.

We will make small changes to E^* and E^*_μ to obtain new generic objects for $\bar{\jmath}(Add(\kappa, \rho))$ and $\bar{\jmath}_\mu(Add(\kappa, \rho))$. The goal is to obtain new lifts of $\bar{\jmath}$ and $\bar{\jmath}_\mu$ onto $V[H][K][E]$, j' and j'_μ, arranging that j'_μ is derived from j' and every ordinal below $\bar{\jmath}_\mu(\kappa)$ is of the form $j'_\mu(h)(\kappa)$ for some $h : \kappa \to \kappa$ in $V[H][K][E]$. Note that $E^*_\mu \subseteq M_\mu$ and similarly, $E^* \subseteq M$.

Since $2^\mu = 2^\kappa = \rho$ in $V[H][K][E]$, we may enumerate the elements of $\bar{\jmath}_\mu(\kappa)$ by $\langle u_\alpha \mid \alpha < \rho\rangle$. Define $F_\mu \subseteq \bar{\jmath}_\mu(Add(\kappa, \rho))$ to be the set of conditions p such that:

(1) $p \restriction \mathrm{dom}(p) \setminus (j_\mu"\rho \times \{\kappa\}) \in E^*_\mu$
(2) For all $\alpha < \rho$, if $(j_\mu(\alpha), \kappa) \in \mathrm{dom}(p)$ then $p(j_\mu(\alpha), \kappa) = u_\alpha$.

Intuitively F_μ is obtained by altering each condition in E^*_μ on the intersection of its domain with $j_\mu"\rho \times \{\kappa\}$.

Routine calculations show that (working in the model $V[H][E][K]$):

- For every $p \in E^*_\mu$, $|p \cap (\bar{\jmath}_\mu"\rho \times \{\kappa\})| \leq \mu$.
- Since $M_\mu[H^*_\mu][K^*_\mu][E^*_\mu]$ is closed under μ-sequences and has the same $< \bar{\jmath}_\mu(\kappa)$-sequences as M_μ, $F_\mu \subseteq \bar{\jmath}_\mu(Add(\kappa, \mu^+))$.
- Since $\bar{\jmath}_\mu(\kappa)$ is inaccessible in $M_\mu[H^*_\mu][K^*_\mu]$, $\bar{\jmath}_\mu(Add(\kappa, \rho))$ is $\bar{\jmath}_\mu(\kappa)$-closed in this model, and $\mu < \bar{\jmath}_\mu(\kappa)$, F_μ is still generic over $M_\mu[H^*_\mu][K^*_\mu]$.

Next we define F by making a small change to E^*. Let $p \in \bar{\jmath}(Add(\kappa, \rho))$ be such that:

- $\mathrm{dom}(p) = \bar{\jmath}``\rho \times \{\kappa\}$,
- for each $\alpha < \rho$, $p(\bar{\jmath}(\alpha), \kappa) = k(u_\alpha)$.

Now let F be the set of $q \in \bar{\jmath}(Add(\kappa, \rho))$ such that $q \restriction (\mathrm{dom}(q) \setminus \mathrm{dom}(p)) \in E^*$ and $q \restriction \mathrm{dom}(p) \subseteq p$. Arguing in the same way as we just did for F_μ, $F \subseteq \bar{\jmath}(Add(\kappa, \rho))$ and F is generic over $M[H^*][K^*]$.

Since $\mathrm{dom}(\bar{\jmath}_\mu(q)) \subseteq \bar{\jmath}_\mu``\rho \times \kappa$ for each $q \in E$, we have that $\bar{\jmath}_\mu``E \subseteq F_\mu$, and similarly $\bar{\jmath}``E \subseteq F$. We claim that also $\bar{k}``F_\mu \subseteq F$, because:

- $\bar{k}``E^*_\mu \subseteq E^*$.
- $\mathrm{crit}(\bar{k}) > \mu$.
- Each condition q in F_μ is obtained by taking a condition $q_0 \in E^*_\mu$, and replacing the value $q_0(\bar{\jmath}_\mu(\alpha), \kappa)$ by u_α.

- $(\bar{\jmath}_\mu(\alpha), \kappa) \in \mathrm{dom}(q) \iff (\bar{\jmath}(\alpha), \kappa) \in \mathrm{dom}(k(q))$, and there are at most μ such α's.
- Each condition r in F is obtained by taking a condition $r \in E^*_\mu$, and replacing the value $r(\bar{\jmath}(\alpha), \kappa)$ by $k(u_\alpha)$.

Now we extend $\bar{\jmath} : V \to M$ to $j' : V[H][K][E] \to M[H^*][K^*][F]$, $\bar{\jmath}_\mu : V \to M$ to $j'_\mu : V[H][K][E] \to M[H^*_\mu][K^*_\mu][F_\mu]$, and $\bar{k} : M_\mu \to M$ to $k' : M_\mu[H^*_\mu][K^*_\mu][F_\mu] \to M[H^*][K^*][F]$. Then $j' = k' \circ j'_\mu$.

Claim 4.1. *In $V[H][K][E]$, j'_μ is the ultrapower by a measure on $P_\kappa(\mu)$.*

Proof. Let $a \in M[H^*_\mu][K^*_\mu][F_\mu]$, so that a is the realization of some term $\dot\tau$ in M by $H^*_\mu * K^*_\mu * F_\mu$. Now $\dot\tau$ is of the form $j_\mu(f)(j_\mu``\mu)$ where $f \in V[H][K][E]$, and $f$ is a function from $P_\kappa(\mu)$ to terms for the forcing $\mathbb{C} * \mathbb{A} * Coll(\mu^{++}, < \lambda)$. If $f^*$ is the function which maps $x \in P_\kappa(\mu)$ to the realization of $f(x)$ by $H * K * E$, then $a = j'_\mu(f^*)(j_\mu``\mu)$. $\square$

Similarly j' is the ultrapower map by an ultrafilter U_λ on $P_\kappa(\lambda)$, j'_μ is the ultrapower map by the projected ultrafilter U_μ on $P_\kappa(\mu)$ and k' is the standard factor map. Since K is generic for μ^{++}-closed forcing, in fact $U_\mu \in V[H][E]$ and j'_μ is a lift of the ultrapower map $j^*_\mu : V[H][E] \to M_\mu[H^*_\mu][F_\mu]$ computed from U_μ in $V[H][E]$.

Now we can compute suitable ultrafilters U_n and filters F_n. Let U_n be the projection of U_μ to $P_\kappa(\kappa_n)$, and $j_n : V[H][E] \to M_n$ the associated ultrafilter map.

Claim 4.2. *There is a $Coll(\kappa^{+\omega+5}, < j_n(\kappa))^{M_n}$-generic filter F_n over M_n.*

Proof. Let $k_n : M_n \to M_\mu[H^*_\mu][F_\mu]$ be the usual factor map, and note that $\mathrm{ran}(k_0) = \{j_\mu(h)(\kappa) : h : \kappa \to \kappa, h \in V[H][E]\}$ and $\mathrm{ran}(k_n) \subseteq \mathrm{ran}(k_{n+1})$. By the construction of F_μ, if $\alpha < \rho$ and h_α is the α^{th} Cohen function added by E, then

$$k_0([h_\alpha]_{U_0}) = j_\mu(h_\alpha)(\kappa) = u_\alpha,$$

so that $j_\mu(\kappa) + 1 \subseteq \mathrm{ran}(k_0) \subseteq \mathrm{ran}(k_n)$ for all n. It follows that $j_n(\kappa) = j_\mu(\kappa)$ and $\mathrm{crit}(k_n) > j_\mu(\kappa)$ for all n.

To finish, let $\mathbb{Q} = Coll(\kappa^{+\omega+5}, < j_\mu(\kappa))^{M_\mu[H^*_\mu][F_\mu]}$. From the point of view of $V[H][E]$, the poset $\mathbb{Q}$ is ρ-closed and the set of its antichains which lie in $M_\mu[H^*_\mu][F_\mu]$ has cardinality ρ, so we may build a generic object F^*. Pulling back along k_n we obtain an M_n-generic filter F_n.

$\square$

The construction of the measures U_n and filters F_n contradicts our choice of λ.

4.2. **The Prikry forcing.** After the work of the previous section, we have in $V[H][E]$ measures U_n and filters F_n such that:

- U_n is a supercompactness measure on $P_\kappa(\kappa_n)$.
- If $j_n : V[H][E] \to M_n = Ult(V[H][E])$ is the ultrapower map, then F_n is $Coll(\kappa^{+\omega+5}, < j_n(\kappa))^{M_n}$-generic over M_n.
- For unboundedly many regular $\lambda > \mu$, there is a $Coll(\mu^{++}, < \lambda)^{V[H]}$-name $\dot U_\lambda$ such that $\dot U_\lambda$ is forced to be a supercompactness measure on $P_\kappa(\lambda)$ whose projection to each $P_\kappa(\kappa_n)$ is U_n.

The forcing $\mathbb{P}$ is a diagonal supercompact Prikry forcing with interleaved collapsing, defined in $V[H][E]$ using the U_n's as the supercompactness measures and the F_n's as "guiding generics". We will suppress many technical details, referring the reader to [8].

Each U_n concentrates on the set of $x \in P_\kappa(\kappa_n)$ such that $x \cap \kappa$ is a cardinal reflecting the properties of κ, and we denote $x \cap \kappa$ by κ_x. A condition p has a *stem* s and a *top part* (A, C) where:

- s has the form $\langle d, x_0, c_0, \dots x_{n-1}, c_{n-1} \rangle$.
- $\langle x_0, \dots x_{n-1} \rangle$ is a stem in supercompact Prikry forcing, that is:
  - $x_i \in P_\kappa(\kappa_i)$.
  - $x_i \subseteq x_{i+1}$
  - $\mathrm{ot}(x_i) < \kappa_{x_{i+1}}$.
- $d \in Coll(\omega, < \kappa_{x_0})$.
- $c_i \in Coll(\kappa_{x_i}^{+\omega+5}, < \kappa_{x_{i+1}})$ for $i + 1 < n$.
- $c_{n-1} \in Coll(\kappa_{x_{n-1}}^{+\omega+5}, < \kappa)$.
- (A, C) has the form $\langle A_k, C_k : n \leq k < \omega \rangle$ where $A_k = \mathrm{dom}(C_k) \in U_k$, $C_k(x) \in Coll(\kappa_x^{+\omega+5}, < \kappa)$ for all $x \in A_k$, and $[C_k]_{U_k} \in F_k$.

The ordering is the usual one for forcings of this type: a condition is extended by strengthening the collapsing conditions in the current stem, adding new x_k's and c_k's to the stem with $c_k \leq C_k(x_k)$, shrinking the remaining A_k's and strengthening the remaining C_k's.

The poset $\mathbb{P}$ satisfies the Prikry lemma and is μ-cc, as any two conditions with the same stem are compatible. So we can easily compute the cardinals in the generic extension. In the extension ν is collapsed to cardinality κ, so that $\mu = \kappa^+$ and $\rho = 2^\kappa = \nu^+$. If $\langle x_n : n < \omega \rangle$ is the diagonal supercompact Prikry sequence added by $\mathbb{P}$, then below κ cardinals in the intervals (ω, κ_{x_0}) and $(\kappa_{x_n}^{+\omega+5}, \kappa_{x_{n+1}})$ are collapsed while the rest are preserved, so that $\kappa = \aleph_{\omega^2}$.

For a stem s and formula ϕ, we define the relations $s \Vdash^* \phi$ in the same way as we did in the preceding sections, that is there exists a condition p with stem s such that $p \Vdash \phi$.

4.3. **The ineffable tree property.**

Theorem 4.3. *If G is $\mathbb{P}$-generic over $V[H][E]$, then ITP_μ holds in $V[H][E][G]$.*

Proof. The argument is similar to that for Theorem 3.1, so we focus on the new points. One of the new features is that when we extend the embedding with critical point κ_m for some $m > 0$, we have to deal with the collapses that made the κ_n's successors of each other. That influences the branch pullback arguments. Another new feature is that we will need some auxiliary poset making λ a finite successor of μ, when we prove $ITP(\mu, \lambda)$. This is necessary in order to carry out the first step: fixing the length of the Prikry conditions.

Suppose for contradiction, that the result fails. Then there is $p \in \mathbb{P}$ forcing that $ITP(\mu, \lambda)$ fails for some λ. Since $\mathbb{P}$ is μ-cc, it is enough to consider lists indexed by $(P_\mu(\lambda))^{V[H][E]}$, and obtain a contradiction by showing that p forces all such lists to have an ineffable branch.

Increasing λ if necessary, we may assume that there is a $Coll(\mu^{++}, < \lambda)^{V[H]}$-name $\dot{U}_\lambda$ such that $\dot{U}_\lambda$ is forced to be a supercompactness measure on $P_\kappa \lambda$ whose projection to each $P_\kappa(\kappa_n)$ is U_n. Let K be $Coll(\mu^{++}, < \lambda)^{V[H]}$-generic over $V[H][E]$, let U^* be the realisation of the name $\dot{U}_\lambda$ and let $j^* : V[H][E][K] \to M^*$ be the associated ultrapower map. Note that $\lambda = \mu^{+++} = \kappa^{+\omega+4}$ in $V[H][E][K]$.

Working in $V[E][H][K]$, we define some auxiliary objects as in Section 3. We let $e : \mathcal{P}_\mu(\lambda) \to \lambda$ be a bijection, which we use to form diagonal intersections of $P_\mu(\lambda)$-indexed sequences of elements of U^*. We let $z^* = \bigcup j^* “P_\mu(\lambda)$, $h(x) = \{z \in P_\mu(\lambda) : e(z) \in x\}$, and $g(x) = \bigcup h(x)$, so that $[g]_{U^*} = z^*$.

We fix a $\mathbb{P}$-name in $V[H][E]$ for a thin list indexed by $P_\mu(\lambda)$, say $\langle \dot{d}_x : x \in P_\mu\lambda \rangle$. For $z \in P_\mu(\lambda)$, $\langle \dot{\sigma}^z_\xi : \xi < \kappa \rangle$ names an enumeration of $\{\dot{d}_y \restriction z \mid z \subseteq y \in \mathcal{P}_\mu(\lambda)\}$. Suppose that p forces that this list has no ineffable branch. Towards a contradiction we will find such a branch in $V[H][E][K][G]$, for some $\mathbb{P}$-generic G with $p \in G$, and then argue that this branch must already exist in $V[H][E][G]$.

Lemma 4.4. *In $V[H][E][K]$ there exist a cofinal set $I \subseteq \mathcal{P}_\mu(\lambda)$, a natural number n^*, and a function $x \mapsto h_x$ with domain $A_0 \in U^*$ such that for all $x \in A_0$, h_x is a stem of length n^* for some condition extending p, and for all $z \in I$ with $e(z) \in x$ there is $\xi < \kappa$ such that*

$$h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_\xi$$

Proof. Let p have a stem of length n, let C_n be the first function in the upper part of p and let $A_n = \mathrm{dom}(C_n)$. Since U_n is the projection of U^*, $j^* “\kappa_n \in j^*(A_n)$. Consider the condition $j^*(p)$, and extend it to a condition $\bar{p} \in j^*(\mathbb{P})$ with a stem of length $n+1$ which forces $j^* “\kappa_n$ to be the next point on the supercompact Prikry sequence. Since $j^* “\kappa_n \cap j^*(\kappa) = \kappa$, it follows from our analysis of the forcing $\mathbb{P}$ that $\bar{p}$ forces all cardinals in the interval $[\kappa, \kappa^{+\omega+5}]$ to be preserved. In particular, since $\mu = \kappa^{+\omega+1}$ and $\lambda = \kappa^{+\omega+4}$ in $V[H][E][K]$ (and hence by closure in M^*), $\bar{p}$ forces that μ and λ remain regular cardinals.

Let G^* be $j^*(\mathbb{P})$-generic with $\bar{p} \in G^*$, and work in $M^*[G^*]$. For all $z \in \mathcal{P}_\mu(\lambda)$ $j^*(z) \subseteq z^*$, so there are $\xi < j(\kappa)$ and $p_z \in G^*$ such that $p_z \leq \bar{p} \leq j^*(p)$ and

$$p_z \Vdash j^*(\dot{d})_{z^*} \restriction j^*(z) = j(\dot{\sigma})^{j^*(z)}_\xi.$$

Then there is some stationary (hence cofinal) $I^* \subset \mathcal{P}_\mu(\lambda)$ in $M[G^*]$ and $n^* < \omega$, such that for all $z \in I^*$, p_z has length n^*. For each $z \in I^*$, denote the stem of p_z by $\langle d^z, x_0, c^z_0, ..., x_{n^*-1}, c^z_{n^*-1} \rangle$. First, by passing to a stationary subset of I^*, we may assume that for some $d, c_0, ..., c_{n-1}$, for all $z \in I^*$, $d = d^z$ and $c_i = c^z_i$ for $i < n$.

Now at the k-th coordinate for $n \leq k < n^*$, by construction each c^z_k is in a generic filter for a collapsing poset that is λ^+-closed, so we can take a lower bound c_k in this generic filter. The key point here is that $\lambda = \kappa^{+\omega+4}$, and we arranged by forcing below $\bar{p}$ that κ is the n^{th} point on the Prikry sequence that $j^*(\mathbb{P})$ adds in $j^*(\kappa)$.

Let $h^* = \langle d, x_0, c_0, ..., x_{n^*-1}, c_{n^*-1} \rangle$. Then for all $z \in I^*$ there is some $\xi < j^*(\kappa)$ such that

$$h^* \Vdash^* j^*(\dot{d})_{z^*} \restriction j^*(z) = j^*(\dot{\sigma})^{j^*(z)}_\xi$$

Working in $V[H][E][K]$, define

$$I = \{z \in P_\mu(\lambda) \mid \exists \xi < j^*(\kappa)\ h^* \Vdash^* j^*(\dot{d})_{z^*} \restriction j^*(z) = j^*(\dot{\sigma})^{j^*(z)}_\xi\}$$

Clearly $I^* \subseteq I$ and hence I is cofinal.

Let $h^* = [x \mapsto h_x]_{U^*}$, where h_x is the stem of an extension of p of length n^*. For all $z \in I$ there is $A_z \in U^*$ with the following property: for all $x \in A_z$ there exists

$\xi < \kappa$ such that
$$h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_\xi.$$
Take $A_0 = \Delta_{z \in I} A_z$, then everything is as required. □

For the next lemma we need the notion of a system and a system of branches on a cofinal subset of $\mathcal{P}_\mu(\lambda)$ with some relations.

Definition 4.5. *Let I be a cofinal subset of $\mathcal{P}_\mu(\lambda)$, ρ be an ordinal, and D be an index set. A* system *on $I \times \rho$ is a family $\langle R_s \rangle_{s \in D}$ of transitive and reflexive relations on $I \times \rho$, so that:*

1. *If $(x, \xi) R_s (y, \zeta)$ and $(x, \xi) \neq (y, \zeta)$ then $x \subsetneq y$.*
2. *If (x_0, ξ_0) and (x_1, ξ_1) are both R_s-below (y, ζ) and $x_0 \subset x_1$, then $(x_0, \xi_0) R_s (x_1, \xi_1)$.*
3. *For every x, y both in I, there are $z \in I$, $s \in D$ and $\xi, \xi', \zeta \in \rho$ so that $x \cup y \subset z$, $(x, \xi) R_s (z, \zeta)$ and $(y, \xi') R_s (z, \zeta)$.*

A branch through *R_s is a partial function $b : I \rightharpoonup \rho$, such that:*

1. *If $x \subset y$ are both in $\mathrm{dom}(b)$, then $(x, b(x)) R_s (y, b(y))$.*
2. *If $y \in \mathrm{dom}(b)$, and $(x, \xi) R_s (y, b(y))$, then $(x, \xi) \in b$, i.e. b is downwards R_s-closed.*

A system of branches through *$\langle R_s \rangle_{s \in I}$ is a family $\langle b_\eta \rangle_{\eta \in J}$ so that each b_η is a branch through some $R_{s(\eta)}$, and $I = \bigcup_{\eta \in J} \mathrm{dom}(b_\eta)$.*

We have the following abstract branch preservation lemma from Lemma 5.11 from [4], which builds on [8]; see for example, Lemma 3.3 of [6].

Lemma 4.6. *Let $V \subset W$ be models of set theory, let W be a τ-c.c. forcing extension of V, and let $\mathbb{Q} \in V$ be τ-closed in V. In W suppose $\langle R_s \rangle_{s \in D}$ is a system on $I \times \rho$, for some cofinal $I \subset \mathcal{P}_\mu(\lambda)$, such that forcing with $\mathbb{Q}$ over W adds a system of branches $\langle b_j \rangle_{j \in J}$ through this system. Finally suppose $\chi := \max(|J|, |D|, \rho)^+ < \tau < \mu$. Then there is a cofinal branch $b_j \in W$.*

Now we are ready for the second step: fixing the stem.

Lemma 4.7. *In $V[H][E][K]$ there are a stem $\bar{h}$ of length n^* and a stationary set $T \subseteq \mathcal{P}_\mu(\lambda)$ such that for all $z_1 \subseteq z_2$ both from T,*
$$\bar{h} \Vdash^* \dot{d}_{z_1} = \dot{d}_{z_2} \restriction z_1.$$

Proof. Let $i : V \to N$ be a λ-supercompact embedding with critical point κ_{n^*+3}. Lift i to $i : V[H][K][E] \to N^*$ in a generic extension of $V_1 := V[H][K][E]$ of the form $V_1[K^* \times F]$, where K^* is generic for a κ_{n^*+2}-closed forcing (in $V[H][K]$) and F is generic for a κ^+-Knaster forcing $\mathbb{A}^*$.

As in Lemma 3.4, let $u^* \in i(I)$ be such that $\bigcup i``\mathcal{P}_\mu(\lambda) \subseteq u^*$, and in $V_1[K^* \times F]$, define partial functions $\langle b_{\delta,h} \mid \delta < \kappa, h \text{ a stem of length } n^* \rangle$ from I to κ by:
$$b_{\delta,h} = \{(z, \xi) \in I \times \kappa \mid h \Vdash^*_{i(\mathbb{P})} i(\dot{\sigma})^{u^*}_\delta \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi\}.$$

Note that $I = \bigcup_{(\delta,h)} \mathrm{dom}(b_{\delta,h})$. Also, the number of such stems is κ_{n^*-1}. Let $W := V[H][K][F]$. First we will show that there are such partial functions in W.

For each h, let R_h be the relation on $I \times \kappa$, given by
$$(z, \xi) R_h (z', \xi') \text{ iff } h \Vdash^* \dot{\sigma}^{z'}_{\xi'} \restriction z = \dot{\sigma}^z_\xi.$$
Then $\langle R_h \rangle_h$ is a system on $I \times \kappa$, and every $b_{\delta,h}$ is a (possibly bounded) branch through R_h. Moreover, the $b_{\delta,h}$'s are a system of branches through the R_h's as in

Definition 4.5. So by the preservation lemma 4.6, at least one of them is cofinal and is in $W = V[H][K][F]$.

Let $D := \{(\delta, h) \mid b_{\delta,h} \in W\}$. Since K^* is generic for a $< \kappa_{n^*+2}$-distributive poset over W, and there are only κ_{n^*-1} relevant stems h, $D \in W$. Similarly, $\langle b_{\delta,h} \mid (\delta, h) \in D\rangle \in W$.

Now continue as in Lemma 3.4:

(1) for every $(\delta, h) \in D$, if $b_{\delta,h}$ is cofinal, then it is in $V[H][K][E]$;
(2) for all pairs (δ, h) define $\mathcal{C}_{\delta,h}$ to be the set of possible values for $b_{\delta,h}$, when (δ, h) is forced to be in D. More precisely, $\mathcal{C}_{\delta,h} = \{C \mid (\exists a \in \mathbb{A}^*) a \Vdash (\delta, h) \in \dot{D}, C = \dot{b}_{\delta,h}\}$[4].

As before we have that $|\mathcal{C}_{\delta,h}| \leq \kappa$ for each (δ, h), and for any two c, c' in $\bigcup_\delta \mathcal{C}_{\delta,h}$ there is some $z \in \mathcal{P}_\mu(\lambda)$, such that for all $z' \supset z$, we cannot have $c(z') = c'(z')$.

Pick $\bar{z} \in \mathcal{P}_\mu(\lambda)$ such that

(1) There is no z such that $\bar{z} \subseteq z$ and $z \in b_{\delta,h}$ with $\mathrm{dom}(b_{\delta,h})$ not cofinal.
(2) There is no z such that $\bar{z} \subseteq z$ and there are distinct $c, c' \in \bigcup_\delta C_{\delta,h}$ with $c(z) = c'(z)$.

Next we want to show an analogue of Claim 3.6. However, we cannot argue exactly as in the claim, for the following reason. Suppose that $z \in I, \bar{z} \subset z$. Then we can still find some (δ, h) and $\xi < \kappa$, such that $b_{\delta,h}(z) = \xi$ and the domain of $b_{\delta,h}$ is unbounded. The problem is that we don't know that $(\delta, h) \in D$ and so cannot conclude that the branch is in $V[H][K][E]$. So, instead, we will show the claim holds on some unbounded subset of I.

We need some definitions. For every $z \in I$ and $x \in A_0$, let $(\dagger)_{x,z}$ be the statement that:

$$\exists c \in \bigcup_{\delta<\kappa} \mathcal{C}_{\delta,h_x} h_x \Vdash^* \dot{d}_{g(x)} \restriction z = \dot{\sigma}^z_{c(z)}$$

Let $A_z := \{x \in A_0 \mid e(z) \in x \text{ and } (\dagger)_{x,z} \text{ holds }\}$.

Claim 4.8. *There is a cofinal $S \subset I$, such that for all $z \in S$, $A_z \in U^*$.*

Proof. Suppose otherwise, i.e. there is some z_0, $\bar{z} \subset z_0$, such that for all $z \supset z_0$, $A_z \notin U^*$. Define $B_z := A_0 \setminus A_z$ and $B := \triangle_{z_0 \subset z \in I} B_z \in U^*$.

Next, in W, we define a subsystem of $\langle R_h\rangle_h$ by "erasing" the branches that are in $V[H][K][E]$. Let $I' := I \cap \{z \mid z_0 \subset z\}$. For every h, let R'_h be the relation on $I' \times \kappa$, given by

$$(z, \xi) R'_h (z', \xi') \text{ iff } (z, \xi) R_h (z', \xi') \text{ and whenever } (h, \delta) \in D \text{ then } (z, \xi) \notin b_{h,\delta}.$$

We claim that $\langle R'_h\rangle_h$ is a system on $I' \times \kappa$. The first two properties are straightforward. For the third property we will use our assumption that $B \in U^*$. Let $z_1, z_2 \in I'$. Let $z \in I'$ be such that $z_1 \cup z_2 \subset z$ and $x \in i(B)$ be such that $i(e)(u^*), i(e(z_1)), i(e(z_2)), i(e(z)) \in x$. By elementarity, applied to the conclusion of Lemma 4.4, there are $\delta, \xi_1, \xi_2, \xi < \kappa$ such that:

- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction u^* = i(\dot{\sigma})^{u^*}_\delta$.
- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction i(z) = i(\dot{\sigma})^{i(z)}_\xi$
- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction i(z_1) = i(\dot{\sigma})^{i(z_1)}_{\xi_1}$
- $i(h)_x \Vdash^* i(\dot{d})_{i(g)(x)} \restriction i(z_2) = i(\dot{\sigma})^{i(z_2)}_{\xi_2}$

[4]Note that $\mathcal{C}_{\delta,h}$ can be empty.

Let $h' := i(h)_x$. Then, we get that:

- $h' \Vdash^* i(\dot\sigma)^{i(z)}_{\xi} \restriction i(z_1) = i(\dot\sigma)^{i(z_1)}_{\xi_1}$
- $h' \Vdash^* i(\dot\sigma)^{i(z)}_{\xi} \restriction i(z_2) = i(\dot\sigma)^{i(z_2)}_{\xi_2}$
- $b_{\delta,h'}(z) = \xi$, $b_{\delta,h'}(z_1) = \xi_1$, $b_{\delta,h'}(z_2) = \xi_2$

Then $(z_1, \xi_1) R_{h'} (z, \xi)$ and $(z_2, \xi_2) R_{h'} (z, \xi)$. We want to show that they are actually $R'_{h'}$-related. Since we are above all the splittings, note that if $z_1 \in \mathrm{dom}(b_{\eta,h'})$, then $b_{\eta,h'} = b_{\delta,h'}$. So it is enough to show that $(\delta, h') \notin D$.

By elementarity of i, there is some $y \in B$, such that $e(z_1), e(z_2), e(z) \in y$ and $h_y = h'$. Then, by definition of B, $(\dagger)_{x,z_2}$, $(\dagger)_{x,z_1}$, $(\dagger)_{x,z}$ all fail. Since $\bar z \subseteq z$, $\mathrm{dom}(b_{\delta,h'})$ is unbounded, so if (δ, h) were in D, then we can take $c = b_{\delta,h'}$ to witness $(\dagger)_{x,z_1}$, $(\dagger)_{x,z_1}$. It follows that $(\delta, h) \notin D$.

So, we have a system.

Now, in $W[K^*]$, for every $(\delta, h) \notin D$, let $b'_{\delta,h}$ be the restriction of $b_{\delta,h}$ to R'_h. Then $\langle b'_{h,\delta} \mid (h, \delta) \notin D \rangle$ is a system of branches through $\langle R'_h \rangle_h$. Then by Lemma 4.6, one of these branches $b'_{h,\delta}$ is in W. But then, so is $b_{h,\delta}$. Contradiction with the assumption that $(h, \delta) \notin D$.

□

The rest is as in lemma 3.4, replacing I with S.

□

Fix $\bar h, T$ as in the conclusion of the above Lemma. Next we want to build top parts for the Prikry conditions, using the following lemma:

Lemma 4.9. *Suppose that h is a stem extending $\bar h$ and T_h is a stationary subset of T such that for all $z_1 \subseteq z_2$ from T_h, $h \Vdash \dot d_{z_1} = \dot d_{z_2} \cap z_1$. Then there are $z_h \in \mathcal{P}_\mu(\lambda)$ and A^h_z, C^h_z for $z \in T_h \cap \{z \mid z_h \subseteq z\}$ such that:*

1. *for each x, $\mathrm{dom}(C^h_z) = A^h_z \in U_n$, $[C^h_z] \in F_n$, and*
2. *for all $z_h \subseteq y \subseteq z$ with $y, z \in T_h$ and all $x \in A^h_y \cap A^h_z$, if $C^h_z(x)$ and $C^h_y(x)$ are compatible, then*

$$h \frown (x, C^h_z(x) \cup C^h_y(x)) \Vdash^* \dot d_y = \dot d_z \cap y.$$

Proof. The proof is as in [8, Lemma 16]. □

Then as before, we can find $z^* \in \mathcal{P}_\mu(\lambda)$ and conditions $\langle p_z \mid z \in T \cap \{z \mid z^* \subseteq z\} \rangle$ with stem $\bar h$, such that if $z^* \subseteq y \subseteq z$ with $y, z \in T$, then $p_y \wedge p_z \Vdash \dot d_y = \dot d_z \cap y$.

As before there is a condition which forces that the set $\{z \mid p_z \in G\}$ is stationary in $\mathcal{P}_\mu(\lambda)$. It follows that there is an ineffable branch b through the list in $V[H][E][K][G]$ for some $\mathbb{P}$-generic G over $V[H][E][K]$ with $p \in G$.

Finally, note that $V[H][K][E][G] = V[H][K \times (E * G)]$. Since $Add(\kappa, \mu^+) * \dot{\mathbb{P}}$ has the μ-chain condition and $Coll(\mu^{++}, < \lambda)$ is μ^{++}-closed (μ-closure suffices), it follows that $Coll^{V[H]}(\mu^{++}, < \lambda)$ has the μ-thin approximation property over $V[H][E * G]$: that is to say, K cannot add a new set a such that every $< \mu$-sized subset c of a is already in $V[H][E][G]$, and the number of possible values for c is less than μ. It follows that the branch b is in $V[H][E][G]$. Since stationarity is downward-absolute, b is an ineffable branch.

□

5. Open problems

Having obtained the failure of SCH together with ITP at the successor of a singular, this opens up the path to forcing ITP at more successive cardinals. The long term project is getting ITP at every regular cardinal greater than $\aleph_1$. The first natural question is if we can get ITP at successive successors of a singular:

Question. *Can we obtain ITP simultaneously at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ where $\aleph_{\omega^2}$ is strong limit?*

We conjecture the answer to be yes. The strategy would be to do an iteration of Mitchell style forcing followed by diagonal Prikry forcing.

The other direction is to combine our result with forcing ITP at successive regular cardinals below the singular cardinal. In 2013, Fontanella [2] and Unger [9] showed independently that it is consistent from large cardinals, to have ITP at $\aleph_n$, for every $n > 1$. More precisely, this happens in the Cummings-Foreman model [1] for the tree property at the $\aleph_n$'s. In that construction SCH does hold at $\aleph_\omega$. This brings up the following old open problem:

Question. *Does ITP at κ (or even just the strong tree property) imply that SCH holds for every strong limit singular cardinal above κ?* [5]

On the positive side of this question, we have Solovay's old theorem that SCH holds above a strongly compact cardinal. Also, in 2008, Viale proved that PFA implies SCH [10], and by a theorem of Weiss [13], ITP at $\aleph_2$ is a consequence of PFA. Viale and Weiss also defined a strengthening of ITP called ISP. ISP is a guessing type principle and at $\aleph_2$ it is also a consequence of PFA. Very recently, Krueger and Hachtman independently showed that ISP at $\aleph_2$ also implies SCH. On the other hand, by Specker's result that $\tau^{<\tau} = \tau$ negates the tree property at τ^+, a negative answer to this question is required to obtain ITP successively across a singular strong limit cardinal.

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