Complex or p-adic wave functions?

This has been an issue from the beginning. Mathematically it is natural to work over the p-adic field. But as there are no Hilbert spaces we have to work in the category of Banach spaces over the field $\mathbf{Q}_p$. There is a beautiful theory of representations over such and so there is no serious problem understanding the role of the symmetry groups.

Already Heisenberg proposed the use of quasi Hilbert spaces where the metric is not positive definite-some people call them *Pontryagin spaces*-in quantum electrodynamics. So it is not inconceivable that p-adic Banach spaces can play a role in arithmetic variants of quantum theory.

Hamiltonians and Feynman-Kac formula

Let G be a locally compact abelian group and $\widehat{G}$ its character group. In applications it will be a vector space over a p-adic division ring or the adelic version of a vector space over a rational division ring. We have the Fourier transform isomorphism

$$\mathcal{F} : L^2(G) \simeq L^2(\widehat{G})$$

For functions (suitably restricted) f on G, $\widehat{g}$ on $\widehat{G}$,

$$M(f) : \psi \longmapsto f\psi, \ (\psi \in L^2(G)) \qquad \widehat{M}(\widehat{g}) = \mathcal{F}^{-1}M(\widehat{g})\mathcal{F}.$$

The $\widehat{M}(\widehat{g})$ are the analogs of the *pseudo differential operators*. For $G = \mathbf{R}^N$,

$$\widehat{M}(\widehat{g}) = -\Delta \quad \widehat{g} = \xi_1^2 + \xi_2^2 + \ldots + \xi_N^2.$$

We first consider the case where only space is non archimedean. The model for space time is $X = G \times \mathbf{R}$. For a quantum system with configuration space G, the $M(f)$ are the *position* and the $\widehat{M}(\widehat{g})$ are the *momentum* operators. The Hamiltonians are of the form

$$H = \widehat{M}(\widehat{g}) + M(V)$$

Here $\widehat{M}(\widehat{g})$ is the free Hamiltonian, V, the potential. If $V \geq 0$ H will be ≥ 0 (modulo some technical issues).

Path integral formalism

In the late 1940's R. Feynman, then a graduate student in Princeton, created in his Ph. D thesis (under John Archibald Wheeler) a new formalism for quantum mechanics that is nowadays called the *path integral formalism*. Assume that we are studying a scalar particle moving in $\mathbf{R}^N$. Given a Hamiltonian H the bilinear form $(e^{tH}\varphi, \psi^{\mathrm{conj}})$ is a distribution on $\mathbf{R}^N \times \mathbf{R}^N$ written as a *generalized kernel* $K_t(x, y)$, called the *propagator*. Feynman exhibited $K_t(x, y)$ as an integral over the space of classical paths of the system starting from x at time 0 and ending at y at time t with respect to a (vaguley defined) measure, and showed that the entire quantum theory can be built from this formula. In spite of the difficulties in making it rigorous this method has been profoundly influential. It has remarkable predictive power that has made QFT a very powerful instrument even in mathematics, at the hands of Witten (for one).

Mark Kac found a very beautiful variant of the Feynman path integral for the propagator in which the vague Feynman measure is replaced by Wiener measure, for a large class of Hamiltonians. This formula is known as the *Feynman-Kac formula.*

Kac on the F-K formula

... It must have been in the spring of 1947 that Feynman gave a talk at the Cornell Physics Colloquium based on some material from his 1942 Ph. D dissertation, which had not yet been published.

... A fundamental concept of quantum mechanics is a quantity called the propagator, and the standard way of finding it (in the nonrelativistic case) is by solving the Schrödinger equation. Feynman found another way based on what became known as the Feynman path integral or "the sum over histories". During his lecture Feynman sketched the derivation of his formula and I was struck by the similarity of his steps to those I had encountered in my work. In a few days I had my version of the formula, although it took some time to complete a rigorous proof. My formula connected solutions of certain differential equations closely related to the Schrödinger equation with Wiener integrals.

... It is only fair to say that I had Wiener's shoulders to stand on. Feynman, as in everything else he has done, stood on his own, a trick of intellectual contortion that he alone is capable of.

... I find Feynman's formula to be very beautiful. It connects the quantum mechanical propagator, which is a twentieth century concept, with the classical mechanics of Newton and Lagrange, in a uniquely compelling way....

From his autobiography *Enigmas of Chance*.

The F-K formula for the propagator in $\mathbf{R}^N$

Kac's big step was to observe that when the potential V is ≥ 0, the Hamiltonian H is ≥ 0, and so, by spectral theory, e^{-zH} makes sense for *complex* z with $\Re(z) \geq 0$. For $z = t$ *real* we get a very nice semigroup whose operators are given by genuine integral operators. For their kernels Kac obtained a formula as a Wiener integral. This transition is then from real time (Feynman) to imaginary time (Kac). The Schrödinger equation becomes the heat equation. The free Hamiltonian is $H_0 = \widehat{M_{\widehat{g}}}$ which is $-\Delta$ in the standard context. The free dynamics is given by e^{-tH_0}. Since $H_0 = \Delta$, this is multiplication by e^{-tr^2} in Fourier transform space and so is convolution by a gaussian density g_t. The g_t are the transition probabilities for the Wiener measure and we write $P_{t,x,y}$ for the *conditional* Wiener measure starting from x at time 0 and ending at y at time t. Then

$$K_t(x,y) = \int_{C[0,t]^N} e^{-\int_0^t V(\omega(s))ds} dP_{t,x,y}.$$

The integration is over the Banach space of continuous functions on $[0,t]$ with values in $\mathbf{R}^N$.

Path integral in the general locally compact case

We assume that $\widehat{g} \geq 0$ and $e^{-t\widehat{g}}$ is *positive definite* in the sense of Bochner for all $t \geq 0$. Then $e^{-t\widehat{g}}$ is the Fourier transform of a *probability measure* on p_t on G and $e^{-t\widehat{M}_{\widehat{g}}}$ is convolution by p_t. The p_t will gve rise to a stochastic process with independent increments, and we write $P_{t,x,y}$ for the corresponding conditional probability measures. Then

$$K_t(x,y) = \int e^{-\int_0^t V(\omega(s))ds} dP_{t,x,y}$$

exactly as before.

We can consider *matrix potentials.* The exponential in the integrand should then be replaced by what are called the *time ordered exponentials*.

p-adic and adelic cases

If $G = \mathbf{R}^d \simeq \widehat{G}$, $g(y) = |y|^2, f = V(x)$ we get

$$H = -\Delta + V$$

the classic form of the usual Hamiltonian. In the general case where there may not be differential operators, the $\widehat{M_g}$ are the *pseudo differential operators.* If

$$G = U \times \mathbf{R}$$
$$U = \text{ a finite dimensional Banach space over } \mathbf{Q}_p$$

we obtain a class of p-adic Hamiltonians

$$H = -\Delta_\alpha + V$$

if we take

$$g(y) = ||y||^\alpha, \quad f = V$$

In the p-adic case the positive definiteness conditions can be verified and so we obtain an analog of the Feynman-Kac formula. The Wiener measure would be replaced by a G-valued stochastic process with independent increments and paths which have only discntinuities of the first kind. There is no difficulty in principle in treating the adelic case. The existence of the F-K formula implies the uniqueness of the ground state if there is a ground state.

QFT

Quantum field theory was begun by Dirac in the 1920's when he worked out his radiation theory, which is the theory of the interaction of the atom with the electromagnetic field. The motivation for his work was the fact that quantum theory does not explain the fact that the atom makes spontaneous transitions between energy levels, and much less, that it does not give the probabilities for such emission and abosrption(derived earlier by Einstein using statistical mecahnics) as well as the Bohr formula (called the magic formula by Weyl)

$$E' - E = \hbar\nu.$$

Dirac realized that these are phenomena arising out of the interaction of the atom with the electromagnetic field, and treated the question by quantizing the electromagnetic field. His theory, which was non relativistic, explained all of the above features and also gave the first explanation of the wave-particle duality. Then Heisenberg and Pauli started the theory of general quantum fields with the additional proviso that the theory be compatible with the theory of relativity (special).

General quantized fields

In QFT the method of procedure was the same as in quantum mechanics: promote the classical observables to operators satisfying suitable commutation rules and assume that the Hilbert space becomes rigid with this kinematics, so that the dynamical operators can then be introduced as functions of the field operators. The classical field $A(x)$ consisting of scalar or vector valued functions on space time satisfying the field equations (Maxwell, for one) now became operator valued; one of the basic commutation rule was

$$[A(x), A(y)] = 0 \quad ((x-y)^2 < 0) \qquad (CR)$$

where $u^2 = (u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2$, explained by the fact that light signals from x will not reach y if $(x-y)^2 < 0$, so that the operators $A(x)$ and $A(y)$ will not interfere with each other.

This procedure was refined to include unobserved classical fields like the Dirac fields and the commutation rule became

$$[A(x), A(y)]_+ = 0 \quad ((x-y)^2 < 0) \qquad (ACR)$$

where $[\cdot, \cdot]_+$ is the *anticommutator*.

Smeared fields

In the 1940's Bohr-Rosenfeld made a systematic analysis of the measurement process in quantum field theory and discovered that the field strengths $A(x)$ are not observable at the quantum level because the measurement used macroscopic test bodies. They showed that only averages

$$\int_D A(x)d^4x$$

over small space time regions D can be observed, i.e., the fields have to be *smeared* before they can be observed. Thus one has a linear map

$$f \longmapsto A(f)$$

from the space of test functions into the space of (unbounded) operators, the field map. In spite of this the formalism was developed as if the fields $A(x)$ themselves made sense, but ran into serious difficulties because of divergences in the various formulae for energy etc. The technique of *renormalization* was created to overcome these difficulties and a very accurate theory of quantum electrodynamics was created by Schwinger, Feynman, Tomanaga.

Constructive (Rigorous) QFT

In the 1950's a number of people (Wightman notably, based on the work of Schwinger) created the general principles of QFT based on the following:

1. Field operators are defined for f in Schwartz space satisying the smeared forms of (CR) and (ACR).
2. There is a unitary representation of the Poincaré group in the Hilbert space $\mathcal{H}$ compatible with the field map
3. The specturm of the UR above is contained in the closed forward light cone (positivity of energy)
4. There is a unique vacuum state and that the application of the field operators on it generate a dense subspace of $\mathcal{H}$.

Such fields exist-certainly the free fields are known to be of this type. But construction of such fields on spacetime even for a limited class of interactions has proved elusive. Glimm and Jaffe were successful in the case when space time dimension was 2 and 3.

Analytic continuation of the vacuum expectation values

Schwinger pioneered the theory of the VEV (*vacuum expectation values*),

$$(\Omega, A(x_1)A(x_2)\ldots A(x_N)\Omega) \quad (\Omega = \text{ vacuum state})$$

where the x_i are points of space time, and noted that they can be analytically continued to *complex space time points*, hence ultimately into points with imaginary time. This space is called *Schwinger space time* and the conditions (CR) and (ACR) remarkably become simplified in the process. Schwinger space time is nothing but the euclidean space time which is a real form of complex space time. (This is a deep generalization of the unitarian trick of Weyl.) For (CR) the condition becomes *total symmetry* and so one could speak of a *probability measure* on the space of fields such that the VEV's (after analytic continuation) are its *moments*.

Remarks

Roughly speaking, even though the VEV's are not functions, the positivity of the spectrum makes the VEV distributions

$$(\Omega, A(f_1)A(f_2)\dots A(f_N)\Omega)$$

the *boundary values* of analytic functions existing in complex space time; the Lorentz invariance now becomes invariance with respect to the *complex Lorentz group* and so the VEV's extend to a huge domain, driven by analyticity and invariance; and this extended domain contains the Schwinger space time). This is very analogous to the analytic continuation from the Feynman to the Feynman-Kac formula discussed earlier.

Probability measures on spaces of tempered distributions

In the case of the F-K formula the probability measure is the Wiener measure or some other auxiliary measures, and these are defined on the space of paths. But in the case of higher space time dimension the measures or *not* defined on the space of fields. They are very singular and they are only defined on the space of tempered distributions on the Euclidean space time.

If this measure is given and satisfies an additional symmetry, the VEV's can be analytically continued backwards into Minkowski space time.

If the measure is gaussian the QFT it generates on Minkowski space time is *free*. To get QFT's with a non trivial scattering matrix the measure has to be non gaussian.

A. N. Kochubei and M. R. Sait-Ametov have extended methods of euclidean quantum field theory to the p-adic case and constructed non gaussian measures corresponding to suitable polynomial interactions. The measures correspond to cut offs, and in some cases the cut off can be removed. These results represent a beginning of p-adic QFT.