Aronszajn Trees and the SCH

Itay Neeman and Spencer Unger

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1 Introduction

These notes are based on results presented by Itay Neeman at the Appalachian Set Theory workshop on February 28, 2009. Spencer Unger was the official note-taker and based these notes closely on Neeman's lectures. The purpose of the workshop was to present a recent theorem due to Neeman [16].

Theorem 1. *From large cardinals, it is consistent that there is a singular strong limit cardinal κ of cofinality ω such that the Singular Cardinal Hypothesis fails at κ and the tree property holds at κ^+.*

The purpose of these notes is to give the reader the flavor of the argument without going into the complexities of the final proof in [16]. Having read these notes, the motivated reader should be prepared to understand the full argument.

We begin with a discussion of trees, which are natural objects in infinite combinatorics. One topic of interest is whether a tree has a cofinal branch. For completeness we recall some definitions.

Definition 2. *Let λ be a regular cardinal and κ be a cardinal.*

1. *A* λ-tree *is a tree of height λ with levels of size less than λ.*
2. *A* cofinal branch *through a tree of height λ is a linearly ordered subset of order type λ.*
3. *A* λ-Aronszajn tree *is a λ-tree with no cofinal branch.*
4. *A κ^+-tree is* special *if there is a function $f : T \to \kappa$ such that for all $x, y \in T$, if $x\,T\,y$ then $f(x) \neq f(y)$.*

Remark 3. *It is easy to see that a special κ^+-tree is in fact a κ^+-Aronszajn tree. Moreover, a special κ^+-tree remains special after cardinal preserving forcing.*

For a regular cardinal λ we can ask whether all λ-trees have a cofinal branch. This leads to the definition of the tree property.

Definition 4. *For a regular cardinal λ, λ has the* tree property *if every λ-tree has a cofinal branch. Equivalently, λ has the tree property if and only if there are no λ-Aronszajn trees.*

We list a few classical results:

- (König [11]) $\aleph_0$ has the tree property.
- (Aronszajn, see [12]) $\aleph_1$ does not have the tree property.
- (Specker [19]) If $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree.
- (Keisler and Tarski [10]) If κ is strongly inaccessible, then κ has the tree property if and only if κ is weakly compact.
- (Jensen [9]) There is special κ^+-Aronszajn tree if and only if the weak square property $\square_\kappa^*$ holds

Forcing is required to answer questions about small cardinals and the tree property. Again we list some results.

- (Mitchell and Silver [15]) Con(ZFC + "There is a weakly compact cardinal") implies Con(ZFC + "$\aleph_2$ has the tree property")
- (Magidor and Shelah [14]) From large cardinals it is consistent with ZFC that $\aleph_{\omega+1}$ has the tree property.

We turn to discussion of the Singular Cardinal Hypothesis.

Definition 5. *The* Singular Cardinal Hypothesis *at a singular cardinal κ (*SCH_κ*) is the assertion, "If $2^{<\kappa} = \kappa$, then $2^\kappa = \kappa^+$." The Singular Cardinal Hypothesis (SCH) is the assertion, "For all singular cardinals κ,* SCH_κ*".*

Easton [4] established that it is consistent that the continuum function ($\kappa \mapsto 2^\kappa$) have any reasonable behavior on regular cardinals. However, the possible behavior of the continuum function on singular cardinals is unclear. In particular, results of inner model theory show that large cardinals are required to construct a model with the failure of SCH. The first such construction started from a supercompact cardinal κ and proceeded in two steps. The first step was forcing to make $2^\kappa > \kappa^+$ while maintaining the measurability of κ. This forcing is due to Silver and an account can be found in [1]. The second step was to use Prikry forcing [17] to make κ singular of cofinality ω without adding bounded subsets of κ or collapsing cardinals. Gitik determined the exact strength of the failure of SCH [6].

Theorem 6. $Con(\neg\,\mathrm{SCH}) \Leftrightarrow Con(\exists \kappa\ o(\kappa) = \kappa^{++})$

We will now argue that there is a κ^+-Aronszajn tree in the Prikry-Silver model for the failure of SCH. By the result of Specker, we have a special Aronszajn tree at the successor of any inaccessible cardinal. In particular there is a special κ^+-tree in the ground model for the construction. By Remark 3 above, special trees are preserved by cardinal preserving forcing. Both steps above are cardinal preserving. So the original model for the failure of SCH_κ has a special κ^+-Aronszajn tree. These ideas can be found in work of Ben-David and Magidor [2].

This brings up the following question, which Woodin asked in the late 80's [5]. For κ singular of cofinality ω, does the failure of SCH imply that there exists a κ^+-Aronszajn tree? The question was intended to test whether the original way to obtain the failure of SCH was the only way. Gitik and Magidor [7] showed that there is a different way to get the failure of SCH. They proved that one can add κ^{++} Prikry sequences without adding bounded subsets of κ. However, Woodin's question remained open. It turns out that there are other ways to make SCH fail, but they all still gave Aronszajn trees.

Many people had hope for a positive answer. The line of thought was that the failure of SCH at κ would imply some intermediate combinatorial principle, like approachability or weak square, and then from this one could construct a κ^+-Aronszajn tree. It turns out that the first step fails. In particular, Gitik and Sharon [8] showed that the failure of SCH does not imply approachability. Cummings and Foreman [3] showed that there is a PCF theoretic object called a bad scale in the Gitik-Sharon model; this implies the failure of approachability.

The purpose of the workshop is to present a proof that the answer to Woodin's question is *no*. By using a variation of the forcing from [8], we construct a model in which we have the failure of SCH_κ at a singular cardinal of cofinality ω and there are no κ^+-Aronszajn trees.

2 The Tree Property

In order to motivate the use of large cardinals in our main result, we outline an application of inner model theory which shows that the tree property at the successor of a singular cardinal has high consistency strength. Let ν be a singular cardinal. As we mentioned in the introduction if the weak square principle $\square^*_\nu$ holds then there is a special ν^+-Aronszajn tree. By inner model theory, if there are no inner models with large cardinals then there is a model $K \subseteq V$ such that

1. $\nu^+ = (\nu^+)^K$, and
2. the weak square principle $\square^*_\nu$ (in fact $\square_\nu$) holds in K.

It follows that in V there is a special ν^+-Aronszajn tree. So large cardinals are required in order to obtain the tree property at the successor of a singular cardinal.

In [14], Magidor and Shelah show that the successor of a singular limit of strongly compact cardinals has the tree property. This result is important because we can view our proof as using the same idea, but with generic large cardinal properties in place of real large cardinal properties. For our context we state Magidor and Shelah's theorem using supercompact cardinals.

Theorem 7. *Let ν be a singular limit of supercompact cardinals, then ν^+ has the tree property.*

We will prove the theorem for a cofinality ω limit of supercompact cardinals, because this reflects the proof of our main theorem. For the proof of this result we will have two lemmas, which we call the Spine Lemma and the Traction Lemma. For the main result of these notes we will have to prove new versions of both lemmas.

Proof of Theorem 7. Let T be a ν^+-tree. Without loss of generality, level α of T is $\{\langle \alpha, \xi \rangle \mid \xi < \nu\}$. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals cofinal in ν.

Spine Lemma 1. *There exist an $n < \omega$ and a cofinal set $C \subseteq \nu^+$ such that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa_n$ such that $\langle \alpha, \xi \rangle \; T \; \langle \beta, \zeta \rangle$.*

We call this Lemma the Spine Lemma, because we can view it as picking out a narrow essential component of the tree. The set $\{\langle \alpha, \xi \rangle \mid \alpha \in C, \xi < \kappa_n\}$ has the property that any two levels from C contain nodes in the spine which are related in the tree. Spine Lemma 1 shows that a set with this spine-like property exists.

To clarify the difference between members of the domain and members of the codomain of our elementary embeddings, we will write ordinals of the domain as the usual Greek letters and ordinals of the codomain as Greek letters with a superscript $*$.

Proof. For convenience we let $\kappa =_{def} \kappa_0$. Let $\pi : V \to M$ be a ν^+-supercompactness embedding with critical point κ; that is $\mathrm{crit}(\pi) = \kappa$, $\pi(\kappa) > \nu^+$ and ${}^{\nu^+}M \subseteq M$. Let $\gamma^* =_{def} \sup \pi``\nu^+$. We claim that $\gamma^* < \pi(\nu^+)$. By elementarity and since ν^+ is regular in V, we have that $\pi(\nu^+)$ is regular in M. So it suffices to show that the cofinality of γ^* is less than $\pi(\nu^+)$ in M. The closure of M under ν^+-sequences implies that $\pi \restriction \nu^+ \in M$. Moreover, $\pi \restriction \nu^+$ is an order preserving bijection from ν^+ to $\pi``\nu^+$. Therefore, the cofinality of γ^* is ν^+ when computed in M. This finishes the claim that $\gamma^* < \pi(\nu^+)$.

We fix some node $\langle \gamma^*, \eta^* \rangle$ on level γ^* of $\pi(T)$. For each $\alpha < \nu^+$, there is a unique $\xi^*_\alpha < \pi(\nu)$ such that

$$\langle \pi(\alpha), \xi^*_\alpha \rangle \; \pi(T) \; \langle \gamma^*, \eta^* \rangle.$$

As $\langle \pi(\kappa_n) \mid n < \omega \rangle$ is cofinal in $\pi(\nu)$, there is $n_\alpha < \omega$ such that $\xi^*_\alpha < \pi(\kappa_{n_\alpha})$. As ν^+ is regular, there exist $n < \omega$ and $C \subseteq \nu^+$ cofinal such that $n_\alpha = n$ for all $\alpha \in C$.

We check that this n and C are as required for Spine Lemma 1. Fix $\alpha < \beta$ both in C. By the choice of C we have

$$\langle \pi(\alpha), \xi^*_\alpha \rangle \; \pi(T) \; \langle \gamma^*, \eta^* \rangle,$$
$$\langle \pi(\beta), \xi^*_\beta \rangle \; \pi(T) \; \langle \gamma^*, \eta^* \rangle.$$

Since $\pi(T)$ is a tree, it follows that $\langle \pi(\alpha), \xi^*_\alpha \rangle \; \pi(T) \; \langle \pi(\beta), \xi^*_\beta \rangle$. We can collect this information in M and then use elementarity to bring the conclusion back to V.

$$M \models \exists \xi^*, \zeta^* < \pi(\kappa_n) \; \langle \pi(\alpha), \xi^* \rangle \; \pi(T) \; \langle \pi(\beta), \zeta^* \rangle.$$

Thus by elementarity

$$V \models \exists\, \xi, \zeta < \kappa_n\, \langle \alpha, \xi \rangle\, T\, \langle \beta, \zeta \rangle.$$

□

Traction Lemma 1. *There exist a cofinal set $J \subseteq C$ and a map $\alpha \mapsto \xi_\alpha$ such that for all $\alpha < \beta$ both in J, $\langle \alpha, \xi_\alpha \rangle\, T\, \langle \beta, \xi_\beta \rangle$.*

The idea is that we have thinned the tree to the spine $\{\langle \alpha, \xi \rangle \mid \alpha \in C, \xi < \kappa_n\}$, which satisfies the conclusion of Spine Lemma 1. Now we take an embedding with a sufficiently high critical point and this embedding will only stretch the spine vertically but not horizontally. In other words we put the spine in traction.

Proof. Let $\pi : V \to M$ be a ν^+-supercompactness embedding with critical point κ_{n+1}. The key point about the new embedding is that $\pi(\kappa_n) = \kappa_n$ and for all sets A of size κ_n, $\pi(A) = \pi``A$. This will be important when we work with the level sets of the spine. As before we can argue that $\sup \pi``\nu^+ < \pi(\nu^+)$. By elementarity, $\pi(C)$ is unbounded in $\pi(\nu^+)$. Working in M, let γ^* be the least element of $\pi(C)$ above $\sup \pi``\nu^+$. By applying elementarity to the conclusion of Spine Lemma 1, for each $\alpha \in C$ there exist $\xi^*_\alpha, \eta^*_\alpha < \pi(\kappa_n)$ such that $\langle \pi(\alpha), \xi^*_\alpha \rangle\, \pi(T)\, \langle \gamma^*, \eta^*_\alpha \rangle$

We claim that there are $\eta^* < \kappa_n$ and $J \subseteq C$ cofinal, such that if $\alpha \in J$, then $\eta^*_\alpha = \eta^*$. In M level γ^* of the spine has size $\pi(\kappa_n) = \kappa_n$. So $\{\eta^*_\alpha \mid \alpha \in C\}$ has at most size κ_n. Therefore by the regularity of ν^+, the map taking $\alpha \in C$ to η^*_α must be constant with some value η^* on an unbounded set $J \subseteq C$. Let $\xi_\alpha = \xi^*_\alpha$ for $\alpha \in J$. Each $\xi_\alpha < \kappa_n$, so $\pi(\xi_\alpha) = \xi^*_\alpha$. We claim that J and $\alpha \mapsto \xi_\alpha$ satisfy the conclusion of Traction Lemma 1. Fix $\alpha < \beta$ both in J. Then

$$\langle \pi(\alpha), \pi(\xi_\alpha) \rangle\, \pi(T)\, \langle \gamma^*, \eta^* \rangle,$$
$$\langle \pi(\beta), \pi(\xi_\beta) \rangle\, \pi(T)\, \langle \gamma^*, \eta^* \rangle.$$

Again using the fact that $\pi(T)$ is a tree, it follows that

$$M \models \langle \pi(\alpha), \pi(\xi_\alpha) \rangle\, \pi(T)\, \langle \pi(\beta), \pi(\xi_\beta) \rangle.$$

Lastly by elementarity $V \models \langle \alpha, \xi_\alpha \rangle\, T\, \langle \beta, \xi_\beta \rangle$. □

To finish the proof, we notice that the map $\alpha \mapsto \xi_\alpha$ for $\alpha \in J$ enumerates a cofinal branch through T. This completes the proof of Theorem 7. □

3 Diagonal Prikry Forcing

In this section we define a forcing due to Gitik and Sharon [8] and prove of some of its properties. The forcing that we define in this section is different from Prikry's diagonal Prikry forcing. Prikry's diagonal Prikry forcing uses an increasing ω-sequence of measurable cardinals $\langle \kappa_n \mid n < \omega \rangle$ where we use one measure on each κ_n. Note that the measures involved become more complete as n increases. The result is that the the forcing adds no bounded subsets of $\sup_{n<\omega} \kappa_n$. The forcing from [8] is quite different. Again we use an ω-sequence of measures but the completeness of each measure is the same. The result is that some cardinals are collapsed, but this is by design. We begin with κ supercompact and $\nu > \kappa$, $cof(\nu) = \omega$. We define a diagonal Prikry forcing to make $cof(\kappa) = \omega$ and $|\nu| = \kappa$ while preserving ν^+ and cardinals less than or equal to κ.

To begin we fix $\langle \kappa_n \mid n < \omega \rangle$ increasing and cofinal in ν with each κ_n regular. Then for each n we let $\mathcal{U}_n$ be a supercompactness measure on $\mathcal{P}_\kappa(\kappa_n)$. The supercompactness measure $\mathcal{U}_n$ can be derived from an embedding j which witnesses that κ is κ_n-supercompact. We define $\mathcal{U}_n =_{def} \{X \subseteq \mathcal{P}_\kappa(\kappa_n) \mid j``\kappa_n \in j(X)\}$. Note that $\mathcal{U}_n$ concentrates on $K_n =_{def} \{a \subseteq \kappa_n \mid |a| < \kappa,\ a \cap \kappa$ inaccessible$\}$. Let $K = \bigcup_{n<\omega} = K_n$. We define an ordering on K as follows. Let $a, b \in K$, $a \prec b$ if and only if $b \supsetneq a$ and $b \cap \kappa > |a|$. For $a \in K$, define $\mathrm{Cone}(a) = \{b \in K \mid a \prec b\}$. Note that if $a \in K_n$, then for all $i \geq n$, $\{b \in \mathcal{P}_\kappa(\kappa_i) \mid b \in \mathrm{Cone}(a)\} \in \mathcal{U}_i$. We use this ordering in the definition of the forcing and it allows us to formulate a notion of diagonal intersection which is crucial in proving that κ is a cardinal in the extension.

We are ready to define our poset $\mathbb{P}$. Conditions are of the form $p = \langle g_p, A_p \rangle$, with

$$\begin{aligned} g_p =& \langle g_p(0), \dots\ g_p(k-1) \rangle, \\ A_p =& \langle A_p(k), A_p(k+1), \dots \rangle, \end{aligned}$$

where for all $n < k$ $g_p(n) \in K_n$ and for all $n \geq k$ $A_p(n) \subseteq K_n$ and has $\mathcal{U}_n$-measure 1. Lastly, we require that $g_p = \langle g_p(0), \dots\ g_p(k-1) \rangle$ satisfy for all $n < m < k$ $g_p(m) \in \mathrm{Cone}(g_p(n))$. In the above condition we call the natural number k the *length of* p and denote it $\ell(p)$. The ordering is defined as follows. $q \leq p$ if and only if

1. g_q extends g_p, that is $g_q \restriction \mathrm{dom}\, g_p = g_p$,
2. $A_q(n) \subseteq A_p(n)$ for all $n \geq \ell(g_q)$,
3. $g_q(n) \in A_p(n)$ for all n such that $\ell(g_p) \leq n < \ell(g_q)$.

Let G be $\mathbb{P}$-generic over V. Define $g = \bigcup\{g_p \mid p \in G\}$. Then $g =_{def} \langle g(n) \mid n < \omega \rangle$ is a sequence with $g(n) \in K_n$ for all n. The intuition is that conditions in $\mathbb{P}$ are a description of g. A condition p specifies a finite initial segment of g, and gives a restriction on later terms of g, namely $g(i) = g_p(i)$ for $i < \ell(p)$ and $g(i) \in A_p(i)$ for $i \geq \ell(p)$.

By genericity $\bigcup_{n<\omega} g(n) = \nu$. So in $V[G]$, $|\nu| = \sum_{n<\omega} |g(n)| \leq \kappa$, because $|g(n)| < \kappa$ for all n. In fact for any τ such that $\kappa \leq \tau \leq \nu$, $\tau = \bigcup_{n<\omega} g(n) \cap \tau$ and $|g(n) \cap \tau| < \tau$. It follows that if τ as above is regular in V, we have $cf(\tau) = \omega$ in $V[G]$.

We show that ν^+ is preserved by an argument that is typical of Prikry forcings.

Lemma 8. *$\mathbb{P}$ is ν^+-cc*

Proof. For $A = \langle A(n) \mid k \leq n < \omega \rangle$ and $B = \langle B(n) \mid k \leq n < \omega \rangle$, define

$$A \cap B = \langle A(n) \cap B(n) \mid k \leq n < \omega \rangle.$$

If p, q are conditions and $g_p = g_q$, then p, q are compatible, because $\langle g_p, A_p \cap A_q \rangle$ is stronger than both. This shows that members of an antichain must have different stems. To determine the chain condition we count the number of stems. To count the number of stems we need a result of Solovay [18].

Theorem 9. *If κ is strongly compact and $\mu > \kappa$ is regular, then $\mu^{<\kappa} = \mu$.*

Therefore we have

$$|\{g_p \mid p \in \mathbb{P}\}| = \sum_{n<\omega} |\mathcal{P}_\kappa(\kappa_n)| = \nu.$$

Assume that there is $\langle p_\alpha \mid \alpha < \nu^+ \rangle$ an antichain in $\mathbb{P}$. Then there are conditions in the antichain with the same stem. This is a contradiction, because we just showed that such conditions are compatible. $\square$

We would like to see that cardinals below κ are preserved. In fact, we show that no bounded subsets of κ are added. To do this we prove that $\mathbb{P}$ has the *Prikry property*, which we will define below. First, we introduce some notation. Let $\varphi = \varphi(\dot{x}_1, \ldots, \dot{x}_n)$ be a statement in the forcing language. Write $h \Vdash \varphi$ (for a finite stem h) if there is some A such that $\langle h, A \rangle$ is a condition forcing φ.

Note 10. *$h \Vdash \varphi$ and $h \Vdash \neg\varphi$ is impossible, as we would have $\langle h, A \rangle \Vdash \varphi$ and $\langle h, B \rangle \Vdash \neg\varphi$, but $\langle h, A \rangle$ and $\langle h, B \rangle$ are compatible.*

We write h decides φ ($h \parallel \varphi$), if $h \Vdash \varphi$ or $h \Vdash \neg\varphi$. Next we define *diagonal intersection* which is an essential concept in the proof of the Prikry property.

Definition 11. *Let $\langle A_s \mid s \text{ is a stem} \rangle$ be a sequence such that for each s, A_s is a sequence of measure one sets such that $\langle s, A_s \rangle$ is a condition in $\mathbb{P}$. Then the* diagonal intersection *of the above sequence, $\triangle_s A_s$, is a sequence of sets whose n^{th} coordinate is the set $\{x \in \mathcal{P}_\kappa(\kappa_n) \mid \text{for all } h \text{ if } h^\frown x \text{ is a stem, then } x \in A_h(n)\}$.*

Fact 12. *The n^{th} coordinate of the diagonal intersection is measure one for $\mathcal{U}_n$.*

Fact 13. *Let $\langle A_s \mid s \text{ is a stem} \rangle$ be a sequence such that $\langle s, A_s \rangle \in \mathbb{P}$ and let A^* be their diagonal intersection. If $\langle s, A_s \rangle$ decides φ, then $\langle s, A^*(\ell(s)), A^*(\ell(s)+1)\ldots \rangle$ decides φ in the same way.*

Proof. Without loss of generality assume that $\langle s, A_s \rangle \Vdash \varphi$. Using the definition of diagonal intersection, any extension of $\langle s, A^*(\ell(s)), A^*(\ell(s)+1)\ldots \rangle$ is compatible with $\langle s, A_s \rangle$. Hence there is a dense set of conditions below $\langle s, A^*(\ell(s)), A^*(\ell(s)+1)\ldots \rangle$ which force φ. $\square$

Prikry Lemma. *For every stem h, and for every formula φ in the forcing language, $h \parallel \varphi$. Equivalently, for every condition p there is a $q \leq p$ such that q decides φ and $g_p = g_q$.*

We begin with a claim that provides the induction step in the proof of the Prikry Lemma.

Claim. *If $h \nparallel \varphi$ with $\ell(h) = k$ implies for $\mathcal{U}_k$ almost every a, $h^\frown a \nparallel \varphi$*

Proof of Claim. We prove that if $B =_{def} \{a \in \mathcal{P}_\kappa(\kappa_k) \mid h^\frown a \text{ decides } \varphi\} \in \mathcal{U}_k$, then h decides φ. Assume that $B \in \mathcal{U}_k$. For each $b \in B$ there is a sequence of measure one sets A_b such that $\langle h^\frown b, A_b\rangle$ decides φ. We can partition B into the set of those b such that $\langle h^\frown b, A_b\rangle \Vdash \varphi$ and the set of b such that $\langle h^\frown b, A_b\rangle \Vdash \neg\varphi$. Exactly one of these sets must be measure one for $\mathcal{U}_k$. Without loss of generality, we let $B' \in \mathcal{U}_k$ such that for all $b \in B'$, $\langle h^\frown b, A_b\rangle \Vdash \varphi$.

Consider the collection of stems f such that $f \restriction k = h$ and $f(k) \in B'$. If there is a sequence of measure one sets C such that $\langle f, C\rangle$ decides φ, then let A_f be one such sequence. Otherwise let $A_f = \langle \mathcal{P}_\kappa(\kappa_{\ell(f)}), \ldots\rangle$. Let A be the diagonal intersection of the sequence $\langle A_f \mid f \text{ extends } h \text{ and } f(k) \in B'\rangle$.

We claim that $\langle h, \langle B'\rangle^\frown A\rangle$ decides φ and hence h decides φ. Suppose $\langle f, C\rangle \leq \langle h, \langle B'\rangle^\frown A\rangle$. By an easy induction using the definition of diagonal intersection, for all m with $k \leq m < \ell(f)$, $f(m) \in A_{f\restriction m}(m)$. Using the fact that $f \restriction (k+1)$ forces φ and another easy induction, we see that f forces φ. So A_f was chosen so that $\langle f, A_f\rangle \Vdash \varphi$. Hence $\langle f, A_f \cap C\rangle \leq \langle f, C\rangle \leq \langle h, \langle B'\rangle^\frown A\rangle$. So there is a dense set of conditions below $\langle h, \langle B'\rangle^\frown A\rangle$ that decides φ. This finishes the claim. $\square$

Proof of the Prikry Lemma. We assume for a contradiction that there is a stem, h, and a statement φ such that $h \nparallel \varphi$. Suppose f is a stem of length $n \geq k$ extending h such that f does not decide φ. Then by the claim there is a $\mathcal{U}_n$ measure one set of extensions of f that do not decide φ. Let $A_f(n)$ be this set and let $A_f(m) = \mathcal{P}_\kappa(\kappa_m)$ for all $m > n$. Let $A_f = \langle A_f(n), A_f(n+1)\ldots\rangle$. Let A be the diagonal intersection of the sequence $\langle A_f \mid f \text{ extends } h\rangle$.

We claim that no extension of $\langle h, A\rangle$ decides φ. This will be our contradiction. Let $\langle f, B\rangle \leq \langle h, A\rangle$ with $\ell(f) = n$. An easy inductive argument using the definition of diagonal intersection shows that for all m with $k \leq m < n$, $f(m) \in A_{f\restriction m}(m)$. So by the choice of $A_{f\restriction m}(m)$ for each m as above, f does not decide φ. So no condition extending $\langle h, A\rangle$ decides φ. However it is a general forcing fact that we can always extend to decide a statement. This is a contradiction. $\square$

Lemma 14. *$\mathbb{P}$ adds no bounded subsets of κ*

Proof. Let τ and θ be cardinals with $\tau \leq \theta < \kappa$. Suppose $\dot{f}$ is a $\mathbb{P}$-name for a function from τ to θ. We assume that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{f} : \check{\tau} \to \check{\theta}$. By the Prikry Lemma, for each $\alpha < \tau$ and $\beta < \theta$ there is $A_{\alpha,\beta}$ such that $\langle \emptyset, A_{\alpha,\beta}\rangle \parallel \dot{f}(\check{\alpha}) = \check{\beta}$. Let

$$A =_{def} \bigcap_{\substack{\alpha<\tau \\ \beta<\theta}} A_{\alpha,\beta}.$$

Then $\langle \emptyset, A \rangle$ is a condition by the κ-completeness of each $\mathcal{U}_n$. Moreover, $\langle \emptyset, A \rangle$ decides $\dot{f}$ completely. So $\dot{f}[G] \in V$. $\square$

It follows easily that cardinals less than κ are preserved. So we have defined a version of diagonal Prikry forcing $\mathbb{P}$. We showed that it singularizes κ while preserving κ as a cardinal, and that it collapses ν to have size κ while preserving ν^+. The main difference in our use of this forcing will be a different choice for the cardinal ν.

3.1 Gitik-Sharon

What follows is a short summary of Gitik and Sharon's [8] use of diagonal Prikry forcing. To begin we start with the statement of a theorem of Laver [13], which is used in their work and which we will use as well.

Theorem 15. *Assuming there is a supercompact cardinal κ, then there is a forcing extension in which κ is still supercompact and remains supercompact under any κ-directed closed forcing. We say κ is* indestructibly supercompact.

For the Gitik and Sharon model, we start from κ indestructibly supercompact, with GCH holding above κ. Take $\nu = \kappa^{+\omega}$, $\kappa_n = \kappa^{+n}$. First, let $\mathbb{A} = Add(\kappa, \nu^{++})$, and take E to be $\mathbb{A}$-generic over V. Note that κ is still supercompact in $V[E]$ by Theorem 15. Now take $\mathbb{P}$ to be diagonal Prikry forcing for κ and G $\mathbb{P}$-generic over $V[E]$. In $V[E][G]$, κ is singular of cofinality ω, SCH fails at κ and the approachability property fails at κ. With more work than we have outlined above, the Gitik-Sharon paper showed that there is a very good scale on κ. Cummings and Foreman extended this to show that there is a bad scale on κ.

4 The Proof of Theorem 1

Recall the question, 'Does the tree property at κ^+ imply SCH_κ?' We will show that the answer is no. We will start with ν as a limit of ω many supercompact cardinals, κ_n for $n < \omega$. Let $\kappa = \kappa_0$ be indestructibly supercompact. Let $\mathbb{A}$ be $\text{Add}(\kappa, \nu^{++})$. Let E be $\mathbb{A}$-generic over V. Let G be $\mathbb{P}$-generic over $V[E]$. We will start by showing that ν^+ has the tree property in $V[E]$ and then we will show that it still has the tree property in $V[E][G]$.

4.1 The Tree Property in $V[E]$

Theorem 16. *ν^+ has the tree property in $V[E]$*

Proof. Let $T \in V[E]$ be a ν^+-tree. Without loss of generality level α of T is $\{\langle \alpha, \xi \rangle \mid \xi < \nu\}$. Fortunately, the Spine Lemma is exactly as before.

Spine Lemma 2. *In $V[E]$, there is a $C \subseteq \nu^+$ cofinal and $n < \omega$, such that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa_n$ such that $\langle \alpha, \xi \rangle \; T \; \langle \beta, \zeta \rangle$*

Using the indestructibility of κ, we have that κ is still supercompact in $V[E]$. Therefore the argument from the proof of Spine Lemma 1 works. Next we reformulate the Traction Lemma. Before we used a supercompactness embedding $\pi : V \to M$ with critical point point κ_{n+1}. This time we will lift the elementary embedding π to the universe of $V[E]$ by passing to a further extension $V[E][F]$. The use of the embedding is the same, but additional work is needed to show that the forcing we used to add the embedding did not add the branch through T.

Traction Lemma 2. *In $V[E]$ there is a $J \subseteq C$ cofinal and a map $\alpha \mapsto \xi_\alpha$ such that for all $\alpha < \beta$ both in J, $\langle \alpha, \xi_\alpha \rangle \, T \, \langle \beta, \xi_\beta \rangle$*

Proof. Let F be $\mathrm{Add}(\kappa, \pi(\nu^{++}))$-generic over $V[E]$. In $V[E][F]$, we claim that π can be extended to an elementary embedding $\pi^* : V[E] \to M[E^*]$ for some $E^* \in V[E][F]$ which is M-generic for $\pi(\mathbb{A})$. By work of Silver, it suffices to arrange that $\pi``E \subseteq E^*$. We can do this by interleaving the generic F with the pointwise image of E under π to create the generic E^*. We do this working in $V[E][F]$. Enumerate F as $\langle f_\alpha : \alpha < \pi(\nu^{++}) \rangle$ where each f_α is a function from κ to 2. Let $\langle g_\alpha : \alpha < \nu^{++} \rangle$ be a similar enumeration of E. Now let $E^* =_{def} \langle h_\beta : \beta < \pi(\nu^{++}) \rangle$, where $h_{\pi(\alpha)} = g_\alpha$ for each $\alpha < \nu^{++}$ and $h_\beta = f_\alpha$ where β is the α^{th} member of $\pi(\nu^{++}) \smallsetminus \pi``\nu^{++}$. It is easy to see that $E^*$ is $M$-generic for $\pi(\mathbb{A})$ and that $\pi``E \subseteq E^*$. Hence we can lift the embedding to the generic extension $V[E]$.

Now we repeat the proof of Traction Lemma 1 using π^*. So in $V[E][F]$ we get that there is a $J \subseteq C$ cofinal and $\alpha \mapsto \xi_\alpha$ for $\alpha \in J$ such that for $\alpha < \beta$ both in J, $\langle \alpha, \xi_\alpha \rangle \, T \, \langle \beta, \xi_\beta \rangle$. So we got a branch through T in $V[E][F]$, call it b. We want to show that $b \in V[E]$. This will follow from the next lemma.

Note 17. *In the statement of the lemma below we use the notation $\mathbb{B}^\lambda$ for a power of the poset $\mathbb{B}$. This notation is ambiguous. However the support of the power that is used is not important so long as we have the hypotheses.*

Lemma S. *Let θ be a cardinal. Let S be a tree of height θ. Let $\mathbb{B}$ be a poset. Assume that*

1. *$\mathbb{B} \times \mathbb{B}$ is $\mathrm{cof}(\theta)$-cc,*
2. *$\mathbb{B}^{|S|^+}$ does not collapse $|S|^+$.*

Then $\mathbb{B}$ does not add cofinal branches through S. More precisely, if F is $\mathbb{B}$-generic over V and $b \in V[F]$ is a branch through S, then $b \in V$.

Note 18. *In the above formulation S need not be a θ-tree.*

We will use this lemma in $V[E]$ with $S = T$, $\theta = \nu^+$ and $\mathbb{B} = Add(\kappa, \pi(\nu^{++}))$. We need to check that the hypotheses hold. $\mathbb{B} \times \mathbb{B}$ is κ^+-cc, because it is $Add(\kappa, \pi(\nu^{++}) + \pi(\nu^{++}))$. Also $\mathbb{B}^{\nu^{++}}$ is $Add(\kappa, \nu^{++} \cdot \pi(\nu^{++}))$ if we use supports of size $< \kappa$. Hence $\mathbb{B}^{\nu^{++}}$ is κ^+-cc and does not collapse ν^{++}. So we have finished with Traction Lemma 2 except for the proof of Lemma S.

Proof of Lemma S. We can assume that θ is regular. If θ were not regular, then we could replace it with $\mathrm{cof}(\theta)$ and S by its restriction to $\mathrm{cof}(\theta)$ many levels cofinal in θ. Let $b \in V[F]$ be a cofinal branch through S. Fix a name $\dot{b}$ such that $\dot{b}[F] = b$. Suppose for a contradiction that $\Vdash_{\mathbb{B}} \dot{b} \notin \check{V}$

We force with $\mathbb{B}^* = \mathbb{B}^{|S|^+}$, letting F^* be $\mathbb{B}^*$-generic over V. We write F^* as a product of generics $\prod_{\delta<|S|^+} F_\delta$. Let $b_\delta = \dot{b}[F_\delta]$. Using the assumption that $\Vdash_{\mathbb{B}} \dot{b} \notin V$, it follows that for all $\delta_1, \delta_2 < |S|^+$, $b_{\delta_1} \neq b_{\delta_2}$. To show this we consider $\mathbb{B} \times \mathbb{B}$. Let $\dot{b}_{left}$ and $\dot{b}_{right}$ be the $\mathbb{B} \times \mathbb{B}$-names for the interpretations of $\dot{b}$ by the left and right generics. The assumption $\Vdash_{\mathbb{B}} \dot{b} \notin \check{V}$ implies $\Vdash_{\mathbb{B}\times\mathbb{B}} \dot{b}_{left} \neq \dot{b}_{right}$. Since $F_{\delta_1} \times F_{\delta_2}$ is generic for $\mathbb{B} \times \mathbb{B}$, we have $b_{\delta_1} \neq b_{\delta_2}$.

Let $H \prec V_\rho$ for a sufficiently large regular cardinal ρ. Since θ is regular, we can arrange that $\{\theta, S, \dot{b}, \mathbb{B}, \mathbb{B}^*\} \subseteq H$, $H \cap \theta$ is an ordinal and $|H| < \theta$. Since $\mathbb{B}$ is θ-cc, each antichain of $\mathbb{B}$ in H, is contained in H. So for all δ, F_δ is $\mathbb{B}$-generic over H, $H[F_\delta] \prec V_\rho[F_\delta]$ and $H[F_\delta] \cap V = H$. We can argue similarly for $\mathbb{B} \times \mathbb{B}$ and $F_{\delta_1} \times F_{\delta_2}$ for any δ_1, δ_2.

Note 19. *We are essentially arguing that $\mathbb{B} \times \mathbb{B}$ satisfies a version of properness for an arbitrary regular cardinal θ.*

Let $\eta = H \cap \theta$. Working in $V[F^*]$, for each δ let β_δ be the node of b_δ on level η. There are $|S|$ possibilities for β_δ and by assumption $|S|^+$ is a cardinal in $V[F^*]$. So there are δ_1 and δ_2 such that $\beta_{\delta_1} = \beta_{\delta_2}$. We will work with $F_{\delta_1} \times F_{\delta_2}$ as a generic for $\mathbb{B} \times \mathbb{B}$. Recall $\Vdash_{\mathbb{B}\times\mathbb{B}} \dot{b}_{left} \neq \dot{b}_{right}$ So by elementarity of $H[F_{\delta_1} \times F_{\delta_2}]$ in $V_\rho[F_{\delta_1} \times F_{\delta_2}]$, there is a condition $\langle p_1, p_2\rangle \in (F_{\delta_1} \times F_{\delta_2}) \cap H$ forcing this. Again using elementarity we can extend $\langle p_1, p_2\rangle$ to $\langle p_1', p_2'\rangle \in F_{\delta_1} \times F_{\delta_2} \cap H$ such that there is $\gamma \in H$ with

$$\langle p_1', p_2'\rangle \Vdash \dot{b}_{left}(\gamma) \neq \dot{b}_{right}(\gamma).$$

This implies that $b_{\delta_1}(\gamma) \neq b_{\delta_2}(\gamma)$. As S is a tree and $\gamma < \eta$ we get $b_{\delta_1}(\eta) \neq b_{\delta_2}(\eta)$, a contradiction. □

This finishes the proofs of both Traction Lemma 2 and Theorem 16. By Lemma S, the branch that we found above is in $V[E]$ and thus the tree property holds in $V[E]$ at ν^+. □

□

4.2 The Tree Property in $V[E][G]$

In this section the proof becomes more difficult. We show that the tree property holds at $\nu^+ = \kappa^+$ in $V[E][G]$. Let $T \in V[E][G]$. Without lost of generality, level α of T is $\{\langle\alpha, \xi\rangle \mid \xi < \kappa\}$. Let $\dot{T} \in V[E]$ such that $\dot{T}[G] = T$. We assume that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{T}$ is a tree with the above form". Working in $V[E]$ we formulate the Spine Lemma.

Spine Lemma 3. *In $V[E]$ there are $n < \omega$ and $C \subseteq \nu^+$ cofinal so that for all $\alpha < \beta$ both in C, there are $\xi, \zeta < \kappa$ and a stem h of length n such that $h \Vdash \langle \alpha, \xi \rangle \, \dot{T} \, \langle \beta, \zeta \rangle$.*

Proof. As in the proof of Spine Lemma 2, we fix $\pi : V[E] \to M$, a ν^+-supercompactness embedding with critical point κ in $V[E]$. Let G^* be $\pi(\mathbb{P})$-generic over M. Let $T^* = \pi(\dot{T})[G^*]$. Let $\gamma^* = \sup \pi``\nu^+$ and fix η^* such that $\langle \gamma^*, \eta^* \rangle$ is a node of T^* on level γ^*.

Working in $M[G^*]$, for each $\alpha < \nu^+$ we fix ξ^*_α such that $\langle \pi(\alpha), \xi^*_\alpha \rangle T^* \langle \gamma^*, \eta^* \rangle$. There is a condition $p_\alpha \in G^*$ forcing this and we let n_α be the length of the stem of p_α. The sequences $\langle \xi^*_\alpha \mid \alpha < \nu^+ \rangle$ and $\langle n_\alpha \mid \alpha < \nu^+ \rangle$ are in $M[G^*]$, since $\pi \restriction \nu^+$ belongs to M. Note that ν^+ is a cardinal in $M[G^*]$ and $\nu^+ < \pi(\kappa)$. So we can find an unbounded set of $\alpha < \nu^+$ and a fixed n such that $n_\alpha = n$. Let h^* be the unique stem of length n of some condition in G^*. Then define C to be the set of $\alpha < \nu^+$ such that there are a condition in $r \in \pi(\mathbb{P})$ with stem h^* and an ordinal ξ^*_α such that $r \Vdash \langle \pi(\alpha), \xi^*_\alpha \rangle \, \pi(\dot{T}) \, \langle \gamma^*, \eta^* \rangle$. Our definition of C does not make use of G^* and hence $C \in M$. Moreover, C is unbounded by the choice of h^*.

Claim. *This C, n satisfy the requirements of Spine Lemma 3*

Note that in this situation we have $M \subseteq V[E]$ and hence $C \in V[E]$. If $\alpha < \beta$ are both in C, then

$$h^* \Vdash \langle \pi(\alpha), \xi^*_\alpha \rangle \, \pi(\dot{T}) \, \langle \gamma^*, \eta^* \rangle \text{ and } \langle \pi(\beta), \xi^*_\beta \rangle \, \pi(\dot{T}) \, \langle \gamma^*, \eta^* \rangle.$$

Since $\Vdash \pi(\dot{T})$ is a tree, $h^* \Vdash \langle \pi(\alpha), \xi^*_\alpha \rangle \, \pi(\dot{T}) \, \langle \pi(\beta), \xi^*_\beta \rangle$.

So $M \models$ "There are a stem h of length n, and $\xi, \zeta < \kappa$ such that $h \Vdash \langle \pi(\alpha), \xi \rangle \, \pi(\dot{T}) \, \langle \pi(\beta), \zeta \rangle$." By elementarity, there is a stem $h \in V[E]$ of length n such that $h \Vdash \langle \alpha, \xi \rangle \, \dot{T} \, \langle \beta, \zeta \rangle$. $\square$

Traction Lemma 3. *In $V[E]$ there are $J \subseteq C$ cofinal, a map $\alpha \mapsto \xi_\alpha (\alpha \in J)$ and a stem $\bar{h}$ such that for all $\alpha < \beta$ both in J, $\bar{h} \Vdash \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$.*

Remark 20. *Our notation is deceptive. This will not finish the proof. We could have $\bar{h}^\frown a \Vdash \neg \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$!*

Proof. Let $\pi : V \to M$ be a ν^+-supercompactness embedding with critical point κ_{n+1}, where n is given by Spine Lemma 3. Let F be $Add(\kappa, \pi(\nu^{++}))$-generic over $V[E]$. Again in $V[E][F]$, π extends to $\pi^* : V[E] \to M[E^*]$. As before let γ^* be least in $\pi^*(C)$ greater than $\sup \pi^* ``\nu^+$. We apply Spine Lemma 3 in $M[E^*]$. For each $\alpha \in C$, we have $\xi^*_\alpha, \eta^*_\alpha$ and h^*_α such that

$$h^*_\alpha \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \, \pi^*(\dot{T}) \, \langle \gamma^*, \eta^*_\alpha \rangle$$

with $\ell(h^*_\alpha) = n$.

We are going to stabilize h^*_α and η^*_α. In $V[E][F]$, there is $J \subseteq C$ cofinal and a fixed stem $\bar{h}$ of length n and an $\eta < \kappa$ such that $\alpha \in J$ implies $\eta^*_\alpha = \eta$ and

$h^*_\alpha = \bar{h}$. In the above we used that $\mathrm{crit}(\pi^*) = \kappa_{n+1}$, to obtain the fact that stems of length n are the same in $\pi^*(\mathbb{P})$ and $\mathbb{P}$. So $\alpha < \beta$ both in J implies

$$\bar{h} \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \, \pi^*(\dot{T}) \, \langle \gamma^*, \eta \rangle,$$
$$\bar{h} \Vdash \langle \pi^*(\beta), \xi^*_\beta \rangle \, \pi^*(\dot{T}) \, \langle \gamma^*, \eta \rangle.$$

So $\bar{h} \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \, \pi^*(\dot{T}) \, \langle \pi^*(\beta), \xi^*_\beta \rangle$.

Set $\xi_\alpha = \xi^*_\alpha$. Note $\pi^*(\xi_\alpha) = \xi^*_\alpha$ and $\pi^*(\bar{h}) = \bar{h}$. So $\bar{h} \Vdash_{\mathbb{P}} \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$ by elementarity. This almost finishes the proof. In $V[E][F]$ we have the map $\alpha \mapsto \xi_\alpha$ for $\alpha \in J$, but we need to pull this back to $V[E]$.

We apply Lemma S. We do this by viewing the above map as a branch through a particular tree, a tree of attempts to create such a map. Without loss of generality, we may assume that J is maximal, by which we mean if $\beta \in J$ and $\alpha < \beta$ such that there is a ξ with $\bar{h} \Vdash \langle \alpha, \xi \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$ then $\alpha \in J$ and $\xi = \xi_\alpha$. Again we are using that $\Vdash \dot{T}$ is a tree. The fact that J is maximal will allow us to code J as a branch through a tree of height nu^+. Let f be the function $i \mapsto (\alpha_i, \xi_{\alpha_i})$ where $i \mapsto \alpha_i$ enumerates J in increasing order. Note that for $i < \nu^+$, $f \restriction i \in V[E]$, because by maximality $f \restriction i$ is determined in $V[E]$ from $(\alpha_i, \xi_{\alpha_i})$ and $\bar{h}$. So f is a branch through a tree in $V[E]$ of length ν^+, namely the tree of attempts to construct such a function. We have already checked that $Add(\kappa, \pi(\nu^{++}))$ satisfies the hypotheses of the poset in Lemma S. Hence $f \in V[E]$ as required. $\square$

Are we done? *No!* Let $\bar{h}$ witness the lemma and let $\ell(\bar{h}) = \bar{k}$. We can assume that $g = \bigcup\{g_p \mid p \in G\}$ extends $\bar{h}$. However $\{\langle \alpha, \xi_\alpha \rangle \mid \alpha \in J\}$ is not necessarily a branch. We know that for all $\alpha < \beta$ both in J, $\bar{h} \Vdash \langle \alpha, \xi_\alpha \rangle \dot{T} \langle \beta, \xi_\beta \rangle$. However there might be an $a \in \mathcal{P}_\kappa(\kappa_{\bar{k}})$ such that $\bar{h}^\frown a \Vdash \neg \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$. The set of such a has measure zero, but need not be empty. For all we know $g(\bar{k})$ is such an a, then we would have $\neg \langle \alpha, \xi_\alpha \rangle \, \dot{T}[G] \, \langle \beta, \xi_\beta \rangle$ in the extension.

4.2.1 The Next Step

The final step in the proof is to get the following.

Fact 21. *In $V[E]$ there are $\rho < \nu^+$ and a sequence $\langle A_\alpha \mid \alpha \in J \setminus \rho \rangle$ with each A_α an ω-sequence of measure one sets, such that for all $\alpha < \beta$ both in $J \setminus \rho$, $\langle \bar{h}, A_\alpha \cap A_\beta \rangle \Vdash \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle$*

Recall that A_α and A_β are sequences of measure one sets and that we intersect them pointwise. We will show that Fact 21 is enough to finish the proof of Theorem 1.

Claim. *If G is $\mathbb{P}$-generic over $V[E]$, then G contains cofinally many of the conditions $\langle \bar{h}, A_\alpha \rangle$.*

Proof. Assume that G only contains boundedly many of the above conditions and fix a condition q_0 forcing this. Note that any condition can be extended to one that satisfies the conclusion of Spine Lemma 3 and Traction Lemma 3. We can assume that q_0 satisfies the conclusions of the lemmas and we let $g_{q_0} = \bar{h}$, J, ρ and $\langle A_\alpha \mid \alpha \in J \setminus \rho \rangle$ witness this. By the ν^+-cc of our forcing, there is an $\alpha_0 < \nu^+$ such that q_0 forces for all $\alpha > \alpha_0$ in $J \setminus \rho$, $\langle \bar{h}, A_\alpha \rangle \notin G$. We take $\alpha \in J$ above α_0. By our choice of q_0, q_0 and $\langle \bar{h}, A_\alpha \rangle$ are compatible. Let r be their common extension. Obviously, $r \Vdash \langle \bar{h}, A_\alpha \rangle \in G$. However, we also have $r \Vdash \langle \bar{h}, A_\alpha \rangle \notin G$, because $r \leq q_0$. This is a contradiction. □

To finish the proof of Theorem 1, we note that if G meets cofinally many of the conditions $\langle \bar{h}, A_\alpha \rangle$, then there is a branch through T in $V[E][G]$. This is easy by the choice of the A_α. So it remains to construct the sets A_α. The idea is to construct $A_\alpha(n)$ by recursion on $n \geq \bar{k}$. In these notes, we will show how to do the construction for $n = \bar{k}$. For the full construction we refer the reader to [16]. The next lemma is the appropriate weakening of Fact 21.

Final Lemma. *In V[E] there are $\rho < \nu^+$ and sets Z_α for $\alpha \in J \setminus \rho$, such that each Z_α has $\mathcal{U}_{\bar{k}}$-measure one and for all $\alpha < \beta$ both in $J \setminus \rho$, for all $a \in Z_\alpha \cap Z_\beta$, $\bar{h}^\frown a \Vdash \langle \alpha, \xi_\alpha \rangle \; \dot{T} \; \langle \beta, \xi_\beta \rangle$.*

Before proceeding with the proof, we will explain a failed attempt which is quite instructive. Fix $\pi : V \to M$ a supercompactness embedding with critical point $\kappa_{\bar{k}+1}$. As before we can lift π to the universe $V[E]$ to get an elementary embedding $\pi^* : V[E] \to M[E^*]$ where E^* is generic for $\pi(Add(\kappa, \nu^{++}))$. As before we can take $\gamma^* > \sup \pi^{*}``\nu^+$, with $\gamma^* \in \pi^*(J)$. Then there is η^* such that for each α, we get A^*_α and ξ^*_α such that for all $x \in A^*_\alpha$, $\bar{h}^\frown x \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \; \pi^*(\dot{T}) \; \langle \gamma^*, \eta^* \rangle$. Here η^* is just $\pi^*(\alpha \mapsto \xi_\alpha)(\gamma^*)$. A^*_α has $\pi^*(\mathcal{U}_{\bar{k}})$-measure one. Note that $\mathcal{P}_\kappa(\kappa_{\bar{k}})$ is the same computed in both $V[E]$ and $V[E][F]$, since $Add(\kappa, \pi(\nu^{++}))$ is κ-closed. However, the powerset of $\mathcal{P}_\kappa(\kappa_{\bar{k}})$ is larger, so $\pi^*(\mathcal{U}_{\bar{k}})$ measures more sets. Also, elements of the powerset are fixed by π^*, so we have $\mathcal{U}_{\bar{k}} \subseteq \pi^*(\mathcal{U}_{\bar{k}})$. It follows that A^*_α need not be in $V[E]$ let alone measure one for $\mathcal{U}_{\bar{k}}$. So this will not work.

Proof of the Final Lemma. The key idea is to work "vertically" instead of "horizontally". A vertical segment will use a version of J for $h^\frown x$. So fix π^*, γ^* and η^* as in the last paragraph. For each $x \in \mathcal{P}_\kappa(\kappa_{\bar{k}})$, let $J_x = \{\alpha \in J \mid \bar{h}^\frown x \Vdash \langle \pi(\alpha), \xi_\alpha \rangle \; \pi^*(\dot{T}) \; \langle \gamma^*, \eta^* \rangle\}$. The "horizontal" sets are $\{x \mid \alpha \in J_x\}$. They have $\pi(\mathcal{U}_{\bar{k}})$-measure one, but they need not be in $V[E]$. So we are going to look at the "vertical" segments J_x.

Claim. *If J_x is unbounded in ν^+, then $J_x \in V[E]$*

Proof. We apply Lemma S in the same way that we did for J. If J_x is unbounded, then it is coded by a branch through a tree of height ν^+ as follows. Again we assume that J_x is maximal in the sense that for all $\alpha \in J_x$ and all $\beta < \alpha$ with $\beta \in J$, if $h^\frown x \Vdash \langle \beta, \xi_\beta \rangle \; \dot{T} \; \langle \alpha, \xi_\alpha \rangle$, then $\beta \in J_x$. We let $f_x \in V[E][F]$ be the map $i \mapsto (\alpha_i, \xi_{\alpha_i})$ that enumerates J_x on the first coordinate in increasing order. Then for all $i < \nu^+$, $f_x \restriction i \in V[E]$, since it is determined from $(\alpha_i, \xi_{\alpha_i})$, $\bar{h}$

and x. Hence f_x is a branch through a tree of attempts to find it in $V[E]$. The other parameters in the application of Lemma S are the same as before. $\square$

Let $\dot{J}_x \in V[E]$ be a name for J_x in $Add(\kappa, \pi(\nu^{++}))$. (Recall that F was the generic object for this poset.) Since $Add(\kappa, \pi(\nu^{++}))$ is κ^+-cc, there is a set $K_x \in V[E]$ of size $\leq \kappa$ such that $\Vdash_{Add(\kappa,\pi(\nu^{++}))}$ "$\dot{J}_x \in K_x$, if $\dot{J}_x$ is unbounded in ν^+". By shrinking K_x, we may assume that for each $I \in K_x$

1. I is unbounded in ν^+
2. $(\beta \in I \wedge \alpha < \beta \wedge \alpha \in J) \Rightarrow (\alpha \in I \Leftrightarrow h^\frown x \Vdash \langle \alpha, \xi_\alpha \rangle \, \dot{T} \, \langle \beta, \xi_\beta \rangle)$

We call condition 2 *maximality.* Note that maximality is essentially the same condition we used in the applications of Lemma S. We can assume that each member of K_x is maximal since any unbounded I as above has a unique extension to an unbounded set which is maximal. This follows from the fact that $\dot{T}$ is forced to be a tree. In $V[E]$ we have the map $x \mapsto K_x$ where K_x is the collection of candidates for J_x. We work to refine our knowledge of each K_x and its members.

Claim. *If $I, I' \in K_x$ are distinct, then they are disjoint on a tail.*

Proof. If there is a place, β, where I, I' agree then by maximality they agree below β. So after the first place where I, I' differ, they are disjoint. $\square$

Corollary 22. *For each x, there is $\rho_x < \nu^+$ such that if $I, I' \in K_x$ are distinct, then they are disjoint above ρ_x.*

Proof. Recall that $|K_x| \leq \kappa$. So to find ρ_x we take a supremum over the least place where any pair from K_x differ. ρ_x is a supremum over just $|K_x|^2$ many ordinals less than ν^+ and hence it is less than ν^+. $\square$

Let $\rho = \sup_{x \in \mathcal{P}_\kappa(\kappa_{\bar{k}})} \rho_x$. Then $\rho < \nu^+$, since $|\mathcal{P}_\kappa(\kappa_{\bar{k}})| = \kappa_{\bar{k}} < \nu^+$. So for any x and for any $I, I' \in K_x$ which are distinct, I, I' are disjoint above ρ. We are going to work above ρ. Define a function f on $\mathcal{P}_\kappa(\kappa_{\bar{k}}) \times (J \setminus \rho)$ by $f(x, \alpha) =$ the unique $I \in K_x$ such that $\alpha \in I$, if such I exists and $f(x, \alpha)$ is undefined otherwise.

Claim. *For $\alpha \in J \setminus \rho$, $\{x \mid f(x, \alpha)$ is defined$\}$ has $\mathcal{U}_{\bar{k}}$-measure one.*

Proof. Fix $\alpha \in J \setminus \rho$. First note that $f \in V[E]$ and hence $\{x \mid f(x, \alpha)$ is defined $\} \in V[E]$. Let Y be its complement. Suppose for a contradiction that Y has $\mathcal{U}_{\bar{k}}$-measure one.

Here we actually need the sets from our failed attempt at a proof of the Final Lemma. Recall, A^*_α was measure one for $\pi^*(\mathcal{U}_{\bar{k}})$. Furthermore, we had the property that there is η^* such that for every α, there is ξ^*_α such that for all $x \in A^*_\alpha$, $\bar{h}^\frown x \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \, \pi^*(\dot{T}) \, \langle \gamma^*, \eta^* \rangle$. As $\mathrm{crit}(\pi^*) = \kappa_{\bar{k}+1}$, $\pi^*(Y) = Y$. By elementarity Y has $\pi^*(\mathcal{U}_{\bar{k}})$-measure one. For every $\beta \in J$, the intersection of measure one sets $A^*_\alpha \cap A^*_\beta \cap Y$ is nonempty. For each β, let $x_\beta \in A^*_\alpha \cap A^*_\beta \cap Y$.

As J is unbounded in ν^+ and $|\mathcal{P}_\kappa(\kappa_{\bar{k}})| = \kappa_{\bar{k}}$, there are a fixed $x \in \mathcal{P}_\kappa(\kappa_{\bar{k}})$ and $U \subseteq J$ unbounded, such that $x = x_\beta$ for all $\beta \in U$. By the construction of the A^*_βs, we have $\bar{h}^\frown x \Vdash \langle \pi^*(\alpha), \xi^*_\alpha \rangle \; \pi^*(\dot{T}) \; \langle \gamma^*, \eta^* \rangle$ and $\langle \beta, \xi^* \beta \rangle \; \pi^*(\dot{T}) \; \langle \gamma^*, \eta^* \rangle$ for all $\beta \in U$. So by the definition of J_x, $\alpha \in J_x$ and $U \subseteq J_x$. But this means that $f(x, \alpha)$ was defined and equal to J_x, a contradiction. □

Claim. *For α, α' both in $J \setminus \rho$, the set $\{x \mid f(x, \alpha) = f(x, \alpha')\}$ has $\mathcal{U}_{\bar{k}}$-measure one.*

Proof. By the previous claim there is a measure one set where both are defined. Fix x and suppose that $f(x, \alpha)$ and $f(x, \alpha')$ are both defined. Without loss of generality $\alpha < \alpha'$. We claim that $\alpha \in f(x, \alpha')$. Using maximality and the fact that $\alpha, \alpha' > \rho$, it suffices to check that $\alpha' \in f(x, \alpha')$, $\alpha < \alpha'$, $\alpha \in f(x, \alpha)$ and $\bar{h}^\frown x \Vdash \langle \alpha, \xi_\alpha \rangle \; \dot{T} \; \langle \alpha', \xi_{\alpha'} \rangle$. The first three are obvious, and the last one follows from the fact that $f(x, \alpha), f(x, \alpha')$ are candidates for J_x and that $\Vdash \dot{T}$ is a tree. This finishes the proof as we have shown that everywhere $f(x, \alpha), f(x, \alpha')$ are defined they are equal. □

We are ready to define the measure one sets Z_α. Let α_0 be the least element of $J \setminus \rho$. Define $Z_\alpha = \{x \mid f(x, \alpha) = f(x, \alpha_0) \text{ where both are defined}\}$. By the previous claims Z_α has $\mathcal{U}_{\bar{k}}$-measure one. If $x \in Z_\alpha \cap Z_\beta$ then let $I = f(x, \alpha) = f(x, \beta) = f(x, \alpha_0)$ Recall, I is maximal so $\bar{h}^\frown x \Vdash \langle \alpha, \xi_\alpha \rangle \; \dot{T} \; \langle \beta, \xi_\beta \rangle$, as required. □

The complete construction of the sequences of measure one sets A_α mentioned above works by recursion on the length of a stem h extending $\bar{h}$. Using suitable inductive hypotheses, Neeman constructs J^h, ρ_h, which are analogs of the J, ρ that we constructed above. From J^h and ρ_h, Neeman obtains measure one sets A^h_α which are analogs of the Z_α that we constructed. To finish the proof, each A_α is essentially the diagonal intersection of the sets A^h_α for h extending $\bar{h}$.

5 Open Problem

We proved that the tree property at κ^+ does not imply SCH_κ.

1. Does ν^+ still have the tree property after cardinal preserving forcing?
2. Can we make κ of the result into $\aleph_\omega$ or some other small cardinal?

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