6. SUPER SPACETIMES AND SUPER POINCARÉ GROUPS

6.1. Super Lie groups and their super Lie algebras. The definition of a super Lie group within the category of supermanifolds imitates the definition of Lie groups within the category of classical manifolds. A *real super Lie group* G is a real supermanifold with morphisms

$$m : G \times G \longrightarrow G, \qquad i : G \longrightarrow G$$

which are multiplication and inverse, and

$$1 : \mathbf{R}^{0|0} \longrightarrow G$$

defining the unit element, such that the usual group axioms are satisfied. However in formulating the axioms we must take care to express then entirely in terms of the maps $m, i, 1$. To formulate the associativity law in a group, namely, $a(bc) = (ab)c$, we observe that $a, b, c \longmapsto (ab)c$ may be viewed as the map $I \times m : a, (b, c) \longmapsto a, bc$ of $G \times (G \times G) \longrightarrow G \times G$ (I is the identity map), followed by the map $m : x, y \longmapsto xy$. Similarly one can view $a, b, c \longmapsto (ab)c$ as $m \times I$ followed by m. Thus the associativity law becomes the relation

$$m \circ (I \times m) = m \circ (m \times I)$$

between the two maps from $G \times G \times G$ to G. We leave it to the reader to formulate the properties of the inverse and the identity. The identity of G is a point of G_{red}. It follows almost immediately from this that if G is a super Lie group, then G_{red}

is a Lie group in the classical sense. Also we have defined real super Lie groups above without specifying the smoothness type. One can define smooth or analytic Lie groups by simply taking the objects and maps to be those in the category of analytic super manifolds. Similarly for complex super Lie groups.

The functor of points associated to a super Lie group reveals the true character of a super Lie group. Let G be a super Lie group. For any supermanifold S let $G(S)$ be the set of morphisms from S to G. The maps $m, i, 1$ then give rise to maps

$$m_S : G(S) \times G(S) \longrightarrow G(S), \qquad i_S : G(S) \longrightarrow G(S), \qquad 1_S : 1_S \longrightarrow G(S)$$

such that the group axioms are satisfied. This means that the functor

$$S \longmapsto G(S)$$

takes values in groups. Moreover, if T is another supermanifold and we have a map $S \longrightarrow T$, the corresponding map $G(T) \longrightarrow G(S)$ is a homomorphism of groups. Thus $S \longmapsto G(S)$ is a *group-valued functor.* One can also therefore define a super Lie group as a functor

$$S \longmapsto G(S)$$

from the category of supermanifolds to the category of groups which is representable by a supermanifold G. The maps $m_S : G(S) \times G(S) \longrightarrow G(S)$, $i_S : G(S) \longrightarrow G(S)$, and 1_S then define, by Yoneda's lemma, maps $m, i, 1$ that convert G into a super Lie group and $S \longmapsto G(S)$ is the functor of points corresponding to G. A morphism of super Lie groups $G \longrightarrow H$ is now one that commutes with $m, i, 1$. It corresponds to homomorphisms

$$G(S) \longrightarrow H(S)$$

that are functorial in S. If G and H are already defined, Yoneda's lemma assures us that morphisms $G \longrightarrow H$ correspond one-one to homomorphisms $G(S) \longrightarrow H(S)$ that are functorial in S.

The actions of super Lie groups on supermanifolds are defined exactly in the same way. Thus if G is a super Lie group and M is a supermanifold, actions are defined either as morphisms $G \times M \longrightarrow M$ with appropriate axioms or as actions $G(S) \times M(S) \longrightarrow M(S)$ that are functorial in S; again Yoneda's lemma makes such actions functorial in S to be in canonical bijection with actions $G \times M \longrightarrow M$.

Sub super Lie groups are defined exactly as in the classical theory. A super Lie group H is a subgroup of a super Lie group if H_{red} is a Lie subgroup of G_{red} and the inclusion map of H into G is a morphism which is an immersion everywhere. One of the most usual ways of encountering sub super Lie groups is as stabilizers of points in actions. Suppose that G acts on M and m is a point of M_{red}. Then, for any supermanifold S, we have the stabilizer $H(S)$ of the action of $G(S)$ on $M(S)$ at the point m_S of $M(S)$. The assignment $S \longmapsto H(S)$ is clearly functorial in S. It can the be shown that this functor is representable, and that the super Lie group it defines is a closed sub super Lie group of G.

As in the classical case products of super Lie groups are super Lie groups. The opposite of a super Lie group is also one.

The super Lie algebra of a super Lie group. In the classical theory the Lie algebra of a Lie group is defined as the Lie algebra of left (or right) invariant vector fields on the group manifold with the bracket as the usual bracket of vector fields. The left invariance guarantees that the vector field is uniquely determined by the tangent vector at the identity; one starts with a given tangent vector at the identity and then translates it to each point to obtain the vector field. In the case of a super Lie group we follow the same procedure, but much more care is required because we have to consider not only the topological points but others also. The main point is that if M is a supermanifold and v is a vector field on M - for instance defined in local coordinates as $\sum_i a_i \partial/\partial x^i + \sum_j b_j \partial/\partial\theta^j$ where a_i, b_j are sections of the structure sheaf locally, then v is *not* determined by the tangent vectors it defines at each point.

For a super Lie group it is now a question of making precise what is a left invariant vector field. If we are dealing with a classical Lie group G, the left invariance of a vector field X is the relation $\ell_x \circ X = X \circ \ell_x$ for all $x \in G$ where ℓ_x is left translation by x, i.e.,

$$X_y f(xy) = (Xf)(xy)$$

for all $x, y \in G$ where X_y means that X acts only on the second variable y. This can also be written as

$$(I \otimes X) \circ m^* = m^* \circ X \tag{1}$$

where m^* is the sheaf morphism from $\mathcal{O}_G$ to $\mathcal{O}_{G\times G}$ corresponding to the multiplication $m : G \times G \longrightarrow G$. Now this definition can be taken over to the super case without change. The following is the basic theorem.

Theorem 6.1.1. *The Lie algebra* $\mathfrak{g}$ *of a super Lie group* G *is the set of all vector fields* X *on* G *satisfying* (1)*. It is a super Lie algebra of the same dimension as* G*. The map* $X \longmapsto X_1$ *that sends* $X \in \mathfrak{g}$ *to the tangent vector at the identity point* 1 *is a linear isomorphism of super vector spaces. If* τ *is a tangent vector to* G *at* 1*, the vector field* $X \in \mathfrak{g}$ *such that* $X_1 = \tau$ *is the unique one such that*

$$Xf = (I \otimes \tau)(m^*(f)) \tag{2}$$

for all (local) sections of $\mathcal{O}_G$. *Finally, the even part of* $\mathfrak{g}$ *is the Lie algebra of the classical Lie group underlying* G, *i.e.,*

$$\mathfrak{g}_0 = \mathrm{Lie}(G_{\mathrm{red}}).$$

Remark. We shall not prove this here. Notice that equation (2) can be interpreted formally as

$$(Xf)(x) = (\tau_y)(f(xy)). \tag{3}$$

Thus the elements of the Lie algebra of G can be obtained by differentiating the group law exactly as we do classically. This will become clear in the examples we consider below. However as a simple example consider $G = \mathbf{R}^{1|1}$ with global coordinates x, θ. We introduce the group law

$$(x,\theta)(x',\theta') = (x + x' + \theta\theta', \theta + \theta')$$

with the inverse

$$(x,\theta)^{-1} = (-x, -\theta).$$

The Lie algebra is of dimension $1|1$. If D_x, D_θ are the left invariant vector fields that define the tangent vectors $\partial_x = \partial/\partial x, \partial_\theta = \partial/\partial\theta$ at the identity element 0, and D^r_x, D^r_θ are the corresponding right invariant vector fields, then the above recipe yields

$$\begin{aligned} D_x &= \partial_x, \qquad & D^r_x &= \partial_x \\ D_\theta &= \theta\partial_x + \partial_\theta, \qquad & D^r_\theta &= -\theta\partial_x + \partial_\theta. \end{aligned}$$

It is now an easy check that

$$[D_x, D_\theta] = 2D_x$$

(all other commutators are zero) giving the structure of the Lie algebra. A similar method yields the Lie algebras of $GL(p|q)$ and $SL(p|q)$; they are respectively $\mathfrak{gl}(p|q)$ and $\mathfrak{sl}(p|q)$.

Theorem 6.1.2. *For morphism* $f : G \longrightarrow G'$ *of super Lie groups* G, G' *we have its differential* Df *which is a morphism of the corresponding super Lie algebras, i.e.,* $Df : \mathfrak{g} \longrightarrow \mathfrak{g}'$. *It is uniquely determined by the relation* $Df(X)_{1'} = df_1(X_1)$ *where* $1, 1'$ *are the identity elements of* G, G' *and* df_1 *is the tangent map* $T_1(G) \longrightarrow T_{1'}(G')$. *Moreover* f_{red} *is a morphism* $G_{\mathrm{red}} \longrightarrow G'_{\mathrm{red}}$ *of classical Lie groups.*

The fundamental theorems of Lie go over to the super category without change. All topological aspects are confined to the classical Lie groups underlying the super

Lie groups. Thus, a morphism $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}'$ comes from a morphism $G \longrightarrow G'$ if and only if $\alpha_0 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ comes from a morphism $G_{\text{red}} \longrightarrow G'_{\text{red}}$. The story is the same for the construction of a super Lie group corresponding to a given super Lie algebra: given a classical Lie group H with Lie algebra $\mathfrak{g}_0$, there is a unique super Lie group G with $\mathfrak{g}$ as its super Lie algebra such that $G_{\text{red}} = H$. The classification of super Lie algebras over **R** and **C** and their representation theory thus acquires a geometric significance that plays a vital role in supersymmetric physics.

Super affine algebraic groups. There is another way to discuss Lie theory in the supersymmetric context, namely as algebraic groups. In the classical theory algebraic groups are defined as groups of matrices satisfying polynomial equations. Examples are $GL(n), SL(n), SO(n), Sp(2n)$ and so on. They are affine algebraic varieties which carry a group structure such that the group operations are morphisms. If R is a commutative k-algebra with unit element, $G(S)$ is the set of solutions to the defining equations; thus we have $GL(n,R), SL(n,R), SO(n,R), Sp(2n,R)$. In general an affine algebraic group scheme defined over k is a functor $R \longmapsto G(R)$ from the category of commutative k-algebras with units to the category of groups which is representable. Representability means that there is a commutative algebra with unit, $k[G]$ say, such that

$$G(R) = \text{ Hom } (k[G], R)$$

for all R. By Yoneda's lemma the algebra $k[G]$ acquires a coalgebra structure, an antipode, and a co unit, converting it into a *Hopf algebra.* The generalization to the super context is almost immediate: a super affine algebraic groups defined over k is a functor

$$R \longmapsto G(R)$$

from the category of supercommutative k-algebras to the category of groups which is representable, i.e., there is a supercommutative k-algebra with unit, $k[G]$ say, such that

$$G(R) = \text{ Hom } (k[G], R)$$

for all R. The algebra $k[G]$ then acquires a super Hopf structure. The theory can be developed in parallel with the transcendental theory. Of course in order to go deeper into the theory we need to work with general super schemes, for instance when we deal with homogeneous spaces which are very often not affine but projective. The Borel subgroups and the super flag varieties are examples of these.

6.2. The Poincaré-Birkhoff-Witt theorem. The analog for super Lie algebras of the Poincaré-Birkhoff-Witt (PBW) theorem is straightforward to formulate. Let

$\mathfrak{g}$ be a super Lie algebra and T the tensor algebra over $\mathfrak{g}$. We denote by I the two-sided ideal generated by

$$x \otimes y - (-1)^{p(x)p(y)} y \otimes x - [x, y]1 \qquad (x, y \in \mathfrak{g})$$

and define

$$\mathcal{U} = \mathcal{U}(\mathfrak{g}) = T/I.$$

Then $\mathcal{U}$ is a super algebra, as I is homogeneous in the $\mathbf{Z}_2$-grading of T inherited from that on $\mathfrak{g}$, and we have a natural map $p : \mathfrak{g} \longrightarrow \mathcal{U}$. The pair $(\mathcal{U}, p)$ has the following universal property: if A is a super algebra with associated super Lie algebra $A_L([x, y] = xy - (-1)^{p(x)p(y)} yx)$ and $f : \mathfrak{g} \longrightarrow A_L$ is a morphism of super Lie algebras, there is a unique morphism $f^\sim : \mathcal{U} \longrightarrow A$ such that $f^\sim(p(X)) = f(X)$ for all $X \in \mathfrak{g}$. It is clear that $(\mathcal{U}, p)$ is uniquely determined by this universality requirement. $(\mathcal{U}, p)$ is called the *universal enveloping algebra of* $\mathfrak{g}$. The PBW theorem below will imply that p is injective. So it is usual to identify $\mathfrak{g}$ with its image by p inside uu and refer to $\mathcal{U}$ itself as the universal enveloping algebra of $\mathfrak{g}$. For simplicity we restrict ourselves to the case when $\mathfrak{g}$ is of countable dimension. Thus bases can be indexed either by the set of integers from 1 to some integer N or by the set of all integers.

Theorem 6.2.1(PBW). *Let k be a commutative ring with unit in which 2 and 3 are invertible. Let $\mathfrak{g}$ be a super Lie algebra over k which is a free k-module with a countable homogeneous basis. Let notation be as above. then the map p of $\mathfrak{g}$ into $\mathcal{U}$ is an imbedding. If $(X_a), (X_\alpha)$ are bases for $\mathfrak{g}_0, \mathfrak{g}_1$ respectively, then the standard monomials*

$$X_1^{a_1} \ldots X_r^{i_r} X_{\alpha_1} \ldots X_{\alpha_s} \qquad (a_1 \leq \ldots \leq a_r, \alpha_1 < \ldots < \alpha_s)$$

form a basis for $\mathcal{U}$. In particular,

$$\mathcal{U} \simeq \mathcal{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$$

as super vector spaces.

Remark. In recent times, as the notion of the Lie algebra has been generalized to include Lie super algebras and quantum groups, the PBW theorem has also been generalized to these contexts. It seems useful to point out that one can formulate and prove a single result from which the PBW theorems in the various contexts follow quite simply. The following treatment is nothing more than a synopsis of a paper by George M. Bergman[1]. See also[2].

We work over a commutative ring k with unit. We wish to construct a basis for an associative k-algebra A given by a set of generators with relations of a special type. Let T be the tensor algebra over k determined by the generators and I the two-sided ideal generated by the relations. In the special contexts mentioned above there is a natural k–module of tensors spanned by the so called *standard monomials* and denoted by S. The problem is to find conditions such that $T = S \oplus I$; then the images of a basis of S in $A = T/I$ will furnish a basis for A. Following Bergman we speak of *words* instead of monomial tensors.

Let X be a set whose elements are called letters and let W be the set of words formed from the letters, i.e., the elements of X, including the null word 1; W is a semigroup with 1 as unit, the product ww' of the words w, w' being the word in which w is followed by w'. T is the free k-module spanned by W whose elements will be called *tensors.* We are given a family $(w_\sigma)_{\sigma\in\Sigma}$ of words and for each $b \in B$ a tensor $f_b \in T$; we assume that for $b\sigma \neq b\sigma'$, $w_\sigma \neq w_{\sigma'}$. Our interest is in the algebra generated by the elements of X with relations

$$w_\sigma = f_\sigma \qquad (\sigma \in \Sigma).$$

A word is called *standard* if it does not contain any of the words $w_\sigma(\sigma \in \Sigma)$ as a subword. Let S be the free k-module spanned by the standard words. Elements of S will be called the *standard tensors.* We write I for the two-sided ideal in T generated by the elements $w_\sigma - f_\sigma$, namely, the k-span of all tensors of the form

$$u(w_b\sigma - f_\sigma)v \qquad (\sigma \in \Sigma, u, v \in W).$$

The theorem sought for is the statement that

$$T = S \oplus I.$$

We shall refer to this as the *basic theorem.* To see how this formulation includes the classical PBW theorem, let $X = (x_i)$ be a basis of a Lie algebra over k where the indices i are linearly ordered. Then B is the set of pairs i, j with $i > j$. The words w_σ are $x_i x_j$ $(i > j)$ and f_σ is $x_j x_i + [x_i, x_j]$ so that the relations defining the universal enveloping algebra are

$$x_i x_j = x_j x_i + [x_i, x_j] \qquad (i > j).$$

A word is then standard if it is of the form $x_{i_1 i_2 \ldots i_r}$ where $i_1 \leq i_2 \leq \ldots \leq i_r$ and S is the usual k-span of standard monomials in the basis elements (x_i).

The natural way to prove the basic theorem is to show that every word is congruent to a standard tensor mod I and that this standard tensor is uniquely determined. We shall say that the standard tensor is a *reduced expression* of the original word and the process of going from the given word to its reduced expression a *reduction procedure.* The procedure of reduction is quite simple. We check if the given word is already standard, and if it is not, then it must contain a subword $w_\sigma(\sigma \in \Sigma)$ which we replace by f_σ; we call this an *elementary reduction.* We repeat this process for the words in the tensor thus obtained. We hope that this process ends in a finite number of steps, necessarily in a standard tensor, and that the standard tensor thus obtained is independent of the reduction algorithm. The ambiguity of the reduction process stems from the fact that a given word may contain several words $w_\sigma(\sigma \in \Sigma)$ as subwords and any one of them can be replaced by f_σ in the next step. If the reduction process exists and is unambiguous, we have an operator R from T to S which is a projection on S. We shall see below that the existence and uniqueness of the reduction to standard form is equivalent to the basic theorem.

Before going ahead let us look at an example where X has three elements $x_i(i = 1, 2, 3)$ and we start with the relations

$$[x_i - x_j] := x_i x_j - x_j x_i = x_k \qquad (ijk) \text{ is an even permutation of } (123).$$

These are the commutation rules of the rotation Lie algebra and we know that the PBW theorem is valid where the standard words are the ones $x_1^{r_1} x_2^{r_2} x_3^{r_3}$. But suppose we change these relations slightly so that the Jacobi identity is not valid, for instance let

$$[x_1, x_2] = x_3, \quad [x_2, x_3] = x_1, \quad [x_3, x_1] = x_3$$

Let us consider two ways of reducing the nonstandard word $x_3x_2x_1$. We have

$$x_3x_2x_1 \equiv x_2x_3x_1 - x_1^2 \equiv x_2x_1x_3 + x_2x_3 - x_1^2 \equiv x_1x_2x_3 - x_1^2 + x_2x_3 - x_1 - x_3^2$$

where we start by an elementary reduction of x_3x_2. If we start with x_2x_1 we get

$$x_3x_2x_1 \equiv x_3x_1x_2 - x_3^2 \equiv x_1x_3x_2 + x_3x_2 - x_3^2 \equiv x_1x_2x_3 - x_1^2 + x_2x_3 - x_1 - x_3^2.$$

Hence we have $x_1 \in I$. The PBW theorem has already failed. From the commutation rules we get that $x_3 \in I$ so that $I \supset I'$ where I' is the two-sided ideal generated by x_1, x_3. On the other hand, all the relations are in I' so that $I \subset I'$. Hence $I = I'$, showing that $T = k[x_2] \oplus I$. Thus $A \simeq k[x_2]$.

We shall now make a few definitions. Words containing a $w_\sigma(\sigma \in \Sigma)$ as a subword are of the form $uw_\sigma v$ where $u, v \in W$; for any such we define the *elementary reduction operator* $R_{uw_\sigma v}$ as the linear operator $T \longrightarrow T$ that fixes any word $\neq uw_\sigma v$ and sends $uw_\sigma v$ to $uf_\sigma v$. If $w_\sigma \neq f_\sigma$, then this operator fixes a tensor if and only if it is a linear combination of words different from $uw_\sigma v$. *We shall assume from now on that $w_\sigma \neq f_\sigma$ for all $\sigma \in \Sigma$.* A finite product of elementary reduction operators is called simply a *reduction operator.* A tensor t is *reduction finite* if for any sequence R_i of elementary reduction operators the sequence $R_1 t, R_2 R_1 t, \ldots, R_k R_{k-1} \ldots R_1 t$ stabilizes, i.e., for some n, $R_k \ldots R_1 t = R_n \ldots R_1 t$ for all $k > n$. Clearly the set T_f of reduction finite tensors is a k-module which is stable under all elementary reduction operators. The set of tensors which is the k-span of words different from any word of the form $uw_\sigma v(u, v \in W, \sigma \in \Sigma)$ is denoted by S and its elements are called *standard.* These are the tensors which are fixed by all the reduction operators. If $t \in T_f$ it is easy to see that there is a reduction operator R such that $Rt = s \in S$; s is said to be a *reduced form of* t. If all standard reduced forms of t are the same, t is called *reduction unique* and the set of all such tensors is denoted by T_u. T_u is also a k-module, $S \subset T_u \subset T_f$, T_u is stable under all reduction operators, and the map that sends $t \in T_u$ to its unique reduced standard form is a well defined linear operator that is a *projection* from T_u to S. We shall denote it by $\mathcal{R}$. Clearly if $t \in T_u$ and L is a reduction operator, $\mathcal{R}(Lt) = \mathcal{R}t$. To see that T_u is closed under addition, let $t, t' \in T_u$ and let t_0, t'_0 be their reduced forms. Then $t + t' \in T_f$; if M is a reduction operator such that $M(t + t') = u_0 \in S$, we can find reduction operators L, L' such that $LMt = t_0, L'LMt' = t'_0$, so that $u_0 = L'LM(t + t') = t_0 + t'_0$, showing that $t + t' \in T_u$ and $\mathcal{R}(t + t') = \mathcal{R}t + \mathcal{R}t'$.

We shall now show that when $T = T_f$, the basic theorem, namely, the assertion that $T = S \oplus I$ is equivalent to the statement that every word is reduction finite, i.e., $T_u = T$. Suppose first that $T = S \oplus I$. If $t \in T$ and R is an elementary reduction operator, it is immediate that $t \equiv Rt \bmod I$. Hence this is true for R any reduction operator, elementary or not, so that any reduced form s of t satisfies $t \equiv s \bmod I$. But then s must be the projection of t on $S \bmod I$. Hence s is uniquely determined by t, showing that $t \in T_u$. Conversely suppose that $T_u = T$. Then $\mathcal{R}$ is a projection operator on S so that $T = S \oplus K$ where K is the kernel of $\mathcal{R}$. It is now a question of showing that $K = I$. Suppose that $t \in K$. Since $t \equiv Rt \bmod I$ for any reduction operator R and $0 = \mathcal{R}t = Rt$ for some reduction operator R, it follows that $t \in I$, showing that $K \subset I$. On the other hand, consider $t = uw_\sigma v$ where $\sigma \in \Sigma$. If R is the elementary reduction operator $R_{uw_\sigma v}$, we know that $\mathcal{R}t = \mathcal{R}(Rt) = \mathcal{R}(uf_\sigma v)$. Hence $\mathcal{R}(u(w_\sigma - f_\sigma)v) = 0$, showing that $\mathcal{R}$ vanishes on I. Thus $I \subset K$. So $K = I$ and we are done.

We now have the following simple but important lemma.

Lemma 6.2.2. *Let $u, v \in W$ and $t \in T$. Suppose that utv is reduction unique and R is a reduction operator. Then $u(Rt)v$ is also reduction unique and $\mathcal{R}(utv) = \mathcal{R}(u(Rt)v)$.*

Proof. It is clearly sufficient to prove this when R is an elementary reduction operator $R_{aw_\sigma c}$ where $a, c \in W$ and $\sigma \in \Sigma$. Let R' be the elementary reduction operator $R_{uaw_\sigma cv}$. Then $R'(utv) = u(Rt)v$. Since $utv \in T_u$, we have $u(Rt)v = R'(utv) \in T_u$ also and $\mathcal{R}(u(Rt)v) = \mathcal{R}(R'(utv)) = \mathcal{R}(utv)$.

The basic question is now clear: when can we assert that every tensor is reduction unique? Since the ambiguities in the reduction process are due to several words w_σ being present in a given word it is reasonable to expect that if we verify that in the simplest possible situations where there are two such words present the reduction is unambiguous, then it will be unambiguous in general. However it is not obvious that the process of reduction of a tensor terminates in a finite number of steps in a standard tensor. To ensure this we consider a partial order on the words such that for any $\sigma \in \Sigma$, f_σ is a linear combination of words *strictly less than* w_σ; it is then almost obvious that any tensor can be reduced to a standard form in a finite number of steps. More precisely let $<$ be a partial order on W with the following properties ($w' > w$ means $w < w'$):

(i) $1 < w$ for all $w \neq 1$ in W.

(ii) $w < w'$ implies that $uwv < uw'v$ for all $u, w, w', v \in W$.

(iii) $<$ satisfies the descending chain condition: any sequence w_n such that $w_1 > w_2 > \ldots$ is finite.

(iv) For any $\sigma \in \Sigma$, f_σ is a linear combination of words $< w_\sigma$.

The descending chain condition implies that any subset of W has minimal elements. *From now on we shall assume that W has been equipped with such a partial order.* If w is a word and t is a tensor, we shall write $t < w$ if t is a linear combination of words $< w$. For any linear space L of tensors we write $L_{<w}$ the subspace of L consisting of elements which are $< w$.

First of all we observe that under this assumption $T_f = T$. For, if some word is not reduction finite, there is a minimal such word, say w. w cannot be standard; if R is an elementary reduction operator with $Rw \neq w$, we must have $w = uw_\sigma v$ for some $\sigma \in \Sigma$ and words u, v, and $R = R_{uw_\sigma v}$. But then $Rw = uf_\sigma v < w$ so that Rw is in T_f. This implies that w is in T_f. We now consider the ambiguities in the reduction process. These, in their simplest form, are of two kinds. The ambiguity of type O, the *overlap ambiguity,* is a word $w_1w_2w_3$ where the w_i are words and there are $\sigma, \tau \in \Sigma$ such that $w_1w_2 = w_\sigma, w_2w_3 = w_\tau$. In reducing such an element we may begin with $w_1w_2 = w_\sigma$ and replace it by f_σ or we may begin with $w_2w_3 = w_\tau$ and replace it by f_τ. The second type is type I, the *inclusion ambiguity,* which is a word $w_1w_2w_3$ where $w_2 = w_\sigma, w_1w_2w_3 = w_\tau$. We shall say that the ambiguities are *resolvable* if there are reduction operators R', R'' such that $R'(f_\sigma w_3) = R''(w_1f_\tau) \in S$ in the type O case and $R'(w_1f_\sigma w_3) = R''(f_\tau) \in S$ in the type I case. The basic result is the following.

Theorem 6.2.3(Bergman). *Assume that W is equipped with an order as above. Then the basic theorem is true if and only if all ambiguities are resolvable.*

Proof. Let us assume that all ambiguities are resolvable and prove that the PBW is valid. As we have already observed, every element of T is reduction finite and so it comes to showing that every word is reduction unique. This is true for the null word 1 and we shall establish the general case by induction. Let w be any word and let us assume that all words $< w$ are reduction unique; we shall prove that w is also reduction unique.

Let R_1, R_2 be two elementary reduction operators such that $R_1w \neq w, R_2w \neq w$. We shall prove that $R - 1w$ and R_2w are reduction unique and have the same reduced form. We must have $R_1 = R_{u_1w_\sigma v_1}$, and $R_2 = u_2w_\tau v_2$ for some $\sigma, \tau \in \Sigma$. We may assume that in w the subword w_σ begins earlier than the subword w_τ. Three cases arise. First we consider the case when w_σ and w_τ overlap. Then $w = uw_1w_2w_3v$ where $w_1w_2 = w_\sigma$ and $w_2w_3 = w_\tau$. By assumption there are reduction operators R', R'' such that $R'(f_\sigma w_3) = R''(w_1f_\tau)$. On the other hand for any elementary reduction operator $R_0 = R_{aw_\theta b}$ ($\theta \in \Sigma$) we have the reduction operator $uR_0v = R_{ua\theta bv}$. So for any reduction operator $R^\sim$, elementary or not, we have a reduction operator $R^\sim_{uv}$ such that for all $t \in T$, $uR^\sim tv = R^\sim_{uv}t$. So if $R'_1 = R'_{uv}, R''_1 = R''_{uv}$, we have $R'_1(uf_\sigma w_3) = R''_1(uw_1f_\tau v)$. But as $f_\sigma < w_\sigma, f_\tau < w_\tau$, we see that $uf_\sigma w_3v < uw_\sigma w_3v = w, uw_1f_\tau v < uw_1w_\tau v = w$ so that $uf_\sigma w_3$ and $uw_1f_\tau v$ are both in $T_{<w}$. Since $\mathcal{R}_{<w}$ is well-defined on $T_{<w}$ and the above two elements can be reduced to the same element in S, they must have the same image under any reduction operators that takes them to reduced form. In other words, R_1w and R_2w have the same reduced form as we wanted to prove. The case when w_σ is a subword of w_τ is similar. The third and remaining case is when w_σ and w_τ do not overlap. This is the easiest of all cases. We can then write $w = uw_\sigma xw_\tau v$. Then $R_1w = uf_\sigma xw_\tau v, R_2w = uw_\sigma xf_\tau v$. We can reduce w_τ in R_1w and w_σ in R_2w to get $uf_\sigma xf_\tau v$ in both cases. This element is in $T_{<w}$ and so has a unique reduced form. So R_1w and R_2w have the same reduced forms under suitable reductions, and as these are in $T_{<w}$, this reduced form is their unique reduced expression. Hence we again conclude that w is reduction unique. Finally, the converse assertion that for

PBW to be valid it is necessary that all ambiguities must be resolvable, is obvious. This proves the theorem.

Proofs of the PBW theorem for Lie algebras and super Lie algebras. The first application is to the classical PBW theorem for the case of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra over a commutative ring k with unit as above which is free as a k-module. Let $(X_i)_{i\in B}$ be a basis for $\mathfrak{g}$ over k. We assume that B has a total order (this is no restriction) so that for any two indices $i, j \in B$ we have one and only one of the following three possibilities: $i < j, j < i, i = j$; we write $i > j$ for $j < i$ and $i \leq j$ if i is either $< j$ or $= j$. W is the set of all words with the letters $X_i (i \in B)$. A word $X_{i_1}X_{i_2}\ldots X_{i_m}$ is standard if $i_1 \leq i_2 \leq \ldots \leq i_m$. Let $[\ ,\ ]$ be the bracket in $\mathfrak{g}$, so that $[X_i, X_j] = \sum_m c_{ijm}X_m, c_{ijm} \in k$. We take Σ to be the set of pairs (i, j) with $i, j \in B, i > j$; and for $(i, j) \in Si$, $w_{i,j)} = X_iX_j$ with $f_{(i,j)} = X_jX_i + \sum_m c_{ijm}X_m$. To define the order in W we proceed as follows. For any word $w = X_{i_1}X_{i_2}\ldots X_{i_m}$ we define its *rank* $rk(w)$ to be m and *index* $i(w)$ to be the number of pairs (a, b) with $a < b$ but $i_a > i_b$. Then a word is standard in our earlier sense if and only if it is standard in the present sense. The ordering of words is by saying that $w < w'$ if either $rk(w) < rk(w')$ or if $rk(w) = rk(w')$ but $i(w) < i(w')$. All the conditions discussed above are satisfied and so to prove the PBW theorem we must check that all ambiguities are resolvable. Since all the words in Σ have rank 2 there are only overlap ambiguities which are words of length 3 of the form $X_rX_jX_i$ where $i < j < r$. We must show that the tensors

$$X_jX_rX_i + [X_r, X_j]X_i, \qquad X_rX_iX_j + X_r[X_j, X_i]$$

have identical reductions to standard forms under suitable reduction operators. The first expression can be reduced to

$$X_iX_jX_r + X_j[X_r, X_i] + [X_j, X_i]X_r + [X_r, X_j]X_i$$

while the second reduces to

$$X_iX_jX_r + [X_r, X_i]X_j + X_i[X_r, X_j] + X_r[X_j, X_i].$$

The quadratic terms in these expressions admit further reduction. For a commutator $[X, Y]$ with $X, Y \in \mathfrak{g}$ and any index $m \in B$ let us write $[X, Y]_{>m}$ to be the expression in terms of the basis containing only the X_a with $a > m$, and similarly when $> m$ is replaced by $<, \leq, \geq$. Notice now that the quadratic terms in the above two expressions differ by the reversal of the multiplications. Now, for any index c the reduction to standard form of $[X, Y]X_c$ and $X_c[X, Y]$ $(X, Y \in \mathfrak{g})$ is given by

$$\begin{array}{l} [X, Y]X_c = [X, Y]_{\leq c}X_c + X_c[X, Y]_{>c} + [[X, Y]_{>c}, X_c] \\ X_c[X, Y] = X_c[X, Y]_{>c} + [X, Y]_{\leq c}X_c + [X_c, [X, Y]_{\leq c}] \end{array}.$$

Hence the difference between these two reduced forms is

$$[X_c, [X, Y]].$$

It follows from this calculation that the two reduced expressions for the word $X_r X_j X_i$ differ by

$$[X_r, [X_j, X_i]] + [X_j, [X_i, X_r]] + [X_i, [X_r, X_j]]$$

which is 0 precisely because of the Jacobi identity.

The second application is when $\mathfrak{g}$ is a Lie super algebra. Recall that $\mathfrak{g}$ is $\mathbf{Z}_2$-graded with a bracket [,] satisfying the skew symmetry condition

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X]$$

and the Jacobi identity which encodes the fact that the adjoint map is a representation; writing as usual $\mathrm{ad}X : Y \longmapsto [X, Y]$, the Jacobi identity is the statement that $\mathrm{ad}[X, Y] = \mathrm{ad}X\mathrm{ad}Y - (-1)^{p(X)p(Y)}\mathrm{ad}Y\mathrm{ad}X$, i.e., for all $X, Y, Z \in \mathfrak{g}$ we have

$$[[X, Y], Z] = [X, [Y, Z]] - (-1)^{p(x)p(y)}[Y, [X, Z]].$$

In these as well as other formulae below $p(X)$ is the parity of X which is 0 for X even and 1 for X odd. If U is the universal enveloping algebra of $\mathfrak{g}$, the skew symmetry becomes, when both X and Y are odd, the relation $2X^2 = [X, X]$. For this to be an effective condition we assume that 2 is invertible in the ring k and rewrite this relation as

$$X^2 = (1/2)[X, X] \qquad (p(X) = 1).$$

Furthermore, when we take $X = Y = Z$ all odd in the Jacobi identity we get $3[X, X] = 0$ and so we shall assume 3 is invertible in the ring k and rewrite this as

$$[[X, X], X] = 0.$$

For the PBW theorem we choose the basis (X_i) to be homogeneous, i.e., the X_i are either even or odd. Let $p(i)$ be the parity of X_i. The set Σ is now the set of pairs (i, j) with either $i > j$ or (i, i) with i odd. The corresponding $w_{(i,j)}$ are

$$w_{(i,j)} = X_i X_j \ (i > j), \qquad w_{(i,i)} = X_i^2 \ (p(i) = -1)$$

and the $f_{(i,j)}$ are given by

$$\begin{array}{ll} f_{(i,j)} = (-1)^{p(i)p(j)} X_j X_i + [X_j, X_i] & (i > j) \\ f_{(i,i)} = (1/2)[X_i, X_i] \qquad (p(i) = -1) & \end{array}.$$

Once again the only ambiguities are of the overlap type. These are the words $X_r X_j X_i$ where now we have to consider $i \leq j \leq r$. We have to consider various case where there may be equalities. The first case is of course when $i < j < r$.

$i < j < r$: We want to show that the reduction to standard form of $X_r X_j X_i$ is the same whether we start with $X_r X_j$ or $X_j X_i$. Starting with $X_r X_j$ we find the expression, with $q = p(i)p(j) + p(j)p(r) + p(r)p(i)$,

$$\begin{array}{r} (-1)^q X_i X_j X_r + [X_r, X_j] X_i + (-1)^{p(r)p(j)} X_j [X_r, X_i] + \\ (-1)^{p(r)p(j)+p(r)p(i)} [X_j, X_i] X_r \end{array}. \tag{1}$$

For the expression starting from $X_j X_i$ we find

$$\begin{array}{r} (-1)^q X_i X_j X_r + X_r [X_j, X_i] + (-1)^{p(i)p(j)} [X_r, X_i] X_j + \\ (-1)^{p(i)p(j)+p(r)p(i)} X_i [X_r, X_j] \end{array}. \tag{2}$$

Apart from the cubic term which is standard these expressions contain only quadratic terms and these need further reduction. For any three indices a, b, c we have, writing $t = p(c)p(a) + p(c)p(b)$,

$$\begin{array}{l} [X_a, X_b] X_c = [X_a, X_b]_{\leq c} X_c + (-1)^t X_c [X_a, X_b]_{>c} + [[X_a, X_b]_{>c}, X_c] \\ X_c [X_a, X_b] = X_c [X_a, X_b]_{>c} + (-1)^t [X_a, X_b]_{\leq c} X_c + [X_c, [X_a, X_b]_{\leq c}] \end{array}.$$

If c is even the two expressions on the right side above are already standard because the term $[X_a, X_b]_{\leq c} X_c$ is standard as there is no need to reduce X_c^2; if c is odd we have to replace X_c^2 by $(1/2)[X_c, X_c]$ to reach the standard form. If E_1, E_2 are the two standard reduced expressions, it follows by a simple calculation that

$$E_1 - (-1)^t E_2 = [[X_a, X_b]_{>c}, X_c] - (-1)^t [X_c, [X_a, X_b]_{\leq c}].$$

Using the skew symmetry on the second term we get

$$E_1 - (-1)^{p(c)p(a)+p(c)p(b)} E_2 = [[X_a, X_b], X_c]. \tag{3}$$

We now apply this result to the reductions of the two expressions in (1) and (2). Let S_1 and S_2 be the corresponding standard reductions. Using (3), we find for $S_1 - S_2$ the expression

$$[[X_r, X_j], X_i] - (-1)^{p(i)p(j)}[[X_r, X_i], X_j] + (-1)^{p(r)p(j)+p(r)p(i)}[[X_j, X_i], X_r].$$

Using skew symmetry this becomes

$$[[X_r, X_j], X_i] - [X_r, [X_j, X_i]] + (-1)^{p(r)p(j)}[X_j, [X_r, X_i]]$$

which is 0 by the Jacobi identity.

$i = j < r, p(i) = -1$ or $i < j = r, p(j) = -1$: These two cases are similar and so we consider only the first of these two alternatives, namely, the reductions of $X_j X_i X_i$ with $i < j$ and i odd (we have changed r to j). The two ways of reducing are to start with $X_j X_i$ or $X_i X_i$. The first leads to

$$X_i X_i X_j + (-1)^{p(j)} X_i [X_j, X_i] + [X_j, X_i] X_i.$$

The second leads to

$$(1/2) X_j [X_i, X_i].$$

We proceed exactly as before. The reduction of the first expression is

$$\frac{1}{2}\Big\{[X_i, X_i]_{\le j} X_j + X_j [X_i, X_i]_{>j} + [[X_i, X_i]_{>j}, X_j]\Big\} + (-1)^{p(j)}[X_i, [X_j, X_i]].$$

The second expression reduces to

$$(1/2)[X_i, X_i]_{\le j} X_j + (1/2) X_j [X_i, X_i]_{>j} + (1/2)[X_j, [[X_i, X_i]_{\le j}.$$

The difference between these two is

$$(1/2)[[X_i, X_i], X_j] + (-1)^{p(j)}[X_i, [X_j, X_i]]$$

which is 0 by the Jacobi identity.

$i = j = r$, i odd: We can start with either the first $X_i X_i$ or the second one. The difference between the two reduced expressions is

$$(1/2)[[X_i, X_i], X_i]$$

which is 0 by the Jacobi identity.

If we order the indices such that all even induced come before the odd ones and we use Latin for the even and Greek for the odd indices, we have a basis (X_i, X_α) and the PBW theorem asserts that the monomials

$$X_{i_1} X_{i_2} \ldots X_{i_r} X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_s} \quad (i_1 \leq i_2 \leq \ldots \leq i_r, \alpha_1 < \alpha_2 < \ldots < \alpha_s)$$

form a basis for the universal enveloping algebra of the super Lie algebra $\mathfrak{g}$. We have thus finished the proof of Theorem 6.2.1.

We note that the ring k has been assumed to have the property that 2 and 3 are invertible in it. In particular this is true if k is a $\mathbf{Q}$-algebra, for instance if k is a field of characteristic 0, or even if its characteristic is different from 2 and 3.

6.3. The classical series of super Lie algebras and groups. Over an algebraically closed field k one can carry out a classification of simple super Lie algebras similar to what is done in the classical theory. A super Lie algebra $\mathfrak{g}$ is *simple* if it has no proper nonzero ideals and $\mathfrak{g} \neq k^{1|0}$. A super Lie algebra $\mathfrak{g}$ is called *classical* if it is simple and $\mathfrak{g}_0$ acts completely reducibly on $\mathfrak{g}_1$, i.e., $\mathfrak{g}_1$ is a direct sum of irreducible $\mathfrak{g}_0$-modules. Then one can obtain a complete list of these. Let us introduce, for any field k the following super Lie algebras.

$\mathfrak{gl}(p|q)$: This is the super Lie algebra $M_L^{p|q}$.

$\mathfrak{sl}(p|q)$: This is given by

$$\mathfrak{sl}(p|q) = \{X \in \mathfrak{gl}(p|q) \mid \mathrm{str}(X) = 0\}.$$

We write

$$A(p|q) = \begin{cases} \mathfrak{sl}(p+1|q+1) & \text{if } p \neq q, p, q \geq 0 \\ \mathfrak{sl}(p+1|q+1)/kI & \text{where } p \geq 1 \end{cases}.$$

For $A(p|q)$ the even parts and the odd modules or as follows.

$$\mathfrak{g} = A(p|q) : \mathfrak{g}_0 = A(p) \oplus A(q) \oplus k, \mathfrak{g}_1 = f_p \otimes f_q' \otimes k$$
$$\mathfrak{g} = A(p|p) : \mathfrak{g}_0 = A(p) \oplus A(p), \mathfrak{g}_1 = f_p \otimes f_p'$$

where the f's are the defining representations and the primes denote duals.

$\mathrm{osp}(\Phi)$: Let $V = V_0 \oplus V_1$ be a super vector space and let Φ be a symmetric nondegenerate even bilinear form $V \times V \longrightarrow k$. Then Φ is symmetric nondegenerate on $V_0 \times V_0$, symplectic on $V_1 \times V_1$, and is zero on $V_i \otimes V_j$ where $i \neq j$. Then

$$\mathrm{osp}(\Phi) = \{L \in \underline{\mathrm{End}}(V) \mid \Phi(Lx, y) + (-1)^{p(L)p(x)} \Phi(x, Ly) = 0 \text{ for all } x, y \in V\}.$$

This is called the *orthosymplectic super Lie algebra associated with* Φ. It is an easy check that this is a super Lie algebra. If k is algebraically closed Φ has a unique standard form and then the corresponding super Lie algebra takes a standard appearance. The series $\mathrm{osp}(\Phi)$ splits into several subseries.

$$\begin{aligned} B(m,n) &= \mathrm{osp}(2m+1|2n)\ (m \geq 0, n \geq 1) \\ D(m,n) &= \mathrm{osp}(2m|2n)\ (m \geq 2, n \geq 1) \\ C(n) &= \mathrm{osp}(2|2n-2)\ (n \geq 2). \end{aligned}$$

The even parts of these and the corresponding odd parts as modules for the even parts are given as follows.

$$\begin{aligned} \mathfrak{g} = B(m,n) &: \mathfrak{g}_0 = B(m) \oplus C(n), \mathfrak{g}_1 = f_{2m+1} \otimes f'_{2n} \\ \mathfrak{g} = D(m,n) &: \mathfrak{g}_1 = f_{2m} \otimes f'_{2n} \\ \mathfrak{g} = C(n) &: \mathfrak{g}_1 = k \otimes f_{2n-2}. \end{aligned}$$

$P(n)(n \geq 2)$: This is the super Lie algebra defined by

$$P(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \middle| \mathrm{tr}(a) = 0, b \text{ symmetric }, c \text{ skew symmetric } \right\}.$$

The Q-series is a little more involved in its definition. Let us consider the super Lie algebra $\mathfrak{gl}(n+1|n+1)$ of all matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and let us define the *odd trace* $\mathrm{otr}(g) = \mathrm{tr}(b)$. Let

$$Q^{\sim}(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| \mathrm{tr}(b) = 0 \right\}$$

and let

$$Q(n) = Q^{\sim}(n)/kI_{2n+2}.$$

For the even parts and the odd modules we have

$$\begin{aligned} \mathfrak{g} = P(n) &: \mathfrak{g}_0 = \mathfrak{sl}(n+1|n+1), \mathfrak{g}_1 = \mathrm{Symm}^2(n+1) \oplus \Lambda^2(n+1) \\ \mathfrak{g} = Q(n) &: \mathfrak{g}_0 = A(n), \mathfrak{g}_1 = \mathrm{ad}A(n). \end{aligned}$$

Theorem 6.3.1(Kac). *Let k be algebraically closed. Then the simple and classical super Lie algebras are precisely*

$$A(m|n), B(m|n), D(m|n), C(n), P(n), Q(n)$$

and the following exceptional series:

$$F(4), G(3), D(2|1,\alpha) \quad (\alpha \in k \setminus (0, \pm 1).$$

Remark. For all of this see[3]. Here is some additional information regarding the exceptional series:

$$\mathfrak{g} = F(4) : \mathfrak{g}_0 = B(3) \oplus A(1), \mathfrak{g}_1 = \text{spin}(7) \otimes f_2, \dim = 24|16$$
$$\mathfrak{g} = G(3) : \mathfrak{g}_0 = G(2) \oplus A(1), \mathfrak{g}_1 = \mathbf{7} \otimes \mathbf{2}, \dim = 17|14$$
$$\mathfrak{g} = D(2|1,\alpha) : \mathfrak{g}_1 = A(1) \oplus A(1) \oplus A(1), \mathfrak{g}_1 = \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}, \dim = 9|8.$$

The interesting fact is that the $D(2|1,\alpha)$ depend on a *continuous parameter.*

The classical super Lie groups. We restrict ourselves only to the linear and orthosymplectic series.

$GL(p|q)$: The functor is $S \longmapsto GL(p|q)(S)$ where S is any supermanifold and $GL(p|q)(S)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a \in GL(p)(\mathcal{O}(S)_0), b \in GL(q)(\mathcal{O}(S)_0)$ while b, c are matrices with entries from $\mathcal{O}(S)_1$. The representing supermanifold is the open submanifold of the affine space of dimension $p^2 + q^2|2pq$ defined by $GL(p) \times GL(q)$.

$SL(p|q)$: The functor is $S \longmapsto SL(p|q)(S)$ where $SL(p|q)(S)$ is the kernel of the Berezinian. The representing supermanifold is the submanifold of $GL(p|q)$ defined by the condition that the Berezinian is 1. One can also view it as the kernel of the morphism Ber from $GL(p|q)$ to $GL(1|0)$.

$\text{osp}(m|2n)$: The functor is $S \longmapsto \text{osp}(m|2n)(S)$ where $\text{osp}(m|2n)(S)$ is the subgroup of $GL(m|2n)(S)$ fixing the appropriate even symmetric bilinear form Φ. The representability criterion mentioned earlier applies.

It is possible to describe the super Lie groups for the P and Q series also along similar lines. See Deligne-Morgan.

6.4. Super spacetimes. Super spacetimes are supermanifolds M such that M_{red} is a classical spacetime. They are constructed so that they are homogeneous spaces for super Poincaré groups which are super Lie groups acting on them.

Super Poincaré algebras. We have seen an example of this, namely the super Lie algebra of Gol'fand-Likhtman. Here we shall construct them in arbitrary dimension and Minkowski signature. Let V be a real quadratic vector space of signature $(1, D-1)$. The usual case is when $D = 4$ but other values of D are also of interest. For conformal theories V is taken to be of signature $(2, D-2)$. We shall not discuss the conformal theories here.

The Poincaré Lie algebra is the semidirect product

$$\mathfrak{g}_0 = V \times' \mathfrak{so}(V).$$

We shall denote by S a real spinorial representation of Spin (V). We know that there is a symmetric nonzero map

$$\Gamma : S \times S \longmapsto V \tag{1}$$

equivariant with respect to Spin (V); Γ is projectively unique if S is irreducible. Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus S.$$

We regard S as a $\mathfrak{g}_0$-module by requiring that V act as 0 on S. Then if we define

$$[s_1, s_2] = \Gamma(s_1, s_2) \qquad (s_i \in S)$$

then with the $\mathfrak{g}_0$-action on $\mathfrak{g}_1$ we have a super Lie algebra, because the condition

$$[s, [s, s]] = -[\Gamma(s, s), s] = 0 \qquad (s \in S)$$

is automatically satisfied since $\Gamma(s, s) \in V$ and V acts as 0 0n S. $\mathfrak{g}$ is a supersymmetric extension of the Poincaré algebra and is an example of a super Poincaré algebra. The Gol'fand-Likhtman algebra is a special case when $D = 3$ and S is the Majorana spinor. As another example we consider the case when $D = 3$. Then Spin (V) is $SL(2, \mathbf{R})$ and $SO(V)$ is its adjoint representation. Let S be the representation of $SL(2, \mathbf{R})$ in dimension 2. We have an isomorphism

$$\Gamma : \mathrm{Symm}^2 V \simeq V$$

and then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus S$$

as before. In the physics literature one takes a basis (Q_a) for S and a basis (P_μ) (linear momenta) for V. Then

$$\Gamma(Q_a, Q_b) = -2\Gamma^{\mu}_{ab} P_\mu \ (\Gamma^{\mu}_{ab} = \Gamma^{\mu}_{ba}).$$

The existence of Γ in (1) and its uniqueness when S is irreducible are thus critical for the construction of super Poincaré algebras.

The fact that Γ takes values in V means that

$$\mathfrak{l} = V \oplus S$$

is also a super Lie algebra. It is a supersymmetric extension of the abelian spacetime translation algebra V; but $\mathfrak{l}$ is *not* abelian as $\Gamma \neq 0$. However it is 2-step nilpotent, namely,

$$[a, [b, c]] = 0 \qquad (a, b, c \in \mathfrak{l}).$$

The corresponding super Lie groups will be the superspacetimes.

The super Lie group L corresponding to $\mathfrak{l}$ will be constructed by the exponential map. We have not discussed this but we can proceed informally and reach a definition which can then be rigorously checked. Using the Baker-Campbell-Hausdorff formula informally and remembering that triple brackets are zero in $\mathfrak{l}$, we have

$$\exp A \exp B = \exp(A + B + (1/2)[A, B]) \qquad (A, B \in \mathfrak{l}).$$

This suggests that we identify L with $\mathfrak{l}$ and *define* the group law by

$$A \circ B = A + B + (1/2)[A, B] \qquad (A, B \in \mathfrak{l}).$$

More precisely let us view the super vector space $\mathfrak{l}$ first as a supermanifold. If $(B_\mu), (F_a)$ are bases for V and S respectively, then Hom $(S, \mathfrak{l})$ can be identified with (β_μ, τ_a) where β_μ, τ_a are elements of $\mathcal{O}(S)$ which are even and odd respectively. In a basis independent form we can identify this with

$$\mathfrak{l}(S) := (\mathfrak{l} \otimes \mathcal{O}(S))_0 = V \otimes \mathcal{O}(S)_0 \oplus S \otimes \mathcal{O}(S)_1.$$

It is clear that $\mathfrak{l}(S)$ is a *Lie algebra*. Indeed, all brackets are zero except for pairs of elements of $S \otimes \mathcal{O}_1$, and for these the bracket is defined by

$$[s_1 \otimes \tau_1, s_2 \otimes \tau_2] = -\Gamma(s_1, s_2)\tau_1\tau_2 \qquad (\tau_1, \tau_2 \in \mathcal{O}(S)_1).$$

Notice that the sign rule has been used since the s_j and τ_j are odd; the super Lie algebra structure of $\mathfrak{l}$ is necessary to conclude that this definition converts $\mathfrak{l}(S)$ into a *Lie algebra.* (This is an example of the *even rules* principle which we have not discussed here.) We now take

$$L(S) = \mathfrak{l}(S)$$

and define a binary operation on $L(S)$ by

$$A \circ B = A + B + (1/2)[A, B] \qquad (A, B \in \mathfrak{l}(S)).$$

The Lie algebra structure on $\mathfrak{l}(S)$ implies that this is a group law. In the bases $(B_\mu), (F_a)$ defined above,

$$(\beta^\mu, \tau^a) \circ (\beta'^\mu, \tau'^a) = (\beta''_\mu, \tau''^a)$$

where

$$\beta''_\mu = \beta_\mu + \beta'_\mu - (1/2)\Gamma^\mu_{ab}\tau_a\tau'_b, \qquad \tau''^a = \tau'^\mu + \tau'^\mu.$$

Symbolically this is the same as saying that L has coordinates $(x^\mu), (\theta^a)$ with the group law

$$(x, \theta)(x', \theta') = (x'', \theta'')$$

where

$$x''^\mu = x^\mu + x'^\mu - (1/2)\Gamma^\mu_{ab}\theta^a\theta'^b, \qquad \theta''^a = \theta^a + \theta'^a \tag{2}$$

(with summation convention). The supermanifold L thus defined by the data V, S, Γ has dimension $\dim(V)|\dim(S)$. It is the underlying manifold of a super Lie group L with $L_{\mathrm{red}} = V$.

Because L is a super Lie group, one can introduce the left and right invariant differential operators on L that make differential calculus on L very elegant, just as in the classical case. Recall that the left invariant vector fields are obtained by differentiating the group law at $x'^\mu = \theta'^a = 0$ and for the right invariant vector fields we differentiate the group law with respect to the unprimed variables at 0. Let D_μ, D_a (D^r_μ, D^r_a) be the left (right) invariant vector fields with tangent vector $\partial/\partial x^\mu, \partial/\partial\theta^a$ at the identity element. Let ∂_μ, ∂_a be the global vector fields on L (the invariant vector fields on the abelian group obtained by identifying L with $\mathfrak{l}$). Then

$$\begin{aligned} D_\mu &= D^r_\mu = \partial_\mu \\ D_a &= (1/2)\Gamma^\mu_{ab}\theta^b\partial_\mu + \partial_a \\ D^r_a &= -(1/2)\Gamma^\mu_{ab}\theta^b\partial_\mu + \partial_a. \end{aligned}$$

It is an easy verification that

$$[D_a, D_b] = \Gamma^\mu_{ab}$$

as it should be.

When $D = 3$ we get the super spacetime $M^{3|2}$. Let $(F_a)_{a=1,2}$ be the standard basis for $S = \mathbf{R}^2$ and let $B_{ab} = F_a F_b$ (symmetric product) so that (B_{ab}) is a basis for V. The super space coordinates are (y^{ab}, θ^a). We have

$$\begin{aligned}\partial_a \theta^b &= \delta_a^b \\ \partial_{ab} y^{a'b'} &= (1/2)(\delta_a^{a'} \delta_b^{b'} + \delta_a^{b'} \delta_b^{a'}) \\ \partial_a y^{bc} &= \partial_{ab} \theta^c = 0.\end{aligned}$$

Also

$$\Gamma(F_a, F_b) = B_{ab} = B_{ba}.$$

The left invariant vector fields are

$$D_{ab} = \partial_{ab}, \qquad D_a = \partial_a + (1/2)\theta^b \partial_{ab}, \qquad D_a^r = \partial_a - (1/2)\theta^b \partial_{ab}.$$

Complex and chiral superspacetimes. The super spacetime constructed when $D = 4$ with S as the Majorana spinor is denoted by $M^{4|4}$. We shall now discuss a variant of the construction above that yields what are called chiral superspacetimes.

We take (F_a) and $(\overline{F}_{\dot{a}})$ as bases for S^+ and S^- so that if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then g acts on $S^\pm$ by

$$g^+ \sim \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad g^- \sim \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix}.$$

If $v = \sum u^a F_a, \overline{v} = \sum_{\dot{a}} \overline{u^{\dot{a}}} \overline{F}_{\dot{a}}$, then

$$\overline{g^+ v} = g^- \overline{v}.$$

On $S = S^+ \oplus S^-$ we define the conjugation σ by

$$\sigma(u, \overline{v}) = (v, \overline{u}).$$

Let

$$V_{\mathbf{C}} = S^+ \otimes S^-, \qquad B_{a\dot{b}} = F_a F_{\dot{b}},\ B_{\dot{a}b} = F_{\dot{a}} F_b \text{ tensor multiplication}).$$

The symmetric nonzero map

$$\Gamma : (S^+ \oplus S^-) \otimes (S^+ \oplus S^-) \longrightarrow V_{\mathbf{C}}$$

is nonzero only on $S^\pm \otimes S^\mp$ and is uniquely determined by this and the relations $\Gamma(F_a, \overline{F_{\dot b}}) = B_{a\dot b}$. So

$$\mathfrak{l}_{\mathbf{C}} = V_{\mathbf{C}} \oplus (S^+ \oplus S^-)$$

is a complex super Lie algebra and defines a complex Lie group $L_{\mathbf{C}}$ exactly as before; the group law defined earlier extends to $L_{\mathbf{C}}$. But now, *because we are operating over* $\mathbf{C}$*, the subspaces*

$$\mathfrak{l}^\pm_{\mathbf{C}} = V_{\mathbf{C}} \oplus S^\pm$$

are super Lie algebras over $\mathbf{C}$ *and determine corresponding complex super Lie groups* $L^\pm_{\mathbf{C}}$. Moreover, as Γ vanishes on $S^\pm \otimes S^\pm$, these are *abelian* and the super Lie algebras $\mathfrak{l}^\pm_{\mathbf{C}}$ are actually *abelian ideals of* $\mathfrak{l}_{\mathbf{C}}$. The $L^\pm_{\mathbf{C}}$ are the *chiral superspacetimes*; actually we define $L^+_{\mathbf{C}}$ as the chiral and $L^-_{\mathbf{C}}$ as the *antichiral* superspacetime. Moreover

$$L_{\mathbf{C}} = L^+_{\mathbf{C}} \times_{V_{\mathbf{C}}} L^-_{\mathbf{C}}$$

where the suffix denotes the fiber product.

We have conjugations on $V_{\mathbf{C}}$ and on $S^+ \oplus S^-$. On $V_{\mathbf{C}}$ the conjugation is given by

$$\sigma : u \otimes \overline{v} \longmapsto v \otimes \overline{u}$$

while the one on $S^+ \oplus S^-$, also denoted by σ, is

$$(u, \overline{v}) \longmapsto (v, \overline{u}).$$

The map Γ is compatible with these two conjugations. Hence we have a conjugation σ on $\mathfrak{l}_{\mathbf{C}}$ and hence on $L_{\mathbf{C}}$. We have

$$L = L^\sigma_{\mathbf{C}}.$$

In other words, L may be viewed as the real supermanifold defined inside $L_{\mathbf{C}}$ as the fixed point manifold of σ. If

$$y^{a\dot b}, \theta^a, \overline{\theta}^{\dot b}$$

are the coordinates on $L_{\mathbf{C}}$, then L is defined by the reality constraint

$$y^{a\dot b} = \overline{y^{b\dot a}}, \qquad \overline{\theta}^{\dot a} = \overline{\theta^a}.$$

The left invariant vector fields on $L_{\mathbf{C}}$ are the complex derivations ∂_μ and the $D_a, \overline{D}_{\dot a}$ with

$$D_a = \partial_a + (1/2)\overline{\theta}^{\dot b}\partial_{a\dot b}, \qquad \overline{D}_{\dot a} = \partial_{\dot a} + (1/2)\theta^b\partial_{b\dot a}$$

where repeated indices are summed over.

Let us now go over to new coordinates

$$z^{a\dot{b}}, \varphi^a, \overline{\varphi}^{\dot{b}}$$

defined by

$$y^{a\dot{b}} = z^{a\dot{b}} - (1/2)\varphi^b\overline{\varphi}^{\dot{a}},\ \theta^a = \varphi^a,\ \overline{\theta}^{\dot{a}} = \overline{\varphi}^{\dot{a}}.$$

Chiral (antichiral) superfields are those sections of the structure sheaf of $L_{\mathbf{C}}$ that depend only on z, φ $(z, \overline{\varphi})$. A simple calculation shows that

$$D_a = \partial/\partial\varphi^{\dot{a}}, \qquad \overline{D}_{\dot{a}} = \partial/\partial\overline{\varphi}^{\dot{a}}.$$

So it is convenient to use these coordinates which we can rename $y, \theta, \overline{\theta}$.

6.5. Super Poincaré groups. The super Poincaré algebra is $\mathfrak{g} = \mathfrak{g}_0 \oplus S$ where $\mathfrak{g}_0 = V \oplus \mathfrak{h}$; here $\mathfrak{h}$ is the Lie algebra $\mathfrak{so}(V)$. The Lie algebra of super spacetime is $\mathfrak{l} = V \oplus S$. Let $H = \mathrm{Spin}(V)$. Then H acts on $\mathfrak{l}$ as a group of super Lie algebra automorphisms of $\mathfrak{l}$. This action lifts to an action of L on the supermanifold L by automorphisms of the super Lie group L. The semidirect product

$$G = L \times' H$$

is the super Poincaré group. The corresponding functor is

$$S \longmapsto G(S)$$

where

$$G(S) = L(S) \times' H(S).$$

This description also works for $L_{\mathbf{C}}, L^{\pm}_{\mathbf{C}}$ with H replaced by the complex spin group.

Super field equations. Once super spacetimes are defined one can ask for the analogue of the Poincaré invariant field equations in the super context. This is a special case of the following more general problem: if M is a supermanifold and G is a super Lie group acting on M, find the invariant super differential operators D and the spaces of the solutions of the equations $D\Psi = 0$ where Ψ is a global section of the structure sheaf. In the case of super spacetimes this means the construction of the differential operators that extend the Klein-Gordon and Dirac operators. The superfields are the sections of the structure sheaf and it is clear that in terms of the components of the superfield we will obtain several ordinary field equations.

This leads to the notion of a *multiplet* and the idea that a super particle defines a multiplet of ordinary particles. We do not go into this aspect at this time.

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