1. Find all the values of $(-8i)^{1/3}$. Sketch the solutions. 1 pt.

**Answer:** We start by writing $-8i$ in polar form and then we’ll compute the cubic root:

$$(-8i)^{1/3} = (8e^{-i\pi/2})^{1/3} = 8^{1/3} \exp \left(i\left(\frac{-\pi}{6} + \frac{2\pi k}{3}\right)\right),$$

Hence $z_0 = 2 \exp(-i\pi/6)$, $z_1 = 2 \exp(i\pi/2) = 2i$, $z_2 = 2 \exp(i7\pi/6)$.

2. Suppose $v$ is harmonic conjugated to $u$, and $u$ is harmonic conjugated to $v$. Show that $u$ and $v$ must be constant functions. 1 pt.

**Answer** By definition, $v$ is harmonic conjugated to $u$ if $\Delta u = \Delta v = 0$ and $u_x = v_y, u_y = -v_x$ hold. On the other hand, as $u$ is harmonic conjugated to $v$, we also have $v_x = u_y$ and $v_y = -u_x$. Then,

$$u_y = -v_x = v_x \text{ implies } v_x = 0 \text{ and } u_y = 0$$

and

$$u_x = v_y = -v_y \text{ implies } v_y = 0 \text{ and } u_x = 0.$$  

Hence, $u(x, y)$ and $v(x, y)$ are constant functions.

3. Show that $f(z) = |z|^2$ is differentiable at the point $z_0 = 0$ but not at any other point. 1 pt.
Figure 1: Roots of $(-8i)^{1/3}$.

**Answer:** We could show that $f$ is not differentiable at any $z_0 \in \mathbb{C} - 0$ by computing different limits for different trajectories. But we’ll use the necessary conditions of differentiability (i.e. the Cauchy-Riemann equations) to do so.

First, we write $f(z) = u(x, y) + iv(x, y)$, so $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Clearly, the first-order partials of $u$ and $v$ are continuous everywhere. But 

$$u_x = 2x \text{ equals } v_y = 0,\text{ and}$$

$$u_y = 2y \text{ equals } -v_x = 0$$

if and only if $x = y = 0$. Then $f$ can only be differentiable at $z_0 = 0$.

4. Construct a branch of $f(z) = \log(z + 4)$ that is analytic at $z = -5$ and takes on the value $7\pi i$ there. Sketch the branch cut in the complex plane.

**1 pt.**

**Answer:** Let $\alpha = 6\pi$. Then define the branch of log as

$$\log_\alpha(z + 4) = \ln|z + 4| + i\theta$$

where $6\pi < \theta \leq 8\pi$. Thus

$$\log_\alpha(-5 + 4) = \log_\alpha(-1) = \ln|-1| + i(\pi + 6\pi)$$

that is $\log_\alpha(-5 + 4) = 7\pi i$.

5. Find

$$\int_C \frac{\exp(z)}{z^n} \, dz$$
\[ \alpha = 6\pi \]

Figure 2: The branch cut of \( \log_{6\pi} \).

where \( C : z(t) = e^{2\pi it}, \ 0 \leq t \leq 1 \) and \( n \) is any non-negative integer.

\[ \text{Answer: Let } f(z) = \exp(z). \text{ Since } z_0 = 0 \text{ lies inside the given contour and } f \text{ is analytic in the interior of } C, \text{ then by Cauchy’s integral formula for derivatives} \]

\[
\int_C \frac{\exp(z)}{z^n} \, dz = \frac{2\pi i f(z_0)}{(n-1)!} = \frac{2\pi i}{(n-1)!}
\]

for \( n = 1, 2, \ldots \). For \( n = 0 \), the integrand reduces to \( f(z) \) which is an entire function. Hence, by Cauchy-Goursat theorem, we conclude

\[
\int_C \frac{\exp(z)}{z^n} \, dz = 0 \quad n = 0.
\]

6. Find two Laurent series expansions for

\[ f(z) = \frac{1}{z^3 - z^4} \]

that involves powers of \( z \). Use the regions \( 0 < |z| < 1 \) and \( |z| > 1 \).

\[ \text{1 pt.} \]

\[ \text{Answer: The given function has two singularities at } z_0 = 0 \text{ and } z_1 = 1. \text{ We’ll proceed by expanding } f \text{ in the regions, } D_0 : 0 < |z| < 1 \text{ and } D_1 : |z| > 1. \]
At $D_0$,
\[ f(z) = \frac{1}{z^3} \frac{1}{1 - \frac{1}{z}} = \frac{1}{z^3} \sum_{n=0}^{\infty} z^n, \]
for $0 < |z| < 1$. Then
\[ f(z) = \sum_{n=0}^{\infty} z^{n-3} \]
\[ = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n, \]
valid at $D_0$.

At $D_1$, $|z| > 1$ implies $1/|z| < 1$. Then we must write $f(z)$ as
\[ \frac{1}{z^3} \frac{1}{1 - \frac{1}{z}} = \frac{1}{z^3} \frac{-1}{1 - \frac{1}{z}}. \]
Hence,
\[ f(z) = -\frac{1}{z^4} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=4}^{\infty} \frac{1}{z^n}, \]
valid at $D_1$.

7. Show
1. \[ \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1, \]
2. \[ \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n, \quad |z - 1| < 1. \]

1 pt.

Answers:
(a) Using the Taylor series for $1/(1 - z) = \sum_{n=0}^{\infty} z^n$, valid at $|z| < 1$, we obtain
\[ \frac{1}{1 + z} = \frac{1}{1 - (-z)} = \sum_{n=0}^{\infty} (-1)^n z^n \]
valid also at $|z| < 1$. 

4
(b) In this case, we’ll use the result from question (a):

\[
\frac{1}{z} = \frac{1}{1 + (z-1)} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n,
\]

valid for \(|z - 1| < 1\).

8. Use Cauchy’s Residue Theorem to evaluate the next integral around the positively oriented contour \(C : \, |z| = 3\)

\[
\int_{C} \frac{1}{1 + z^2} \, dz.
\]

1 pt.

**Answer** Let \(p(z) = 1\) and \(q(z) = 1 + z^2\), so \(f(z) = p(z)/q(z)\) represents the integrand given. Notice that \(q(z)\) has two zeros at \(z_0 = i\) and \(z_1 = -i\), and both zeros are simple since \(q'(z) = 2z = 0\) if and only if \(z = 0\). Then \(f\) has two simple poles at \(z_0\) and \(z_1\), as \(p\), being a constant function, has no zeros at all. By Cauchy’s residue theorem, we obtain

\[
\int_{C} \frac{1}{1 + z^2} \, dz = 2\pi i \left( \text{Res}_{z=z_0} \frac{1}{1 + z^2} + \text{Res}_{z=z_1} \frac{1}{1 + z^2} \right)
= 2\pi i \left( \frac{1}{2i} + \frac{1}{-2i} \right)
= 0
\]

after evaluating \(q'(z)\) at \(z_0\) and \(z_1\).

9. State and prove Liouville’s theorem.

1 pt.

**Answer:** Let \(f \) be an entire function. If \(f\) is bounded, then \(f\) reduces to a constant function.

Assume there exists \(M > 0\) such that \(|f| \leq M\) in the whole complex plane. Fix any point \(z_0 \in \mathbb{C}\). For any \(R > 0\), Cauchy’s inequality states

\[
|f'(z_0)| \leq \frac{M}{R}
\]

when we restrict \(f\) to the circular contour \(C_R : \, |z - z_0| = R\). As \(R\) tends to \(\infty\), the absolute value of \(f'(z_0)\) becomes zero. That implies

\[
u_x = v_x = 0 \text{ and } u_y = v_y = 0,
\]
at \((x_0, y_0)\). Since \(z_0\) was arbitrary, it follows that \(u\) and \(v\) are constant functions in the whole complex plane, and \(f\) reduces to a constant.

10. Compute

\[
\int_0^\infty \frac{dx}{x^4 + 1}.
\]

1 pt.

Let \(p(x) = 1\) and \(q(x) = x^4 + 1\). Clearly, \(q\) has complex roots, so \(q(x) \neq 0\) for all \(x \in \mathbb{R}\). Since the degree of \(q\) is larger than 2, we can apply the theorem of indefinite integrals. We first compute the singularities of the integrand in complex form. The zeros of \(q(z) = z^4 + 1\) are given by the fourth roots of \(-1,\)

\[
z_k = \exp \left( i \left( \frac{\pi}{4} + \frac{2\pi k}{4} \right) \right)
\]

for \(k = 0, 1, 2, 3\). Only the roots \(z_0 = (1 + i)/\sqrt{2},\ z_1 = (-1 + i)/\sqrt{2}\) lie in the upper half plane, so we shall only compute the residues for these two singularities. Also notice that, since \(q'(z) = 4z^3 = 0\) if and only if \(z = 0\), we only have simple zeros. It follows that

\[
\text{Res}_{z=z_k} \frac{p(z)}{q(z)} = \frac{1}{4z_k^3}.
\]

Hence

\[
\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2i\pi \left( e^{-\frac{3\pi}{4}} + e^{-\frac{9\pi}{4}} \right) = \frac{i\pi}{2} \left( -\frac{1}{\sqrt{2}}(1 + i) + \frac{1}{\sqrt{2}}(1 - i) \right) = \frac{i\pi}{2\sqrt{2}} (-1 - i + 1 - i) = \frac{\pi}{\sqrt{2}}
\]

Since the given integrand is an even function, we conclude

\[
\int_0^\infty \frac{dx}{x^4 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}}.
\]

A final question

What is the relationship between Complex Analysis and the animated gif of a solar flare found in the course webpage?